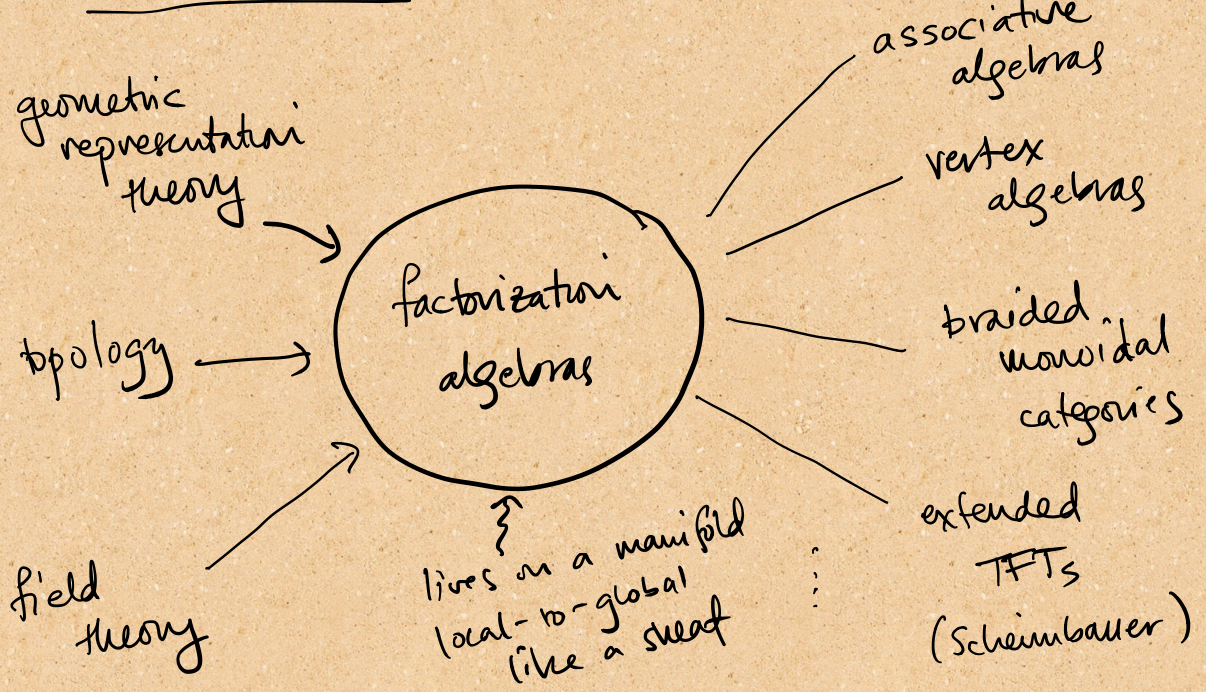
Bulk-boundary correspondences & factorization algebras

Owen Gwilliam
UMass Amherst

Big picture



Slogans:

- Every Lagrangian field theory T on a manifold X has a fact. algebra Obs^{cl}_T encoding its classical observables

  commutative

- If a perturbative quantization exists, it induces a deformation

  $\mathrm{Obs}^{cl} \rightsquigarrow \mathrm{Obs}^q$

as a factorization algebra

BV quantization of field theories $\longrightarrow$ deformation quantization of factorization algebras

Examples

dim 1 :

symplectic σ-model $\rightsquigarrow$ Fedosov quantization

$\mathbb{R} \xrightarrow{\sigma} (X, \omega)$

(Grady-Li-Li)

dim 2 :

holomorphic σ-model $\rightsquigarrow$ sheaf of vertex algs on X known as chiral diff. operators

$\mathbb{C} \xrightarrow{\sigma} X$

(Gorbounov-G.-Williams)

dim 3

Chern-Simons theory $\rightsquigarrow$ $\mathrm{Rep}(U_\hbar \mathfrak{g})$

"quantum group"

$\vdots$

there are many more examples

You can interpret important phenomena in this framework:

asymptotic freedom, Higgs mechanism,...

