Skein Lasagna Modules for 2-handlebodies

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Joint with C. Manolescu

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$\mathcal{S}^N_{*,i,j}(B^4; L) = \mathrm{KhR}^{i,j}_N(L)$, supported in $* = 0$

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"Direct sum of the Khovanov-Rozansky homology groups of an infinite family of cables of K, modulo relations coming from cobordism maps between these cables."

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where $\underline{\mathrm{KhR}}_{N,\alpha}(K)$ is the cabled Khovanov-Rozansky homology of K.

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$\mathcal{S}_0^N(W_1 \natural W_2; L_1 \cup L_2; \Bbbk) \cong \mathcal{S}_0^N(W_1; L_1; \Bbbk) \otimes_{\Bbbk} \mathcal{S}_0^N(W_2; L_2; \Bbbk)$

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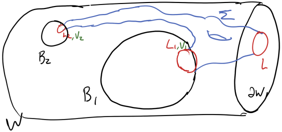

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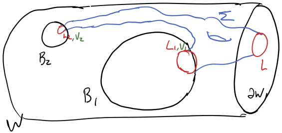

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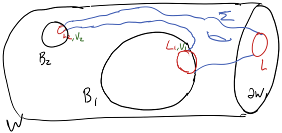

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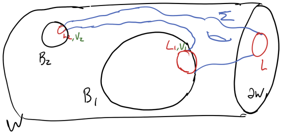

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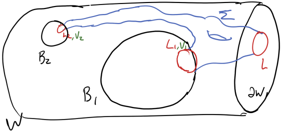

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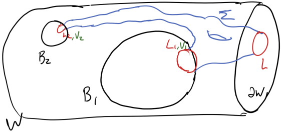

If $W = B^4$, can define map $\text{KhR}_N(\Sigma) : \otimes \text{KhR}_N(L_i) \to \text{KhR}_N(L)$ and an *evaluation* $\text{KhR}_N(F) = \text{KhR}_N(\Sigma)(\otimes v_i) \in \text{KhR}_N(L)$

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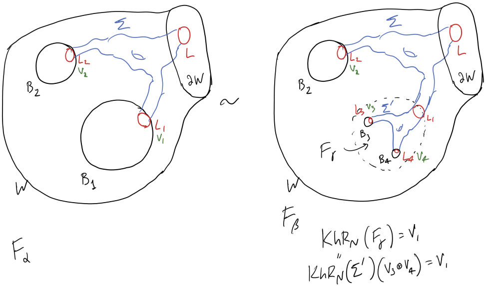
- $F_\alpha \sim F_\beta$ if F_β is obtained from F_α by inserting a filling F_γ of $(B^4; L_1)$ into an input ball B_1 in F_α and $\mathrm{KhR}_N(F_\gamma) = v_1$



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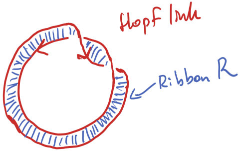

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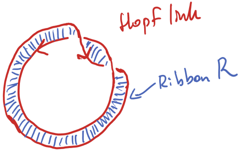

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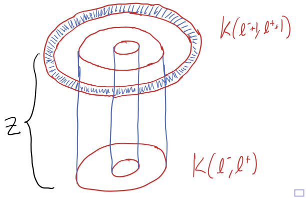

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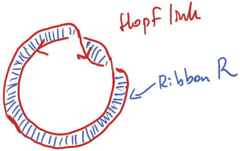

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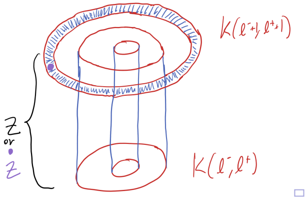

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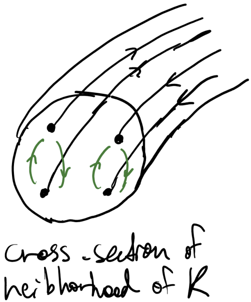

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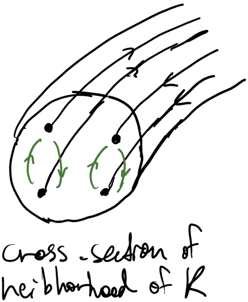
$\beta : B_{\ell^-,\ell^+} \to \mathrm{Aut}(\mathrm{KhR}_N(K(\ell^-, \ell^+)))$



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Cabled Khovanov-Rozansky Homology - Definition

The cabled Khovanov homology at level $\alpha \in \mathbb{Z}$ is

$$\underline{\mathrm{Kh}}_{2,\alpha}(K) := \bigoplus_{r \geq 0} \mathrm{Kh}\left(K(r - \alpha^-, r + \alpha^+)\right)\{-(2r + |\alpha|)\}/\sim$$

where $\alpha^\pm$ are the negative/postive parts of α, $\{-\}$ denotes a quantum degree shift and $\sim$ is generated by:

- $\beta(b)v \sim v$
- $\mathrm{Kh}(Z)v \sim 0$
- $\mathrm{Kh}(\dot{Z})v \sim v$

Proof of Main Theorem: Part I

Theorem

Let W be the 2-handlebody associated to K. Then we have an isomorphism,

$$\Phi : \underline{\mathrm{KhR}}_{N,\alpha}(K) \cong \mathcal{S}_0^N(W; \emptyset, \alpha)$$

Proof of Main Theorem: Part I

Take $|K| = 1, N = 2, \alpha = 0$ for simplicity.

