A groupoid-based perspective on Quantum Mechanics

$$\begin{tikzcd}
\text{Wave functions} \arrow[r, dashed, bend right] & \text{Hilbert spaces} \arrow[l, bend right, "\text{C.S.C.O.}"'] \arrow[r, dashed, bend left] & C^*\text{-algebras} \arrow[l, bend left, "\text{GNS}"] \arrow[r, dashed, bend right] & \text{Groupoids} \arrow[l, bend right, "\text{Convolution}"']
\end{tikzcd}$$

SOME OF THE FRUITS OF ONGOING DISCUSSIONS BETWEEN:

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SCHRÖDINGER'S WAVE MECHANICS

$$-i\hbar \frac{d}{dt}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(x,t)\right]\Psi(x,t)$$

$\Psi(x,t)$ is a wave function defined on Physical space

The Born Rule $\longrightarrow$ $|\Psi|^2$ is a probability distribution defined on Physical space

HILBERT SPACE QUANTUM MECHANICS

$\Psi(x,t) \longrightarrow |\Psi_t\rangle \in \mathcal{H}$ abstract Hilbert space

·) Dirac's algebraic harmonic oscillator

··) von Neumann's theory of self-adjoint linear operators

We can go back to Hilbert space Quantum Mechanics:

$$\mathcal{H} \longrightarrow L^2(\mathcal{M}, \mu)$$

$$|\Psi_t\rangle \xrightarrow{\text{C.S.C.O.}} \Psi(x,t)$$

ALGEBRAIC QUANTUM THEORIES

$\hat{X}, \hat{P}, \hat{L}, \hat{H}, \hat{S} \ldots \longrightarrow a \in \mathcal{A}$ $\begin{cases} C^*\text{-algebras} \\ W^*\text{-algebras} \end{cases}$

- Heisenberg matrix mechanics

- Haag-Kastler's algebraic Quantum field theory

We can go back to Hilbert space Quantum Mechanics:

$$(\mathcal{A}, \rho) \xrightarrow[\text{G.N.S. construction}]{} \mathcal{H}_\rho = \overline{\mathcal{A}/N_\rho}^{\rho}$$

state on $\mathcal{A}$

$\rho : \mathcal{A} \to \mathbb{C}$, $\rho(a^+ a) \geq 0$

Gel'fand ideal

$N_\rho = \{ a \in \mathcal{A} \,/\, \rho(a^+ a) = 0 \}$

GROUPOID-BASED QUANTUM MECHANICS

- Schwinger's seminal work on Symbolism of atomic measurements
- Ritz-Rydberg combination principle

$\longrightarrow$ Quantum systems are described through Groupoids

We can go back to algebraic quantum theories:

$$\mathcal{G} \xrightarrow{\text{convolution product among functions}} \nu(\mathcal{G})$$

$\mathcal{G}$: groupoid of a quantum system

$\nu(\mathcal{G})$: groupoid von Neumann algebra

WHAT IS A GROUPOID?

A groupoid $\mathcal{G}=(G_0, G_1, s, t, I, i, \circ)$ is composed by:

•) two sets G_0 and G_1

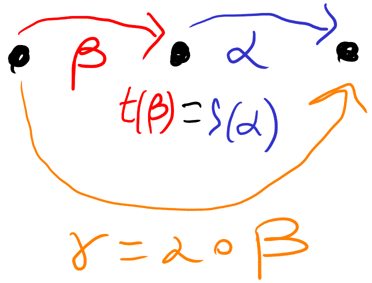
•) two surjections $s, t : G_1 \longrightarrow G_0$

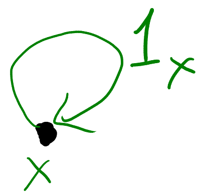
•) An idempotent map $I : G_1 \longrightarrow G_1$

•) An injective map $i : G_0 \longrightarrow G_1$

•) A partial composition law

$$\begin{array}{l} \circ : G_2 \longrightarrow G_1 \\ \quad\;\; \cap \\ \quad G_1 \times G_1 \end{array}$$





Source Target

i) $G_2 = \{(\alpha,\beta) \mid s(\alpha) = t(\beta)\}$

$\alpha \circ \beta \in G_1$

ii) $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$

iii) $x \in G_0 \longrightarrow i(x) \equiv 1_x \in G_1$

$\alpha \circ 1_x = \alpha$

$1_x \circ \beta = \beta$

iv) $I(\alpha) \equiv \alpha^{-1}$

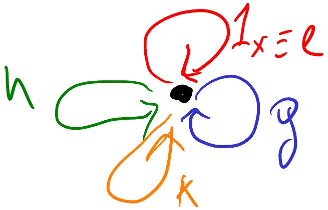
$s(\alpha^{-1}) = t(\alpha)$ $\qquad \alpha^{-1} \circ \alpha = 1_{s(\alpha)}$

