TQFT Seminar

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# Universal Symmetries of Gerbes and Smooth Higher Group Extensions

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[1804.08953[1], 2004.13395[1], 2007.06039, 2008.12263]

[1]joint work with Lukas Müller and Richard J. Szabo

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# Motivating Example

## Nonassociative magnetic translations

# Observables in 2-form background

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- **Translations act non-commutatively and non-associatively**.
- Motivating question: Can we find a **geometric explanation** for this?

# Lifting translations to geometry

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- $(\mathcal{V}\mathrm{ect}, \oplus, 0, \otimes, \mathbb{C})$ is a categorified ring, which replaces $(\mathbb{C}, +, 0, \times, 1)$.
- The **linear automorphisms** of $\mathbb{C}$ are multiplication by $z \in \mathbb{C}^\times$; the linear automorphisms of $\mathcal{V}\mathrm{ect}$ are multiplication by **complex lines** $L \in \mathcal{V}\mathrm{ect}^\times$. (The inverse of $L$ is the dual line $L^\vee$.)

# Gerbe holonomy

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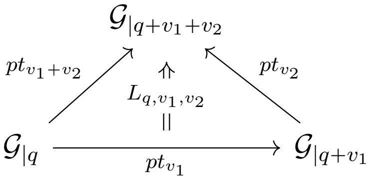
**Example:** For $\mathcal{G}$ a gerbe with connection on $\mathbb{R}^n$, any pair of translation vectors $v_1, v_2 \in \mathbb{R}^n$ gives rise to a holonomy:




$L_{q,v_1,v_2}$ a complex line

Letting $q$ vary, this yields a functor

$$L\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathcal{LB}\mathrm{un}(\mathbb{R}^n)\,.$$

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- **Observation:** the construction $L\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{LB}\mathrm{un}(\mathbb{R}^n)$ from a gerbe with connection on $\mathbb{R}^n$ is a **categorified group cocycle**.
- This should equivalently be encoded in an extension of $(\mathbb{R}^n, +)$ by the 2-group $(\mathcal{LB}\mathrm{un}(\mathbb{R}^n), \otimes)$ (we'll unravel this in a minute).

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- For the gerbe connection determined by $B \in \Omega^2(\mathbb{R}^n)$, we have

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$\Rightarrow$ Gerbe parallel transport **explains nonassociative magnetic translations** [SB, Müller, Szabo].

## Symmetries of gerbes

## Smooth 2-group extensions

Bundle gerbes on general manifolds $M$:

# Bundle gerbes and their morphisms

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Bundle gerbes on general manifolds $M$:

- $\{U_a\}_{a\in A}$ an open covering of $M$.
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Gerbes describe B-fields in string theory, twists in K-theory, smooth 2D (invertible) field theories [Kapustin; BCMMS; BTW; Picken; SB, Waldorf], ...

- A morphism $\mathcal{E} \colon \mathcal{G} \to \mathcal{G}'$ is a **twisted hermitean vector bundle**:
  (Requires common refinements of covers in general.)

$$E_a \in \mathcal{HVB}\mathrm{un}(U_a)\,, \qquad \alpha_{ab} \colon L_{ab} \otimes E_b \xrightarrow{\cong} E_a \otimes L'_{ab}$$

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One can also understand bundle gerbes as **categorified principal bundles** with **structure group** $\mathrm{BU}(1)$

[Baez, Huerta, Schreiber; MacKaay, Picken; Martins; Bullivant; Waldorf; ...]

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- In particular, $(\mathrm{Isom}_{\mathcal{G}rb(M)}(\mathcal{G}, \mathcal{G}), - \circ -)$ is canonically equivalent to $(\mathcal{HLB}\mathrm{un}(M), \otimes)$ as 2-groups.