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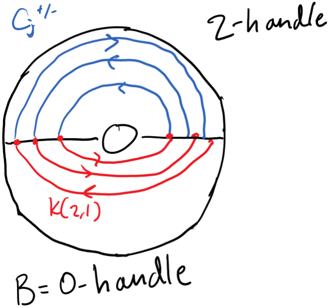
$\widetilde{\Phi} : \bigoplus_{r \geq 0} \mathrm{Kh}(K(r,r))\{-\} \twoheadrightarrow \mathcal{S}_0^2(W; \emptyset, 0)$:



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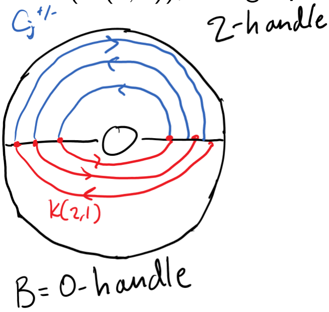

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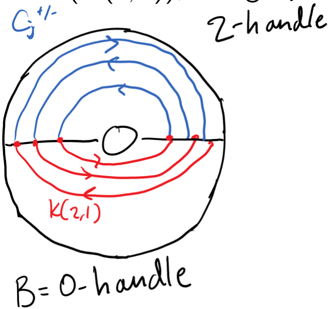

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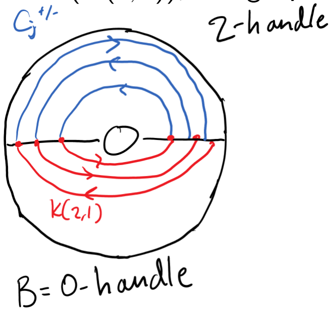

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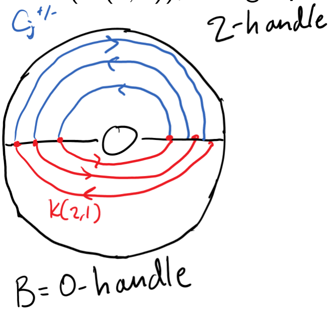

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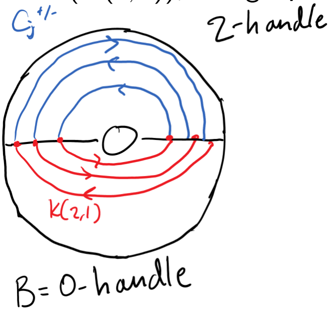
Set $\widetilde{\Phi}(v) = [F_v]$



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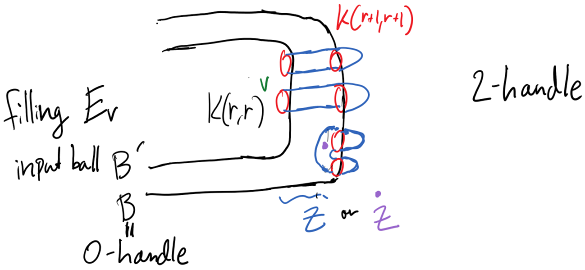

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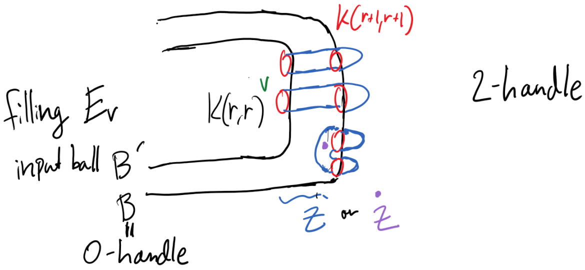

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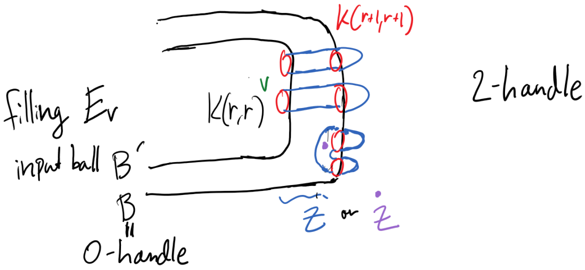

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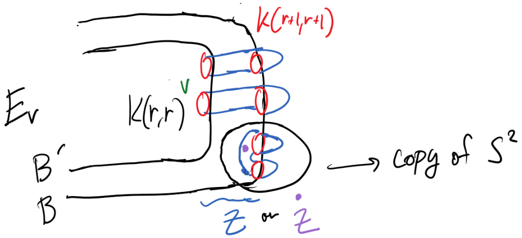

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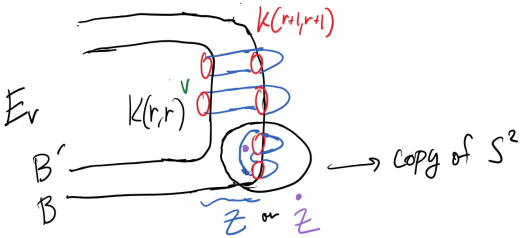

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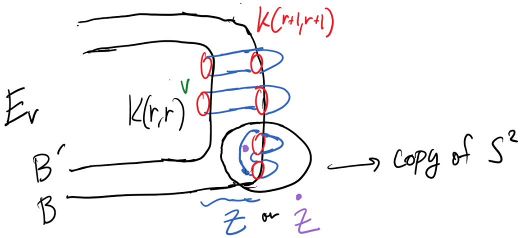

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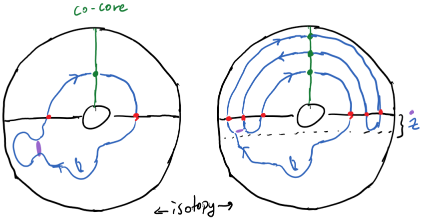
Consider different choices of isotopies. While the number of intersection points of Σ with the cocore remains constant, the motion of these points is described by a braid group element. When we introduce/cancel two intersection points, we are pushing a disc through the cocore, corresponding to the cobordism Z.



Thank you!