# Intro to Hecke Category and diagonalization of the full twist

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featuring joint work with Matt Hogancamp (Northeastern U.)

Diagonalization: Given an operator $f \in \mathrm{End}(V)$ satisfying $(f-\kappa_0)(f-\kappa_1)\cdots(f-\kappa_r)=0$, split $V$ into eigenspaces, i.e. construct idempotents $p_i \in \mathrm{End}(V)$ s.t.

$$1_V = \sum_i p_i \quad , \quad p_i p_j = \delta_{ij} p_i \quad , \qquad f p_i = p_i f = \kappa_i f .$$

(Std construction: $p_i$ is a poly in $f$, coeffs are rational functions in $\kappa_i$)

Utility in Rep thry: Let $A$ be semisimple alg, $z \in Z(A)$. Then $z$ acts by a scalar on any irrep.

Artin-Wedderburn $\Rightarrow$ Simult. espaces for $Z(A)$ $=$ Isotypic components.

How does it play out for $\mathbb{Q}[S_n]$?

**Classic:** $\left\{ \begin{array}{c} \text{Irred Reps of } S_n \\ \text{over } \mathbb{Q} \end{array} \right\} / \cong \ \xleftrightarrow{\sim} \ \{ \lambda \vdash n \}$

$$S_\lambda \longleftrightarrow \lambda$$

**Classical construction:** (Specht)

Given a $\lambda$-tableau $T = \begin{array}{|c|c|c|c|} \hline 2 & 1 & 8 & 6 \\ \hline 5 & 4 & 7 & \\ \cline{1-3} 3 & & & \\ \cline{1-1} \end{array}$ have poly $p_T = \prod (x_i - x_j)$

in example, $p_T = (x_2 - x_5)(x_2 - x_3)(x_5 - x_3)(x_1 - x_4)(x_8 - x_7)$

**Claim:** Let $S_\lambda = \operatorname{Span}_{\mathbb{Q}} \{ p_T \mid T \in \operatorname{Tab}(\lambda) \}$. Then $\{ p_T \mid T \in SYT(\lambda) \}$ is a basis.

Moreover, $S_\lambda$ is an irred $\mathbb{Q}[S_n]$ repn

**More modern approach:** (Okounkov-Vershik. Young)

One can find a better basis $\{ e_T \mid T \in SYT(\lambda) \}$, an eigenbasis for a large commutative subalgebra of $\mathbb{Q}[S_n]$

**More modern approach:** One can find a better basis $\{e_T \mid T \in SYT(\lambda)\}$, an eigenbasis for a large commutative subalgebra of $\mathbb{Q}[S_n]$

(Okounkov-Vershik. ) Young

**Note:** $Z(\mathbb{Q}[S_n])$ spanned by sym polys in $\{y_k\}$.

**Def:** Let $y_k := (1\ k)+(2\ k)+\dots+(k{-}1\ k)$.

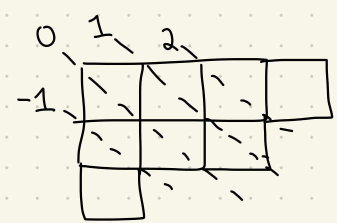
Then $y_k \in \{z \in \mathbb{Q}[S_k] \mid z \text{ commutes with } \mathbb{Q}[S_{k-1}]\} \Rightarrow [y_k, y_{k'}]=0$.