$t(\alpha^{-1}) = s(\alpha)$ $\qquad \alpha \circ \alpha^{-1} = 1_{t(\alpha)}$

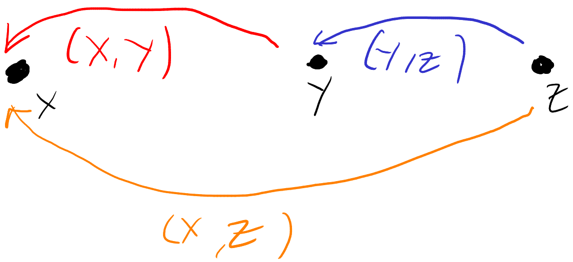
EXAMPLES OF GROUPOIDS

Trivial groupoid : $\mathcal{G} = (M, M, s(x) = t(x) = I(x) = i(x) = x \, , \, x \circ x = x)$ [Classical systems]

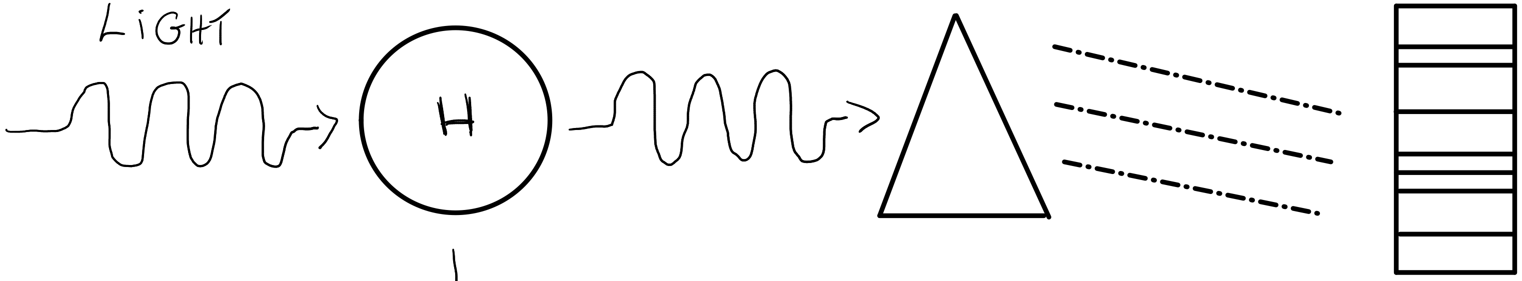
Groups : $\mathcal{G} = \left(G, \{x\}, \; s(g) = t(g) = x \, , \, I(g) = g^{-1} , \, i(x) = \underbrace{e}_{\text{identity element}} , \; \overbrace{g \circ h}^{\text{group law}} = g h \right)$



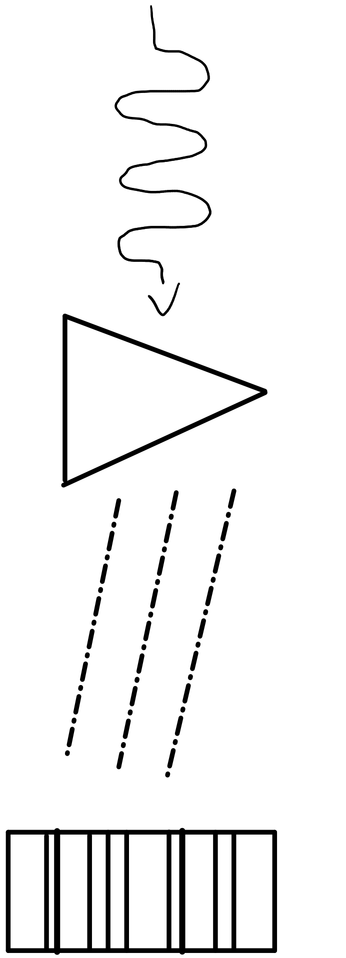
Pair groupoid : $P(M) = \left(M, M \times M \, , \, s(x,y) = t(y,x) = y \, , \, I(x,y) = (y,x) , \, i(x) = (x,x) \, , \, (x,y) \circ (y,z) = (x,z) \right)$