Goal today: how to understand bulk-boundary correspondences in this framework

(field theory on manifolds with boundary e.g. CS/WZW)

Outline

0. Intro
1. factorization algebras
2. examples $\longrightarrow$ abelian CS/WZW correspondence
3. nonabelian version

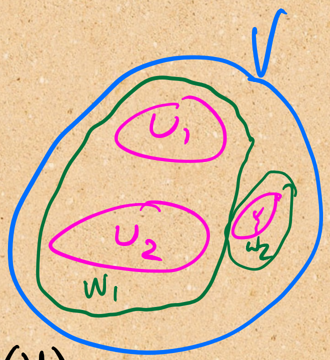
§1 Fact. Algs

Def A <u>prefactorization algebra</u> $\mathcal{F}$ on a manifold X with values in $Ch^{\otimes}$ consists of

- for each open set $U \subset X$, a cochain cplx $\mathcal{F}(U) \in Ch$
- for each inclusion $U \hookrightarrow V$, a cochain map

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

- for a finite collection of pairwise disjoint opens $U_1, \ldots, U_k \subset V$, a cochain map



$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \longrightarrow \mathcal{F}(V)$$

such that

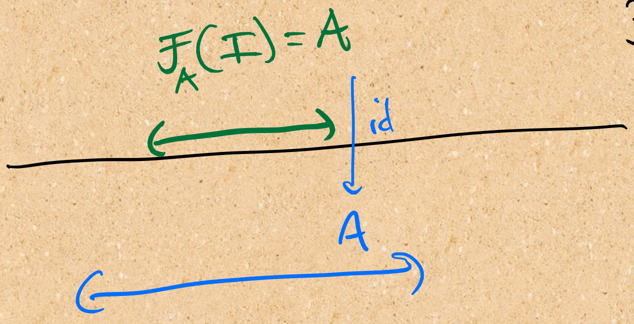
• the maps are "associative"

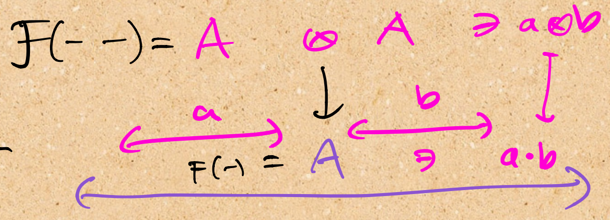
$$\begin{tikzcd}[column sep=small]
& \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \otimes \mathcal{F}(U_3) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\circlearrowright"] & \\
\mathcal{F}(W_1) \otimes \mathcal{F}(W_2) \arrow[rr] & & \mathcal{F}(V)
\end{tikzcd}$$

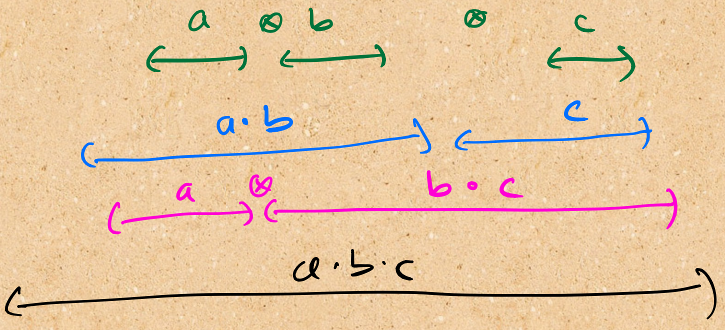
• the maps are equivariant under relabeling