**Ex:**

$e_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} = x_1 - x_2 \qquad e_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} = x_1+x_2-2x_3$

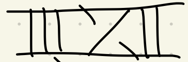
| eigenvalue for | $e_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}$ | $e_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}$ |
|---|---|---|
| $y_1 = 0$ | $0$ | $0$ |
| $y_2 = (12)$ | $-1$ | $+1$ |
| $y_3 = (13)+(23)$ | $+1$ | $-1$ |

Evalue for $y_k$ on $e_T$ is $x(\boxed{k})$ "content"




O-V explains why
- $\{y_k\}$ diag'le
- Spectrum of $\{y_k\}$ corresponds to $SYT(\lambda)$

# Deformation

$t_i =$   $Br_n$

$s_i = (i\ \ i{+}1)$  $S_n$  $\mathbb{H}_n = \mathbb{Z}[v^{\pm}][Br] \big/ (t_i+v)(t_i-v^{-1})=0$

evalues: $+1, -1$

$S_n \xleftarrow{v=1} \mathbb{H}_n$

evalues: $+v^{-1}, -v$

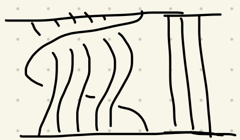
Repns deform.

$y_k \xleftarrow{v=1} y_k \in \mathbb{H}_n$

$y_k$ deforms in two ways.

"deriv. at $v=1$"

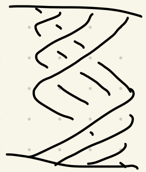
$j_k \in \mathbb{H}_n$  (multiplicative) Young-Jucys-Murphy operators

Def: $j_k =$  $\in Br_n$. As before, $j_k$ commutes with $Br_{k-1}$

Thm: $\{\mathrm{Irr}\,\mathbb{H}_n\}/\cong \;\longleftrightarrow\; \{\lambda \vdash n\}$ where $V_\lambda$ has basis $\{e_T \mid T \in SYT(\lambda)\}$ and

$V_\lambda \longleftrightarrow \lambda$  $j_k \cdot e_T = v^{2\chi(\boxed{k})}\, e_T$.

**Note:** $Z(\mathbb{H}_n)$ spanned by symmetric polys in $\{j_1, \dots, j_n\}$

$ft_n =$  $= (ht_n)^2 = j_1 j_2 \cdots j_n \in Z(Br_n)$. Acts on $V_\lambda$ by scalar $v^{2\chi(\lambda)}$.

**Rmk:** $ft_n$ almost distinguishes b/w irreps, but $\chi\left(\ydiagram{2,2}\right) = \chi\left(\ydiagram{3,1,1}\right)$.

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General idea: Structure should be reflected in/arise from the regular repn

What's the eigenbasis? Relation to Artin-Wedderburn?

Deformation makes it easier. mysterious!!!

**Thm** (Kazhdan-Lusztig): $\exists!$ basis $\{b_w\}_{w \in S_n}$ of $\mathbb{H}_n$ with awesome properties.

**Ex:** $b_1 := b_{s_1} = t_1 + v \xrightarrow{v=1} 1 + s_1$.

**Ex:** $b_{w_0} = \sum_{w \in S_n} v^{\ell(w_0) - \ell(w)} t_w$

This is <u>not</u> an ebasis but it's good enough

<u>Recall</u>: Robinson-Schensted Correspondence $\quad S_n \overset{\sim}{\longleftrightarrow} \left\{ (P,Q,\lambda) \;\middle|\; \begin{matrix} \lambda \vdash n \\ P,Q \in SYT(\lambda) \end{matrix} \right\}$

Let $b_{(P,Q,\lambda)} := b_w$ for correspondent. $\quad$ <u>Ex</u>: $w_0 \longleftrightarrow \left( \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} , \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \right)$

<u>Thm</u> (K-L): Let $I_\lambda = \mathrm{Span}\{ b_{(P,Q,\mu)} \mid \mu \leq \lambda \}$.

1) Then $I_\lambda$ is an ideal! 2) Moreover, for any $x \in H$, $x\, b_{(P,Q,\lambda)} \in \mathrm{Span}\{ b_{(P',Q,\lambda)} \} + I_{<\lambda}$.