FROM RYTZ-RYDBERG COMPOSITION PRINCIPLE TO GROUPOIDS



ABSORPTION LINES



EMISSION LINES

$$j,k \in M = \{ j \in \mathbb{N}_0 \mid j = n^2 , n \in \mathbb{N}_0 \}$$

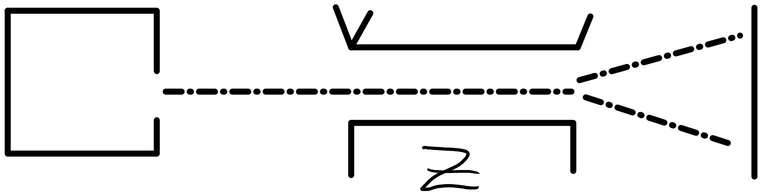
$$\nu_{jk} = cR\left(\frac{1}{j} - \frac{1}{k}\right) , \quad \nu_{jk} \in M \times M$$

"negative frequencies" account for emission

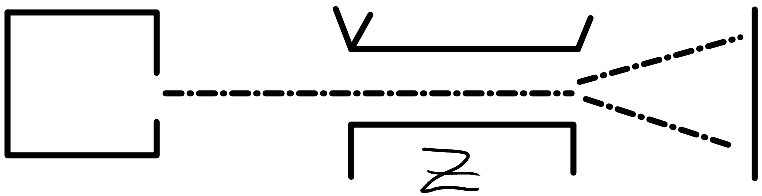
$$\nu_{jl} = \nu_{jk} + \nu_{kl} \equiv \nu_{jk} \circ \nu_{kl}$$

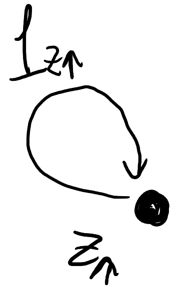
Partial Composition

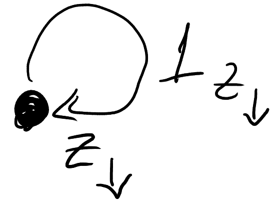
FROM STERN-GERLACH EXPERIMENTS TO GROUPOIDS

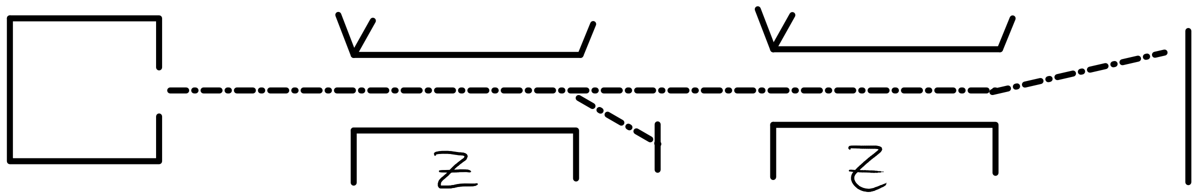


FROM STERN-GERLACH EXPERIMENTS TO GROUPOIDS









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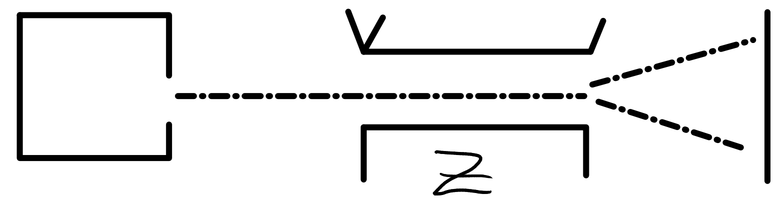
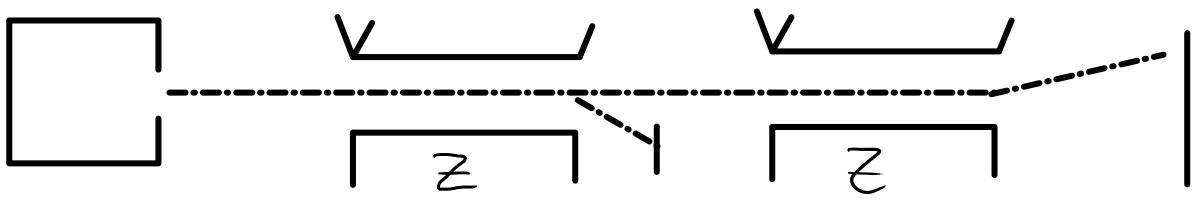
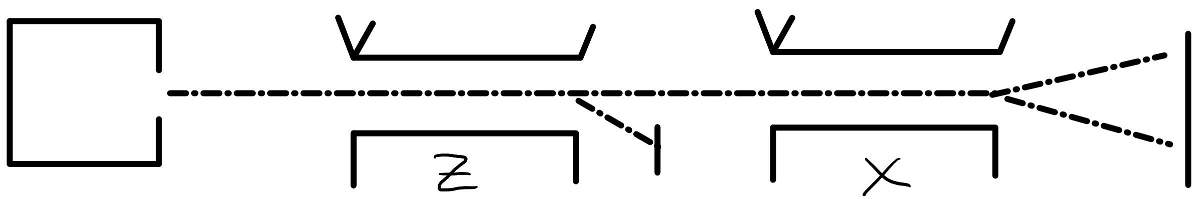
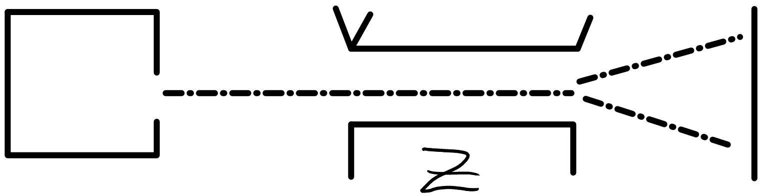
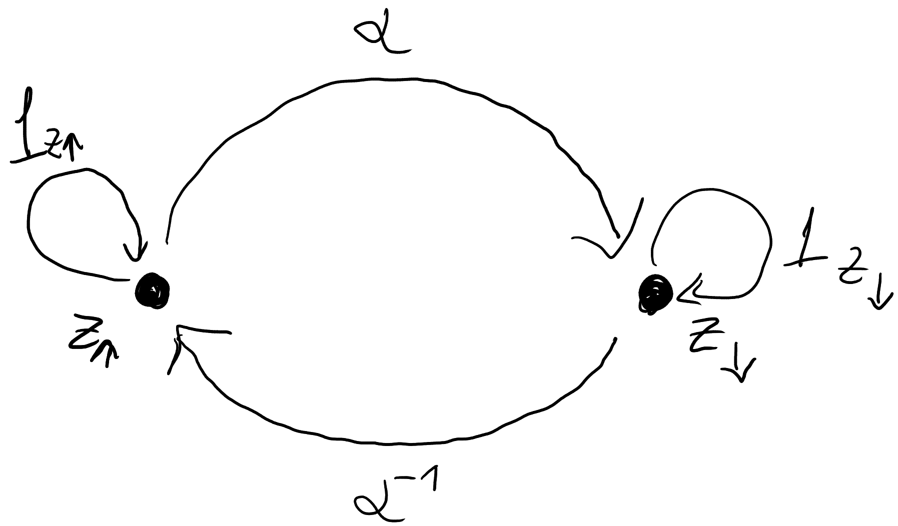


