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Semisimple Topological Quantum Field Theories in even dimensions

based on arXiv:2001.02288
and joint upcoming work with
Chris Schommer-Pries

David Reutter, Max-Planck-Institute Bonn

Lisbon TQFT club

→ Fix K: algebraically closed field.

→ All manifolds appearing here are $\left\{\begin{array}{l}\text{compact}\\ \text{smooth}\end{array}\right.$

Def [Atiyah '88, Segal, Witten, ...]

An n-dimensional top. quantum field theory (TQFT) is a symmetric monoidal functor

$$\mathrm{Cob}_n^X \longrightarrow \mathrm{sVect}_K$$

X: tangential structure, e.g. orientation, spin, ...

Cob_n^X: X-structured closed, smooth $(n-1)$-manifolds; structure preserving diffeomorphism classes of compact n-dim'l cobordisms; $\sqcup$

sVect_K: $\mathbb{Z}/2$-graded K-vector space; grading preserving linear map; $\otimes_K$, Koszul sign rule

In short: An assignment of vector spaces to closed $(n-1)$-manifolds and linear maps to n-manifolds compatible with $\left\{\begin{array}{l}\text{gluing}\\ \text{disjoint union}\end{array}\right\}$.

TQFT = symmetric monoidal functor $Z: \mathrm{Cob}_n^X \longrightarrow \mathrm{sVect}_K$

Original motivation: M^n closed n-manifold

$\Rightarrow Z(M^n) = Z(\phi_{n-1} \xrightarrow{M^n} \phi_{n-1}) \in K$ is a diffeomorphism invariant.

Immediate question: How much manifold topology can TQFTs see?

Thm [R'20 in 4d, R–Schommer-Pries in preparation $\forall$ even dim.]

Semisimple even-dimensional TQFTs only see stable diffeomorphism type.

For appropriately finite manifolds (e.g. finite fundamental group in 4d), the converse holds:

M, N not stably diffeomorphic $\Rightarrow$ distinguishable by semisimple TQFTs

(Includes all currently known >2d TQFTs & all TQFTs arising from once extended TQFTs with appropriate "higher linear algebraic" target)

TQFT = symmetric monoidal functor $Z: Cob_n^X \longrightarrow sVect_K$

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Immediate question: How much manifold topology can TQFTs see?

Thm [R'20 in 4d, R-Schommer-Pries in preparation $\forall$ even dim.]

Semisimple even-dimensional TQFTs only see stable diffeomorphism type.

For sufficiently "finite" manifolds (e.g. finite fundamental group in 4d), this "upper bound" is optimal:

M, N not stably diffeomorphic $\Rightarrow$ distinguishable by semisimple TQFTs

Consequences in 4d:

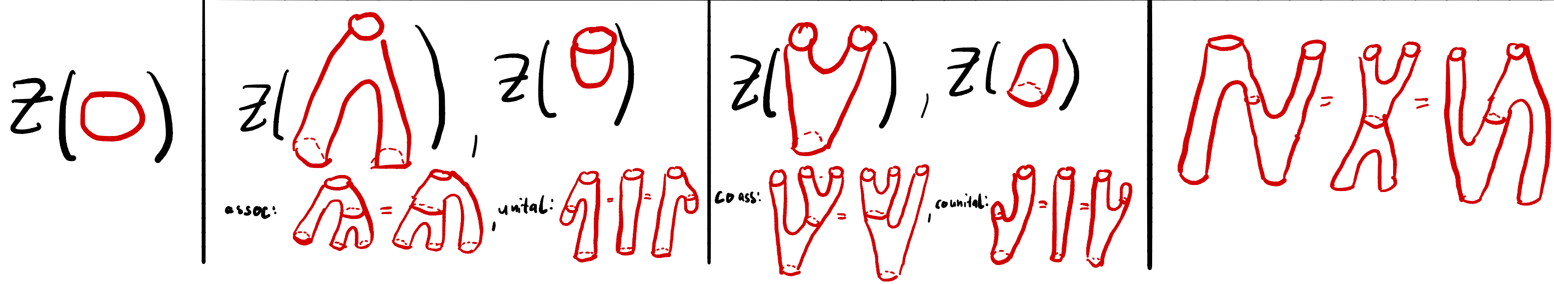
→ unoriented ss. TQFTs can sometimes detect exotic smooth structure.