3) $\mathrm{Span}\{ b_{(-,Q,\lambda)} \} / I_{<\lambda} \cong V_\lambda$

(Schützenberger involution)

<u>Thm</u> (Graham, Mathas): $ht_n \cdot b_{(P,Q,\lambda)} = (-1)^{c(\lambda)} v^{x(\lambda)} b_{(P^\vee,Q,\lambda)} + I_{<\lambda}$

$\Rightarrow ft_n \cdot b_{(P,Q,\lambda)} = (-1)^{2c(\lambda)} v^{2x(\lambda)} b_{(P,Q,\lambda)} + I_{<\lambda}$

Here $c(\boxed{k})$ is the "column number"

$$\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 2 & \\ \cline{1-3} 0 & & & \\ \cline{1-1} \end{array}$$

So $\{b_w\}$ is not an ebasis, but $ft_n$ is upper-triangular.

Let's categorify everything!

Let $R = \Bbbk[x_1, \ldots, x_n] \circlearrowleft S_n$. $\deg x_i = 2$

To categorify a ring we need an (additive) monoidal category. We'll find one inside $(R\text{-bim}, \otimes_R)$.

<u>Thm</u> (Chevalley): $R$ is free over $R^{S_n}$ w/ rank $n!$.
$R^{S_n}$ another poly ring.

<u>Thm</u> (Demazure): $R$ is a graded <u>Frobenius extension</u> of $R^{S_n}$. Equiv, Ind $\vdash$ Rest $\vdash$ Ind (up to shift)

This applies to parabolic subgps too.

<u>Ex</u>: $R$ is free over $R^{s_i}$ of rank 2. Basis: $\{1, x_i - x_{i+1}\}$

Roughly, Soergel bimodules are "generated" by induction + restriction between $R$ and $R^I := R^{W_I}$.

<u>Ex</u>: $I = \{s_2, s_3, s_5\}$ $W^I = S_1 \times S_3 \times S_2$ $R^I = \Bbbk\left[x_1, \begin{matrix} x_2+x_3+x_4 \\ x_2x_3+x_2x_4+x_3x_4 \\ x_2x_3x_4 \end{matrix}, \begin{matrix} x_5+x_6 \\ x_5x_6 \end{matrix}\right]$

**Def:** $B_i = B_{s_i} := R \otimes_{R^{s_i}} R(1)$  So $B_i \otimes (-) : R\text{-mod} \to R\text{-mod}$ agrees with $\cong$ $\mathrm{Ind} \circ \mathrm{Res}$ so its self-adjoint.

**Recall** $b_i \xrightarrow{v=1} 1+s_i$  $(1+s_i)^2 = 2(1+s_i)$  $b_i^2 = (v+v^{-1})b_i$.

Thus $B_i \otimes B_i = R \otimes_{R^{s_i}} R \cancel{\otimes_R} R \otimes_{R^{s_i}} R(2) \cong R \otimes_{R^{s_i}} \left(R^{s_i} \oplus R^{s_i}(-2)\right) \otimes_{R^{s_i}} R(2) \cong$

$$R \otimes_{R^{s_i}} R(2) \oplus R \otimes_{R^{s_i}} R(0) \cong B_i(1) \oplus B_i(-1).$$

**Def:** Soergel bimodules are $\otimes, \oplus, (1), \overset{\oplus}{\subset}$ of $B_i$

**Ex:** $s = X|$  $B_s B_t$ is cyclic, indecomp. $B_t B_s$ too.
$t = |X$

$\quad\;\; \shortparallel$
$B_s \otimes_R B_t$

Hard exercise: $B_s B_t B_s \cong B_s \oplus \left(R \otimes_{R^{s,t}} R(3)\right)$

Thm (Soergel): 1) If $\underline{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_d})$ is a red. exp. for $w \in S_n$ then

$B_{\underline{w}} := B_{i_1} B_{i_2} \cdots B_{i_d}$ has a ! "top summand" not seen in shorter expressions.

2) Two red exp for $w$ give isom. top summands. Call $B_w$.