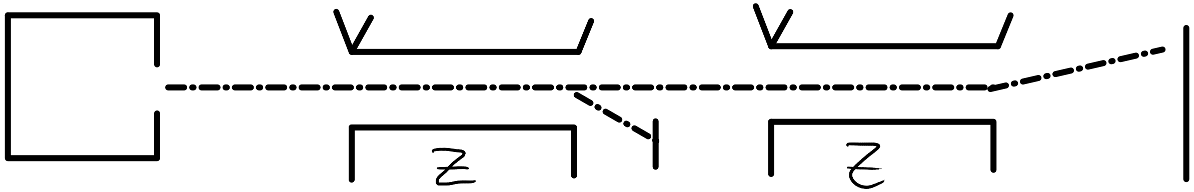


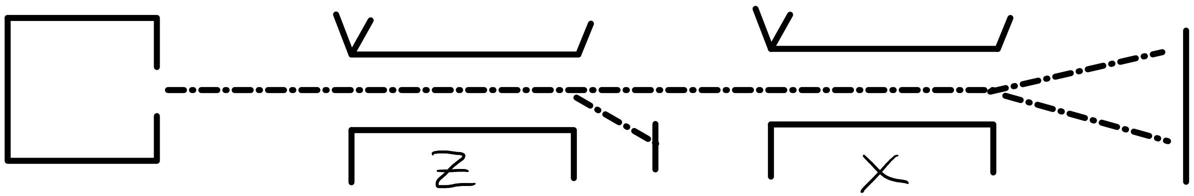


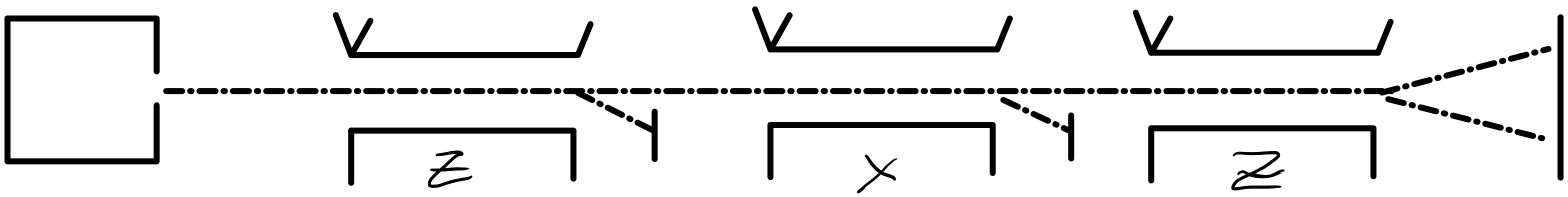
FROM STERN-GERLACH EXPERIMENTS TO GROUPOIDS