<u>Ex</u> $X = \mathbb{R}$, $\mathrm{Vect}^{\otimes}$

associative alg A $\rightsquigarrow$ $\mathcal{F}_A$ on $\mathbb{R}$

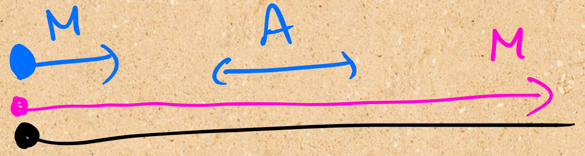






$$\begin{array}{ccc} I_1 & I_2 & I_3 \\ J_2 & J_3 & J_1 \end{array}$$

Ex bis assoc alg A & right A-module M $\rightsquigarrow \mathcal{F}_{A,M}$

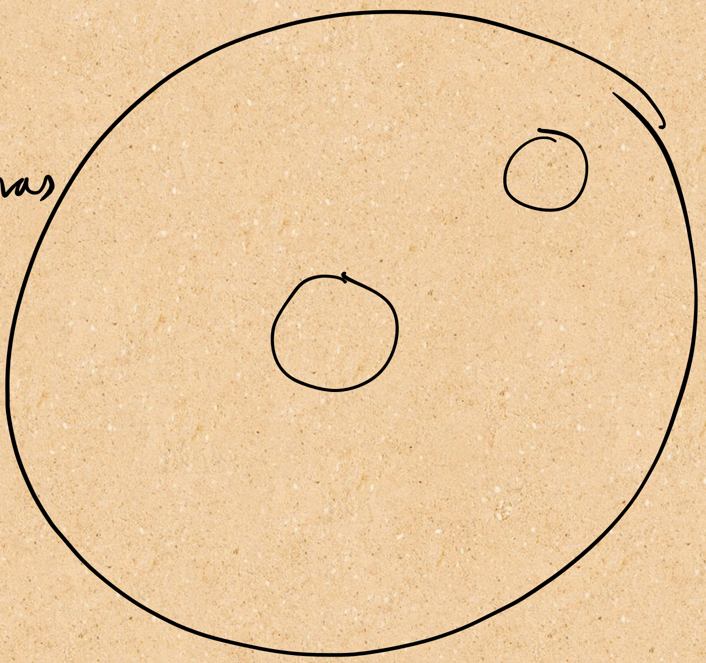


$\mathbb{R}_{\geq 0}$

$$\begin{array}{ccc} m \otimes a & \longmapsto & m \cdot a \\ \in & & \in \\ M \otimes A & & M \end{array}$$

Ex 2, 2d

$\mathcal{E}_2$ algebras



$$\begin{array}{c} A \otimes A \\ \downarrow \\ A \end{array}$$

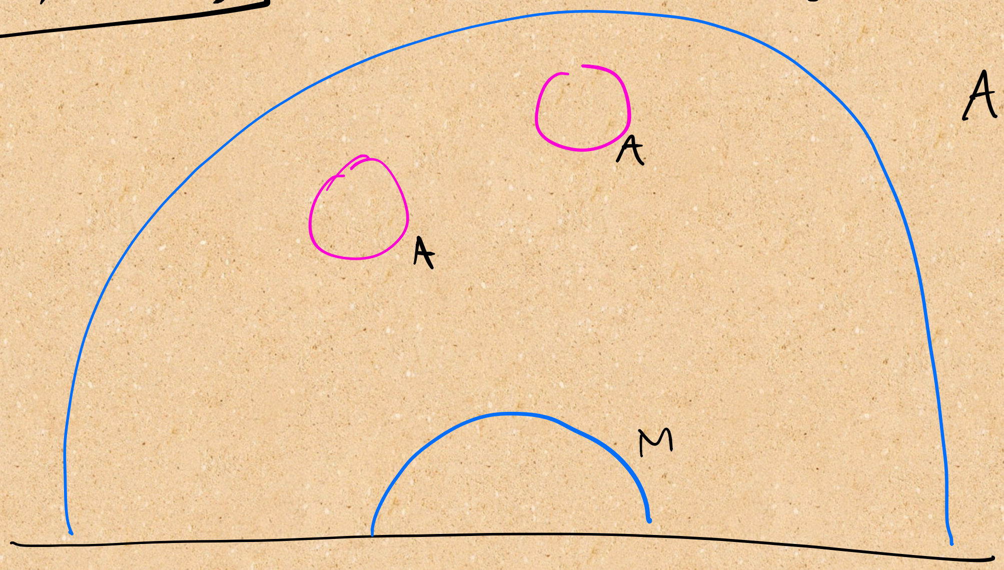
Thm (Lurie, Ayala-Francis)

$$\mathrm{Alg}_{E_n} \xrightarrow{\ \simeq\ } \mathrm{PreFactAlg}(\mathbb{R}^n)^{\text{locally constant}}$$

$$D \hookrightarrow D' \quad \text{disks}$$

$$\mathcal{F}(D) \xrightarrow{\ \simeq\ } \mathcal{F}(D')$$

Ex 2 bis) swiss cheese algebras



$$\begin{array}{c} A\otimes A\otimes M \\ \downarrow \\ M \end{array}$$

<u>Rmk</u> A <u>factorization algebra</u> is a prefact. alg. whose value on "big opens" is determined by its behavior on "small opens"

(Weiss cosheaf / excision)

---

② <u>How to construct examples</u>

We'll use concrete, diff. geometric methods

<u>Inputs</u>

$\Omega^*_c : \text{Opens}(X)^{\sqcup} \longrightarrow \text{Ch}^{\oplus}$

compactly supported de Rham cplx

$C^{Lie}_* : \text{Alg}_{Lie}(\text{Ch}_k)^{\times} \longrightarrow \text{Ch}_k^{\otimes}$