3) $\{B_w\}_{w \in S_n}$ parametrize indecomp. up to isom, $(1)$

4) $[SBim] \cong H(S_n)$ with $[B_s] = b_s$ $[R] = 1$ $[R(1)] = v$

5) (For $S_n$ in char 0) $[B_w] = b_w$ ← MUCH HARDER

$\Rightarrow$ if $b_w b_x = \sum c_{wx}^y b_y$ then $B_w B_x \cong \bigoplus B_y^{\oplus c_{wx}^y}$

Let's write $B_{(P,Q,\lambda)}$ for $B_w$ under RSK. Then

$\{B_{(P,Q,\mu)}\}_{\mu\leq\lambda} = I_{\leq\lambda}$ is a <u>monoidal ideal</u>

$\{B_{(-,Q,\lambda)}\}/I_{<\lambda}$ categorifies $V_\lambda$.

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Now the braid group: $\exists$ nat'l bimodule maps

$$B_s \longrightarrow R(1) \qquad\qquad R(-1) \longrightarrow B_s$$
$$f\otimes g \longmapsto fg \qquad\qquad 1 \longmapsto \tfrac{1}{2}(\alpha_s\otimes 1 + 1\otimes\alpha_s)$$

dual bases for Frobenius extension $R^s \subset R$

<u>Def</u>: Let $\mathcal{H}$ be $K^b(SBim)$. Inside $\mathcal{H}$, let $T_s := (\underline{B_s} \to R(1))$

$T_s^{-1} := (R(-1) \to \underline{B_s})$

underline is hom degree zero

<u>Thm</u> (Rouquier): For any braid word $\beta$, $T_\beta = \bigotimes$ of various $T_s^{\pm}$.

Then if $\beta = \beta'$ in braid group, $T_\beta \cong T_{\beta'}$ <u>canonically</u>. (Br acts strictly)

<u>Ex</u>: $T_s \otimes T_s^{-1} = \left( B_s(-1) \to \begin{matrix} B_sB_s \\ \oplus \\ \underline{R} \end{matrix} \to B_s(+1) \right) \simeq (\underline{R})$

$\rightsquigarrow$ we get canonical complexes $HT_n$, $FT_n$, $J_n$, etc. etc. They're not easy.

$HT_3 =$  $\rightsquigarrow T_s T_t T_s \simeq \left( \underline{B_{sts}} \to \begin{matrix} B_{st}(1) \\ \oplus \\ B_t(1) \end{matrix} \to \begin{matrix} B_s(2) \\ \oplus \\ B_t(2) \end{matrix} \to \mathbb{1}(3) \right)$

Ex:

$$HT_4 = \left( \underline{B_{w_0}} \to \underset{\ell(w)=5}{\oplus} B_w(1) \to \underset{\substack{\ell(w)=4 \\ \, \\ \oplus \\ B_{su}(2)}}{\oplus} B_w(2) \to \underset{\substack{\ell(w)=3 \\ \, \\ \oplus \\ B_t(3)}}{\oplus} B_w(3) \to \underset{\ell(w)=2}{\oplus} B_w(4) \to \underset{\ell(w)=1}{\oplus} B_w(5) \to R(6) \right)$$

Ex:

$$FT_3 = \left( \underline{B_{sts}}(-3) \to \begin{array}{c} B_{sts}(-1) \\ \oplus \\ B_{ts}(-1) \end{array} \to \begin{array}{c} B_{sts}(1) \\ \oplus \\ B_{sts}(1) \\ \oplus \\ B_s(1) \\ \oplus \\ B_t(1) \end{array} \to \begin{array}{c} B_{sts}(3) \\ \oplus \\ B_{st}(2) \\ \oplus \\ B_{ts}(2) \end{array} \to \begin{array}{c} B_{st}(4) \\ \oplus \\ B_{ts}(4) \end{array} \to \begin{array}{c} B_s(5) \\ \oplus \\ B_t(5) \end{array} \to 1(6) \right)$$