FROM GROUPS TO VON NEUMANN'S ALGEBRAS

Lie group G with a left-Haar measure ν:

$$f_1 * f_2(g) := \int_G f_1(gh)\, f_2(h^{-1})\, d\nu(h) \qquad [f_1, f_2 \in C_c^\infty(G)]$$

Convolution product

On $\mathcal{H} = L^2(G,\nu)$, define $T_f : \mathcal{H} \longrightarrow \mathcal{H}$

$$T_f \underline{\Psi} := (D^{-\frac{1}{2}} f) * \underline{\Psi} \qquad D \text{ is the modular function}$$

The von Neumann algebra $\mathcal{V}(G)$ is the weak closure of:

$$T_c^\infty(G) := \{ T_f \in \mathcal{B}(\mathcal{H}) \mid f \in C_c^\infty(G) \}$$

FROM GROUPOIDS TO VON NEUMANN'S ALGEBRAS

The left-Haar measure is replaced by a left-Haar system $\{\nu^x\}_{x \in G_0}$

•) ν^x is a Radon measure, $\mathrm{supp}(\nu^x) \subseteq t^{-1}(x)$ $\forall x \in G_0$

••) $\displaystyle\int_{t^{-1}(s(\alpha))} f(\beta)\, d\nu^{s(\alpha)}(\beta) = \int_{t^{-1}(t(\alpha))} f(\alpha \circ \beta)\, d\nu^{t(\alpha)}(\beta)$ $\quad \forall f \in C_c(G_1)$

•••) $\displaystyle x \longmapsto F_f(x) := \int_{t^{-1}(x)} f(\alpha)\, d\nu^x(\alpha)$

is a continuous function on G_0

FROM GROUPOIDS TO VON NEUMANN'S ALGEBRAS

To define $\mathcal{H} = L^2(G_1, \nu)$ we need a Radon measure μ on G_0:

$$\nu(A) := \int_{G_0} \int_{t^{-1}(x)} \chi_A(\alpha)\, d\mu^x(\alpha)\, d\mu(x)$$

$$\nu^{-1}(A) := \nu(A^{-1})$$

$\{\mu^x\}_{x \in G_0}$ and μ must be such that ν is absolutely continuous w.r.t. ν^{-1}

$D := \dfrac{d\nu}{d\nu^{-1}}$ is the modular function of $(\mathcal{G}, \nu)$

$$D(\alpha^{-1}) = D^{-1}(\alpha)$$

$$D(\alpha \circ \beta) = D(\alpha)\, D(\beta)$$

FROM GROUPOIDS TO VON NEUMANN'S ALGEBRAS

The convolution product is:

$$f_1 * f_2(\alpha) := \int_{t^{-1}(s(\alpha))} f_1(\alpha\circ\beta)\, f_2(\beta^{-1})\, d\mu^{s(\alpha)}(\beta) \qquad \left[f_1, f_2 \in C_c(G_1)\right]$$

There is an involution map:

$$f^+(\alpha) := \overline{f(\alpha^{-1})}, \qquad (f^+)^+ = f$$

We obtain an involutive algebra:

$$T_c(\mathcal{G}) := \left(C_c(G_1), *, +\right)$$

FROM GROUPOIDS TO VON NEUMANN'S ALGEBRAS

We represent $T_c(G)$ on $\mathcal{H} = L^2(G_1, \nu)$

$$\pi(f) \equiv T_f \,, \quad T_f \Psi := (D^{-\frac{1}{2}} f) * \Psi \,, \quad T_f^{+} = T_{f^{+}} \,, \quad T_{f_1} T_{f_2} = T_{f_1 * f_2}$$

The groupoid von Neumann algebra is:

$$\mathcal{V}(G) := \overline{\pi(T_c(G))}^{w}$$

We already have the modular operator Δ and the modular conjugation J:

Tomita-Takesaki theory

$$\begin{cases} \Delta \Psi(\alpha) := D(\alpha) \, \Psi(\alpha) \\ J \Psi(\alpha) := D^{-\frac{1}{2}}(\alpha) \, \overline{\Psi(\alpha^{-1})} \end{cases}$$

$$\Delta^{it} \, \mathcal{V}(G) \, \Delta^{-it} = \mathcal{V}(G)$$

$$J \, \mathcal{V}(G) \, J = \left(\mathcal{V}(G)\right)'$$

A CLASSICAL EXAMPLE

Trivial groupoid: $\mathcal{G} = (M, M, s(x) = t(x) = I(x) = i(x) = x \;, \; x \circ x = x)$