→ oriented ss. TQFTs cannot detect exotic smooth structure but can sometimes distinguish homotopy equivalent 4-manifolds.

I) Definitions & Examples

Recall: 2D oriented TQFTs $\overset{1-1}{\longleftrightarrow}$ commutative Frobenius algebras

a vector space A	an algebra structure $m: A\otimes A\to A$, $u: k\to A$	a coalgebra structure $\Delta: A\to A\otimes A$, $\varepsilon: A\to k$	Frobenius relation: Δ is an A-A-bimodule map



More generally: Z oriented n-dim. TQFT $\Rightarrow Z(S^{n-1})$ is a comm. Frob. alg.

"higher pair of pants" $\rightsquigarrow$ $m = D^n \setminus (D^n \sqcup D^n)$, $u = D^n$, ...

Also $Z(S^k \times M^{n-1-k})$ is a comm. Frob. alg $\forall\ 1 \le k \le n-1$, M^{n-1-k} arbitrary closed $(n-1-k)$-mfd.

$m = D^{k+1} \setminus (D^{k+1} \sqcup D^{n+1}) \times M$, $u = D^{k+1} \times M$, ...

Notation: $\underline{S^k} \times S^{n-1-k}$ underline = use algebra structure coming from this sphere.

Fix tangential structure $X \in \{\text{unoriented}, \text{oriented}, \text{spin}\}$.

Def A $2n$-dimensional TQFT Z is semisimple if the (super) algebras $Z(S^{2n-1})$ and $Z(\underline{S^n} \times S^{n-1})$ are semisimple.

Physics aside in 4d:

$\to Z(S^3)$ = algebra of "local operators"

$\sim Z(S^3)$ is vector space of labels of 0d submanifolds in 4d manifolds

$\left[Z(M^4 \text{ with (normally framed) point } p \in M^4 \text{ labelled by } v \in Z(S^3)) := K \xrightarrow{v} Z(S^3) \xrightarrow{Z(M^4 \setminus D^4)} K \right]$

$\to Z(S^2 \times S^1) \sim$ controls "fusion of point particles"

$\sim Z(S^2)$ is "category" of labels of 1d submanifolds in 4d manifolds

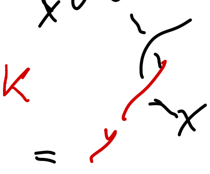
$\sim$ world lines of point particles.

$Z(S^2 \times S^1)$ its trace decategorification / Labels of normally framed knots in M^4

$K \xrightarrow{v} Z(S^2 \times S^1) \xrightarrow{Z(M^4 \setminus D^3 \times S^1)} K$

Exm: Crane–Yetter–Kauffman: For every ribbon fusion category $\mathcal{C}$ there is an associated semisimple 4d oriented TQFT $Z_{\mathcal{C}} : Cob_4^{or} \to Vect_K$

$x \in \mathcal{C}$:



$Z_{\mathcal{C}}(S^3) = K$

$Z_{\mathcal{C}}(S^2 \times S^1) = K_0(Z_{sym}(\mathcal{C})) \otimes_{\mathbb{Z}} K$

Thm [R`20]: All of the following TQFTs are semisimple:

→ invertible TQFTs [here: $Z(M^{2n-1})$ 1-dim'l, $Z(W^{2n}) \neq 0$]

→ unitary TQFTs [$Z: \mathrm{Cob}_{2n} \to \mathrm{Hilb}_{\mathbb{C}}$, s.t. $Z(M \xrightarrow{\overline{W}} N) = Z(N \xrightarrow{W} M)^{+}$] (or extended down to (2n-2)-manifolds; $+$: Hilbert space adjoint)

→ once-extended TQFTs $Z: \mathrm{Cob}_{2n,2n-1,2n-2} \to B$ where B is one of

{ k-algebras, bimodules, bimodule maps } or { (Karoubi-complete) k-linear categories, k-linear functors, natural transformations } or ... or their super versions

Exm: Crane-Yetter-Kauffman $Z_{\mathcal{C}}$ [is invertible $\Leftrightarrow$ $\mathcal{C}$ modular]

Rmk: [Freedman-Kitaev-Nayak-Slingerland-Walker-Wang `05] show that unitary 4d TQFTs cannot detect smooth structure.

[↳ We see: really due to semisimplicity

↳ Also: FKNSWW show invariance under s-cobordism.

Later: Really invariant under much coarser relation of stable diffeomorphism.]