$\mathfrak{g} \longmapsto C^{Lie}_*(\mathfrak{g}, k)$

Key example of a factorization alg:
Pick a Lie alg $\mathfrak{g}$, manifold X

$$U_{X,\mathfrak{g}} : \mathrm{Open}(X)^{\sqcup} \longrightarrow \mathrm{Ch}^{\otimes}$$
$$U \longmapsto C_*^{\mathrm{Lie}}(\Omega_c^*(U) \otimes \mathfrak{g})$$

Prop For $X = \mathbb{R}$,

$U_{\mathbb{R},\mathfrak{g}}$ is equivalent to $\mathcal{F}_{U\mathfrak{g}}$

$\rightsquigarrow$ This construction recovers the enveloping algebra $U\mathfrak{g}$

Thm $X = S^1$

$$C_*^{\mathrm{Lie}}(\Omega^*(S^1) \otimes \mathfrak{g}) \simeq \mathrm{Hoch}_*(U\mathfrak{g})$$

On $\mathbb{C}$ (or any Riemann surface), consider the dg Lie algebra

$$\Omega^{0,*} \otimes \mathfrak{g}$$

with $\quad [\alpha \otimes x, \beta \otimes y] = \alpha \wedge \beta \otimes [x,y]$

and $\quad \bar{\partial} \otimes \mathrm{id}$

You can extend this:

$$\mathbb{C}\cdot \underbrace{c}_{\deg 1} \longrightarrow \hat{\mathfrak{g}}_\kappa \longrightarrow \Omega^{0,*}_c \otimes \mathfrak{g}$$

$$[\alpha \otimes x, \beta \otimes y]_{\wedge} = \alpha \wedge \beta \otimes [x,y] + c \cdot \underbrace{\int \alpha \wedge \partial \beta \cdot \kappa(x,y)}_{\text{Schwinger term}}$$

Curr

$\parallel$

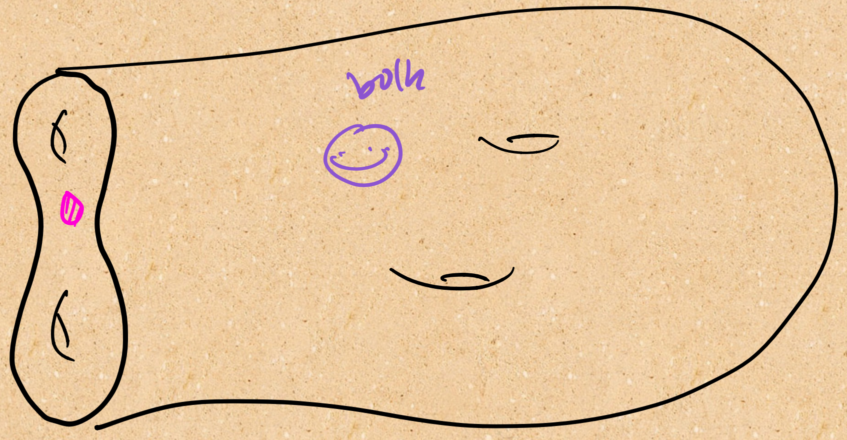
<u>Prop</u> $U_{\hat{\mathfrak{g}}_\kappa} = C_*^{Lie}(\hat{\mathfrak{g}}_\kappa)$ encodes the vertex algebra of ~~the~~ KM Lie alg.

Abelian Chern-Simons theory can be treated by the same method:

Prop The quantum of ab. CS is encoded by

$$\mathrm{Obs}^q_{CS} = \left(U \longmapsto C^{Lie}_*\Big(\underbrace{\Omega^*_c(U) \oplus \overbrace{\mathbb{C}\hbar}^{\deg 1}}_{\text{Heisenberg}}\Big) \right)$$

$$[\alpha, \beta] = \hbar \int \alpha \wedge \beta$$



$\Sigma = \partial X$ $\qquad$ X^3

If you pick a cplx str. on Σ,

it determines a fact alg on Σ

~~gssoc~~ $\mathcal{H}_p$ ~ Heisenberg VOA

But you can also put ab CS as

a fact alg on $X^3 \setminus \partial X$

<u>Thm</u> There is a stratified fact alg

on all of X s.t.

① in bulk: it ~~recovers~~ CS:

$U \cap \partial X = \emptyset, \quad \mathcal{F}(U) \simeq \mathrm{Obs}^q_{CS}(U)$

② on ∂, it recovers current alg:

$U \subset \partial X$

$\mathcal{F}(U \times [0, \varepsilon)) \simeq \underbrace{\mathrm{Curr}}_{\text{"currents"}}(U)$

see paper w/
Eugene Rabinovich &
Brian Williams
for a precise statement