μ is a Radon measure on M

$\{\mu^x\}_{x \in M}$, $\mu^x := \delta_x \quad \forall x \in M$ Dirac measure at $x \in M$

If $M = T^*Q$ and μ is the symplectic measure, we obtain Statistical Mechanics

$$\nu(A) := \int_M \int_M \chi_A(y)\, d\delta_x(y)\, d\mu(x) = \mu(A)$$

$$\nu^{-1}(A) = \nu(A) \implies D := \frac{d\nu}{d\nu^{-1}} = id_M$$

$$\mathcal{H} = L^2(M, \mu)$$

$$T_f \Psi(x) := \left[(D^{-\frac{1}{2}} f) * \Psi \right](x) = \int_M f(x \circ y)\, \Psi(y)\, d\delta_x(y) = f(x)\, \Psi(x)$$

$$T_c(\mathcal{G}) = C_c(M) \implies \nu(\mathcal{G}) = L^\infty(M, \mu)$$

A QUANTUM EXAMPLE

Pair groupoid: $P(M)=\big(M, M\times M, S(x,y)=t(y,x)=y, I(x,y)=(y,x), i(x)=(x,x), (x,y)\circ(y,z)=(x,z)\big)$

μ is a Radon measure on M

$\{\mu^x\}_{x\in M}$, $\mu^x := \delta_x \times \mu \quad \forall x \in M$ Dirac measure at $x\in M$

$$\nu(A) := \int_M \int_{\{x\}\times M} \chi(y,z)\, d\delta_x(y)\, d\mu(z)\, d\mu(x) = \mu\times\mu(A)$$

$$\nu^{-1}(A) = \nu(A) \Longrightarrow D = id_{M\times M}$$

$$L^2(M\times M, \mu\times\mu) = L^2(M,\mu)\otimes L^2(M,\mu) \equiv \mathcal{H}\otimes\mathcal{H}$$

$$f_1 * f_2(x,y) = \int_{\{y\}\times M} f_1((x,y)\circ(z,w))\, f_2(w,z)\, d\delta_y(z)\, d\mu(w) =$$

$$= \int_M f_1(x,w)\, f_2(w,y)\, d\mu(w)$$

A QUANTUM EXAMPLE

The representation of $T_c(P(M))$ on $\mathcal{H} \otimes \mathcal{H} = L^2(M \times M, \nu \times \nu)$ reads:

$$T_f \Psi(x,y) = \int_M f(x,z)\, \Psi(z,y)\, d\nu(z)$$

Take $\Psi = \varphi \otimes \eta$ so that

$$T_f \Psi(x,y) = \left(\int_M f(x,z)\, \varphi(z)\, d\nu(z) \right) \eta(y)$$

$$T_f(\Psi) = T_f(\varphi \otimes \eta) \equiv (K_f(\varphi)) \otimes \eta$$

$$K_f : \mathcal{H} = L^2(M, \nu) \longrightarrow L^2(M, \nu) = \mathcal{H}$$

$$\mathcal{V}(P(M)) \simeq \mathcal{B}(\mathcal{H})$$

FROM MEASURES ON $\mathcal{G}$ TO STATES ON $\mathfrak{v}(\mathcal{G})$

Physical states are states on $\mathfrak{v}(\mathcal{G})$:

$$\cdot) \quad \rho : \mathfrak{v}(\mathcal{G}) \longrightarrow \mathbb{C} \,, \quad \text{norm-continuous}$$

$$\cdot\cdot) \quad \rho\left(T_{f^\dagger \star f}\right) \geq 0 \qquad \forall\, f \in T_c(\mathcal{G})$$

$$\cdot\cdot\cdot) \qquad \rho(\mathbb{1}) = \|\rho\| = 1$$

Take a measure $\underline{P}_\nu$ on G_1 absolutely continuous w.r.t. ν:

$$\langle f \rangle_{\underline{P}_\nu} = \int_{G_1} f(\alpha)\, P(\alpha)\, d\nu(\alpha) =: \rho_{\underline{P}_\nu}\left(T_f\right)$$

Then $\rho_{\underline{P}_\nu}$ is a normal state on $\mathfrak{v}(\mathcal{G})$ iff:

$$P = \Psi^\dagger \star \Psi, \quad \Psi \in L^2(G_1, \nu), \quad \|\Psi\| = 1$$

FROM GROUPOIDS TO...... OTHER THINGS

i) Composite systems and subsystems:

- Direct composition of groupoids and entanglement
- Free composition of groupoids and free probability theory