# Recall

**Thm** (Graham, Mathas): $ht_n \cdot b_{(P,Q,\lambda)} = (-1)^{c(\lambda)} v^{x(\lambda)} b_{(P^\vee,Q,\lambda)} + I_{<\lambda}$

$$\Rightarrow ft_n \cdot b_{(P,Q,\lambda)} = (-1)^{2c(\lambda)} v^{2x(\lambda)} b_{(P,Q,\lambda)} + I_{<\lambda}$$

**Thm** (E-H): $$HT \otimes B_{(P,Q)} \cong \left(\text{lower cells} \ldots \longrightarrow B_{(P^\vee,Q)}(x(\lambda))\langle c(\lambda)\rangle\right)$$

$$FT \otimes B_{(P,Q)} \cong \left(\text{lower cells} \ldots \longrightarrow B_{(P,Q)}(2x(\lambda))\langle 2c(\lambda)\rangle\right)$$

$HT_3 \otimes B_{sts} \cong B_{sts}(-3)\langle 0\rangle \quad c(\ydiagram{1,1,1}) = 0 \quad x(\ydiagram{1,1,1}) = -3$

$HT_3 \otimes B_s \cong \left(\underline{B_{sts}(-1)} \longrightarrow B_{ts}(0)\right) \quad c(\ydiagram{2,1}) = 1 \quad x(\ydiagram{2,1}) = 0$

$HT_3 \otimes \mathbb{1} \cong \left(\text{lower cells} \ldots \longrightarrow \mathbb{1}(3)\right) \quad c(\ydiagram{3}) = x(\ydiagram{3}) = 3$

NOT JUST LOWER CELLS BUT LOWER HOMOLOGICAL DEGREE TOO!

**Implication**: Let $K_\lambda := \mathbb{1}(2x(\lambda))\langle 2c(\lambda)\rangle$. Then $\exists$ chain map $K_\lambda B_{(P,Q,\lambda)} \longrightarrow FT_n B_{(P,Q,\lambda)}$ whose cone lives in $I_{<\lambda}$.

# Categorical Diagonalization

**Thm:** In fact, $\exists$ chain map $\alpha_\lambda : K_\lambda \longrightarrow FT_n$ s.t. $\alpha_\lambda \otimes B_{(P,Q,\lambda)} : K_\lambda B \xrightarrow{\sim} FT_n \otimes B$ modulo $\mathcal{I}_{<\lambda}$
(E-H) for everything in cell $\lambda$ at once! "Functorially isomorphic."

$\alpha_\lambda$ is the natural transformation which relates the cat'f'd eigenvalue to the cat'f'd operator.
We call $\alpha_\lambda$ an <u>eigenmap</u>. Note: $B_{(P,Q,\lambda)}$ is not an <u>eigenobject</u>, but descends to an eigenobject in the module $\mathcal{I}_{\leq\lambda} / \mathcal{I}_{<\lambda}$.

**Think:** $\mathrm{Cone}(\alpha_\lambda)$ categorifies $(ft_n - K_\lambda)$. To be diag'le need $\prod (ft_n - K_\lambda) = 0$.

**Thm** (E-H): $\bigotimes \mathrm{Cone}(\alpha_\lambda) \simeq 0$.

Amusing Key Step: $\mathcal{I}_\lambda \cap \mathrm{End}(R)$ is ideal generated by $S_\lambda$ !!

Ex:

$$FT_3 = \Big( B_{sts}(-3) \to \begin{matrix} B_{sts}(-1) \\ \oplus \\ B_{sts}(-1) \end{matrix} \to \begin{matrix} B_{sts}(1) \\ \oplus \\ B_{sts}(1) \\ \oplus \\ B_s(1) \\ \oplus \\ B_t(1) \end{matrix} \to \begin{matrix} B_{sts}(3) \\ \oplus \\ B_{st}(2) \\ \oplus \\ B_{ts}(2) \end{matrix} \to \begin{matrix} B_{st}(4) \\ \oplus \\ B_{ts}(4) \end{matrix} \to \begin{matrix} B_s(5) \\ \oplus \\ B_t(5) \end{matrix} \to \mathbb{1}(6) \Big)$$

$$\mathbb{1}(-6)\langle 0 \rangle \xrightarrow{\alpha} B_{sts}(-3)$$

$$\mathbb{1}(0)\langle 2 \rangle \xrightarrow{\alpha} \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}$$

$$\mathbb{1}(6)\langle 6 \rangle \xrightarrow{\mathrm{id}} \mathbb{1}(6) \qquad \alpha$$