II The Theorem

Def: Two connected, compact, $2n$-cobordisms $W, \tilde{W}$ are stably diffeomorphic if there is a $k \geq 0$ s.t.

$$W \#^k S^n \times S^n \cong \tilde{W} \#^k S^n \times S^n$$

interior connected sum $M \# N := (M \setminus D^{2n}) \cup_{S^{2n-1}} (N \setminus D^{2n})$

Thm [R'20 in 4d, R-SP in higher dim.]

Z semisimple, $2n$-dimensional TQFT.
$W, \tilde{W}$ stably diffeomorphic cobordisms $\Rightarrow Z(W) = Z(\tilde{W})$

Upshot: Stable diffeo classes very well understood [Wall, Kreck, ...]

Consequences in 4d: Z semisimple oriented 4D TQFT.

M, N closed oriented smooth 4-manifolds.

1) M, N (orient. pres.) homeomorphic $\Rightarrow Z(M) = Z(N)$

2) M, N simply connected & homotopy equivalent $\Rightarrow Z(M) = Z(N)$

Gompf '84

Wall '64

Cor: Z semisimple, oriented, indecomposable 4D TQFT.
M simply connected closed 4-manifold.

global dimension of $\mathcal{Z}_{sym}(\mathcal{C})$

	M spinnable	M non-spinnable
Z has fermions	$\widetilde{Z}(K3)^{-\frac{\sigma(M)}{16}}\ \widetilde{Z}(S^2\times S^2)^{\frac{1}{2}\left(\chi(M)-2+\frac{11}{8}\sigma(M)\right)}$	0
Z has no fermions	$\widetilde{Z}(\mathbb{C}P^2)^{\frac{1}{2}(\chi(M)+\sigma(M)-2)}\ \widetilde{Z}(\overline{\mathbb{C}P}^2)^{\frac{1}{2}(\chi(M)-\sigma(M)-2)}$	

Gauss sums $\sum_i \theta_i^{\pm 1} \dim(X_i)^2$

Moreover, all entries of this table (except for top right) are $\neq 0$.

In green: value of CYK field theory $Z_{\mathcal{C}}$

Special case: $\mathcal{C}$ modular ($\Leftrightarrow Z_{\mathcal{C}}$ invertible) $\Rightarrow Z_{\mathcal{C}}(M^4) = (\dim \mathcal{C})^{\frac{1}{2}\chi(M)} \cdot \left(e^{2\pi i c/8}\right)^{\sigma(M^4)}$

Proof sketch:

To show: Z semisimple $\Rightarrow Z(M) = Z(N)$ if $M \#^k S^n \times S^n \cong N \#^k S^n \times S^n$

1) Semisimplicity of $Z(S^{2n-1}) \Rightarrow Z(S^{2n-1}) \underset{\substack{\uparrow \\ \text{algebra iso}}}{\cong} \bigoplus_i k \overset{\text{Sawin}}{\Longrightarrow} Z = \bigoplus_i Z_i$ with $Z_i(S^{2n-1}) \cong k$

2) WLOG assume $Z(S^{2n-1}) \cong k \Rightarrow Z(M \# N) = k \xrightarrow{Z(M \setminus D^{2n})} Z(S^{2n-1}) \xrightarrow{Z(N \setminus D^{2n})} k = Z(S^{2n})^{-1} Z(M)\, Z(N).$

3) Remains to show: $Z(\underline{S}^d \times S^{d-1})$ semisimple $\Rightarrow Z(S^d \times S^d) \neq 0.$

Note: A Frobenius algebra $(A, m: A \otimes A \to A, \Delta: A \to A \otimes A)$ over an alg. closed field k is semisimple $\Leftrightarrow m \circ \Delta : A \to A$ is invertible

For the Frobenius algebra $\underline{S}^d \times S^{d-1}$:

$$\underset{\substack{\cap \\ \mathrm{End}(S^d \times S^{d-1})}}{(m \circ \Delta)}\ \underset{\substack{\cap \\ \mathrm{Hom}(\emptyset, S^d \times S^{d-1})}}{S^d \times D^d} \cong \underset{\substack{\cap \\ \mathrm{End}(\emptyset)}}{(S^{2d-1} \times S^1) \# (S^d \times S^d)} \# \underset{\substack{\cap \\ \mathrm{Hom}(\emptyset, S^d \times S^{d-1})}}{S^d \times D^d}$$

An eigenvalue equation in the cobordism category!