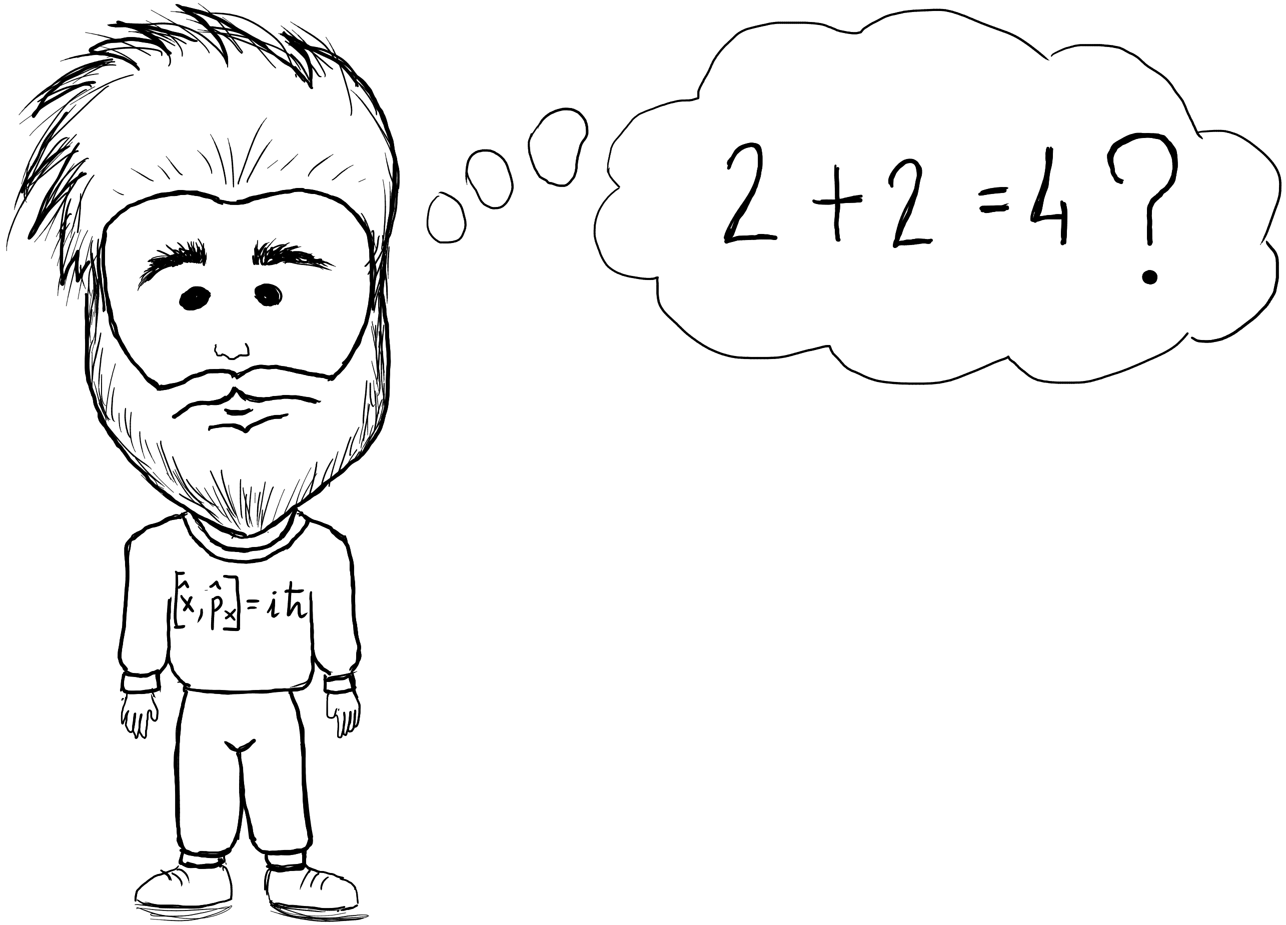
ii) Classical systems and the quantum-to-classical transition