**Diagonalization:** Given an operator $f \in \mathrm{End}(V)$ satisfying $(f-\kappa_0)(f-\kappa_1)\cdots(f-\kappa_r)=0$,

split $V$ into eigenspaces, i.e. construct idempotents $p_i \in \mathrm{End}(V)$ s.t.

$$1_V = \sum p_i \ , \quad p_i p_j = \delta_{ij} p_i \ , \qquad f p_i = p_i f = \kappa_i f.$$

(Std construction: $p$ is a poly in $f$ coeffs are rational functions in $\kappa_i$)

**Thm (E-H):** $\exists$ (infinite) complexes $P_\lambda$ satisfying:

$$\mathbb{1}_H = \left(\bigoplus P_\lambda, d\right) \qquad P_\lambda \otimes P_\mu \simeq 0 \qquad P_\lambda \text{ projects to } \alpha_\lambda\text{-eigencategory}$$

"$\mathbb{1}$ is filtered by $P_\lambda$"

$$P_\lambda \otimes P_\lambda \simeq P_\lambda$$

Videos of previous talks are available – Learning Seminar on Cat'fn

Technology of cat'l diag'n connected to projective alg. geom., "$\alpha_\lambda$ are sections of an ample line bundle."

Gorsky–Negut–Rasmussen

THANKS FOR
LISTENING!

**Ex:** $$FT_2 = \left( \underline{B_s(-1)} \longrightarrow B_s(1) \longrightarrow \mathbb{1}(2) \right)$$

$$\begin{tikzcd} \underline{B_s(-1)} & & \mathbb{1}(2) \\ \mathbb{1}(-2)\langle 0\rangle \arrow[u, "\alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}"] & & \mathbb{1}(2)\langle 2\rangle \arrow[u, "\alpha_{\square\square}"'] \end{tikzcd}$$

$$(f-K_1)(f-K_2)=0$$

$$\Rightarrow P_{K_1} = \frac{f-K_2}{K_1-K_2} = K_1^{-1}(f-K_2)\left(1+\frac{K_2}{K_1}+\frac{K_2^2}{K_1^2}+\dots\right)$$

$$\begin{aligned}
&\mathbb{1} \xrightarrow{\alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}} (B_s \to B_s \to \underline{\mathbb{1}}) \\
&\qquad \searrow^{\alpha_{\square\square}} \\
&\mathbb{1} \xrightarrow{\alpha_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}} (B_s \to B_s \to \mathbb{1}) \\
&\qquad \searrow^{\square\square} \\
&\mathbb{1} \xrightarrow{\begin{smallmatrix}\square\\\square\end{smallmatrix}} (B_s \to B_s \to \mathbb{1}) \\
&\qquad \searrow^{\square\square} \\
&\dots
\end{aligned}$$