$\Rightarrow Z(S^d \times S^d)$ eigenvalue of invertible endo $Z(m \circ \Delta) \Rightarrow Z(S^d \times S^d) \neq 0.$ $\square$

Have seen:

Z semisimple even TQFT, M, N stably diffeo $\Rightarrow Z(M) = Z(N)$

The converse is also true, provided M, N are sufficiently finite to avoid certain pathologies.

[technical condition, can probably be weakened]

Def: A connected, closed $2n$-dimensional manifold M is sufficiently finite if the homotopy fiber of the classifying map $M \to BO(2n)$ of its tangent bundle has finite $\pi_1, \ldots, \pi_n$.

E.g: In 4d, a manifold is sufficiently finite iff it has finite π_1.

Thm [R-SP upcoming]: Let M, N be sufficiently finite closed $2n$-manifolds. If M, N are not stably diffeomorphic $\Rightarrow$ $\exists$ semisimple TQFT distinguishing M and N.

These TQFTs are built using a generalized Dijkgraaf–Witten construction.

Examples in 4d:

$\exists$ [Kreck] unoriented homeomorphic but not stably diffeomorphic closed 4-manifolds

$\Rightarrow$ can be distinguished by unoriented semisimple TQFTs.

$\exists$ [Teichner, ...] oriented homotopic but not stably diffeomorphic closed 4-manifolds

$\Rightarrow$ can be distinguished by oriented semisimple TQFTs

Summary

Assuming appropriate finiteness:

Semisimple even-dim. TFTs are "complete" stable diffeo invariants.

Thanks for listening!

Extra material: Stable diffeos and generalized DW theories

Tangential n-type of a $2n$-manifold M: Factor (uniquely up to homotopy)

$$\begin{tikzcd}[column sep=large] & \tau^{\mathrm{tang}}_{\le n} M \arrow[dr, dashed, "\substack{\text{mono on } \pi_n \\ \text{iso on } \pi_{>n} \\ \Downarrow \\ n\text{-truncated}}"] & \\ M \arrow[ur, dashed, "\substack{\text{epi on } \pi_n \\ \text{iso on } \pi_{<n} \\ \Downarrow \\ n\text{-connected}}"] \arrow[rr, "\text{tang. bundle}"'] & & BO(2n) \end{tikzcd}$$

Thm [Kreck]: M^{2n}, N^{2n} stably diffeomorphic $\iff$

(i) M, N have homotopic tangential n-type $X \to BO(2n)$

(ii) they are in the same $\mathrm{Aut}(X \to BO(2n))$ orbit in the X-structured bordism group Ω^X_{2n}.

Generalized DW theories:

$f: X \to BO(2n)$ with π-finite fiber.

invertible TFT $W: \mathrm{Bord}^X_{2n} \to s\mathrm{Vec}$ $\rightsquigarrow$ non-invertible TFT $\int_{\mathrm{fib} f} W: \mathrm{Bord}_{2n} \to s\mathrm{Vec}$.

$\mathrm{Bord}^X_{2n,\dots,0} \to Ik^X$

Extra material: Stable diffeos and generalized DW theories

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group homomorphism $\Omega^X_{2n} \to k^\times$

$\rightsquigarrow$

non-invertible TFT

$\int_{fib f} W : \mathrm{Bord}_{2n} \to s\mathrm{Vec}.$

Extra material: Stable diffeos and generalized DW theories

Tangential n-type of a $2n$-manifold M: Factor (uniquely up to homotopy)

epi on π_n, iso on $\pi_{<n}$ $\Leftrightarrow$ n-connected

mono on π_n, iso on $\pi_{>n}$ $\Leftrightarrow$ n-truncated

$$\begin{tikzcd} & \tau^{\mathrm{tang}}_{\leq n} M \arrow[dr, dashed, "n\text{-truncated}"] & \\ M \arrow[ur, dashed, "n\text{-connected}"] \arrow[rr, "\text{tang. bundle}"'] & & BO(2n) \end{tikzcd}$$

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group homomorphism $\Omega^X_{2n} \to k^\times$ $\qquad$ $\int_{\mathrm{fib} f} W: \mathrm{Bord}_{2n} \to s\mathrm{Vec}$.

Slogan: DW theories are Pontryagin dual to stable diffeo classes.

Conjecture: Every semisimple $2n$-dim. TQFT is equivalent to a DW theory.