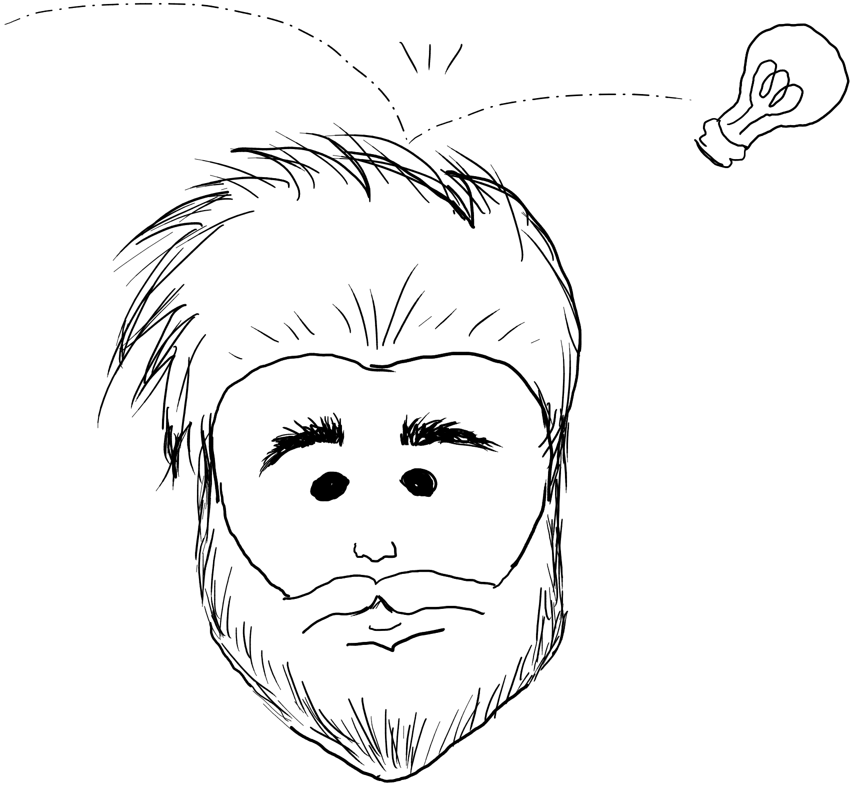
iii) Symmetries and Wigner's theorem in the groupoid formalism

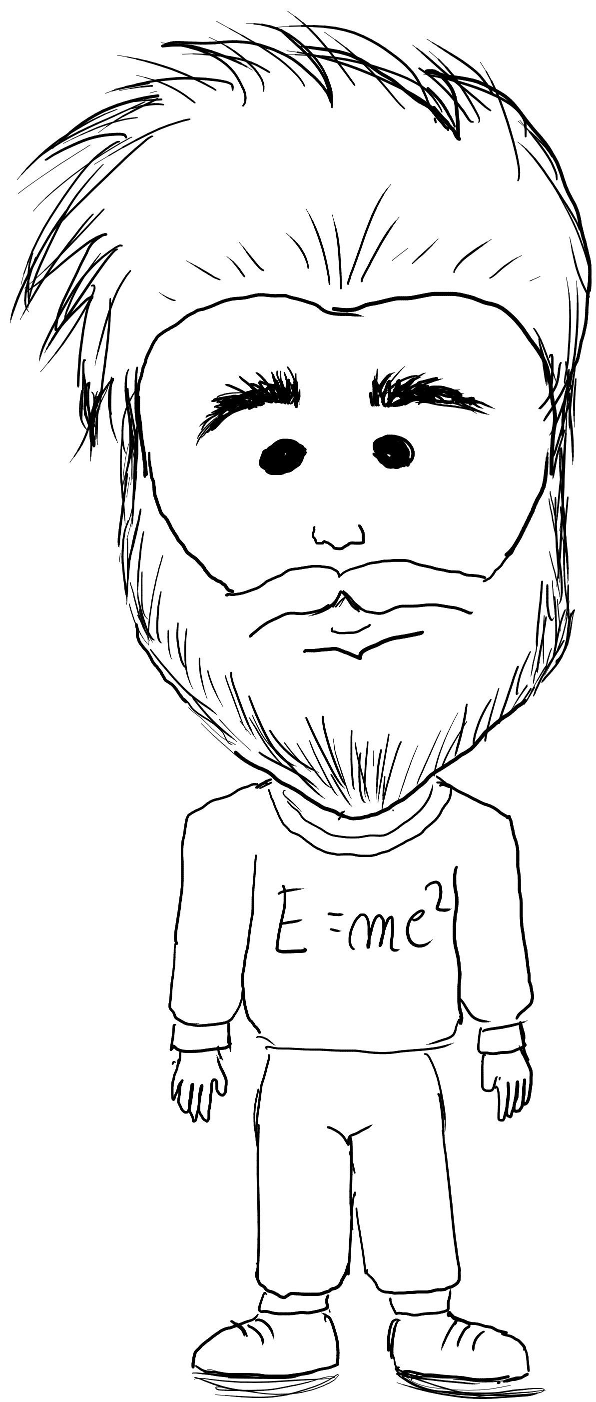
iv) Factorizable states and Sorkin's Quantum Measures

THANK YOU


LOOK FOR:
SCHWINGER'S PICTURE OF Q.M.
ON RESEARCH GATE
FOR MORE DETAILS
$[\hat{x},\hat{p}_x]=i\hbar$
4 8 15
16 23 42


$[\hat{x}, \hat{p}_x] = i\hbar$
2 + 2 = 4 ?




$E = mc^2$