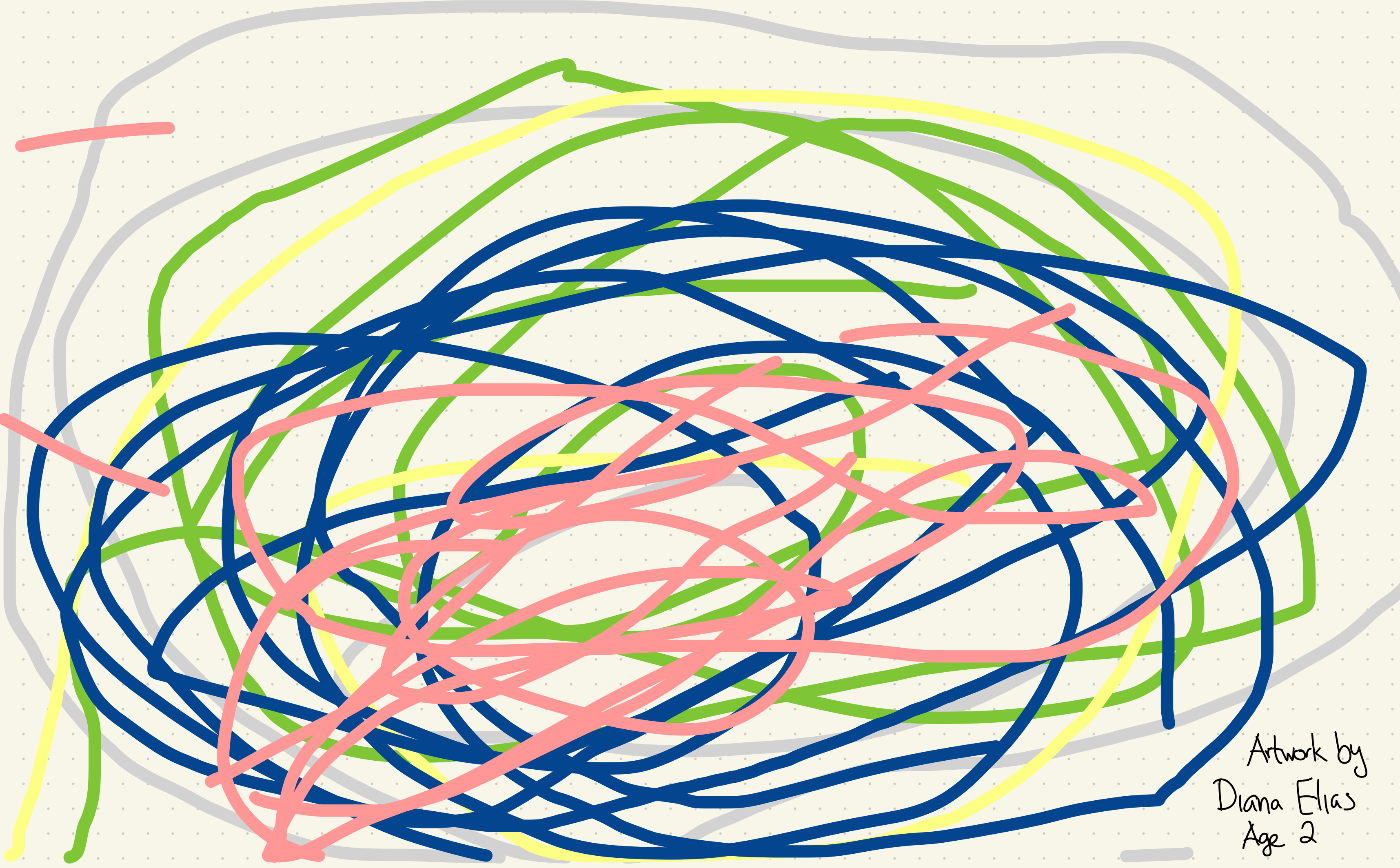
$$\simeq \left( \dots \to B_s \to B_s \to B_s \to B_s \to \underline{\mathbb{1}} \right) =: P_{\square\square}$$

Similarly (w/ appropriate power series expansion) get

$$\left( \dots \to B_s \to B_s \to B_s \to \underline{B_s(-1)} \right) =: P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

**Observe:** $$\mathbb{1} \cong \mathrm{Cone}\left(P_{\square\square} \longrightarrow P_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right)$$

Artwork by
Julian Elias
Age 5



Artwork by
Diana Elias
Age 2