# Gerbes and group actions—algebra

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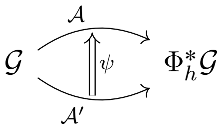

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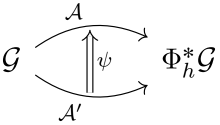


**$\mathrm{Sym}(\mathcal{G})$ has a product:**

$$(h_1, \mathcal{A}_1) \otimes (h_0, \mathcal{A}_0) := \Big(h_1 h_0,\ \mathcal{G} \xrightarrow{\mathcal{A}_0} \Phi_{h_0}^* \mathcal{G} \xrightarrow{\Phi_{h_0}^* \mathcal{A}_1} \Phi_{h_0}^* \Phi_{h_1}^* \mathcal{G} = \Phi_{h_1 h_0}^* \mathcal{G}\Big)$$

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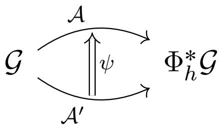


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Extending this product to morphisms, **$\mathrm{Sym}(\mathcal{G})$ becomes a 2-group!**

- $\mathrm{Sym}(\mathcal{G})$ comes with a surjective monoidal projection

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In summary, there is an equivalence **of 2-groups**

$$\pi^{-1}\{e_H\} = \mathrm{Isom}_{\mathcal{G}rb(M)}(\mathcal{G}, \mathcal{G}) \simeq \mathcal{HLB}\mathrm{un}(M)\,,$$

- $\mathrm{Sym}(\mathcal{G})$ comes with a surjective monoidal projection
  $\pi\colon \mathrm{Sym}(\mathcal{G}) \to H\,, \qquad (h, \mathcal{A}) \mapsto h\,, \quad (h, \psi) \mapsto 1_h.$
- The (homotopy) fibre of $\pi$ has:
  **objects:** $(e_H, \mathcal{A})$, where $\mathcal{A} \in \mathrm{Isom}_{\mathcal{G}rb(M)}(\mathcal{G}, \mathcal{G}) \simeq \mathcal{HLB}\mathrm{un}(M)$,
  **morphisms:** $(e_H, \psi)$, where $\psi\colon \mathcal{A} \to \mathcal{A}'$ with $\mathcal{A}, \mathcal{A}'\colon \mathcal{G} \to \mathcal{G}$.
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i.e. we have a **2-group extension**

$$\mathcal{HLB}\mathrm{un}(M) \longrightarrow \mathrm{Sym}(\mathcal{G}) \longrightarrow H\,.$$

# Gerbes and group actions—geometry

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- There is a **smooth version $\mathcal{HLB}\mathrm{un}^M$ of $\mathcal{HLB}\mathrm{un}(M)$:** to $c \in \mathcal{C}\mathrm{art}$, assign the groupoid of hermitean line bundles on $c \times M$. This describes smooth $c$-parameterised families of hermitean line bundles on $M$.

- A **smooth groupoid** is a presheaf of groupoids $\mathcal{H} \colon \mathcal{C}\mathrm{art}^{\mathrm{op}} \to \mathcal{G}\mathrm{rpd}$.
- A **smooth 2-group** is a presheaf of 2-groups on $\mathcal{C}\mathrm{art}$.
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Let $H$ be a connected Lie group acting on $M$, let $\mathcal{G} \in \mathcal{G}rb(M)$.

There is an **extension of smooth 2-groups**

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**Applications:** Nonassociative magnetic translations, QFT anomalies, ...

## Smooth extensions of $\infty$-groups

## Smooth String group models

# String group models

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A **string group extension** of $H$ is an extension

$$A \longrightarrow \mathrm{String}(H) \xrightarrow{\pi} H$$

such that $\pi_i(p)$ is an isomorphism $\forall\, i \neq 3$ and $\pi_3(\mathrm{String}(H)) = 0$, i.e.

- $A$ is homotopy equivalent to $\mathrm{BU}(1) \simeq K(\mathbb{Z};2)$, and
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**Variants:** ambient (higher) category: top. groups, crossed modules, smooth 2-groups, . . . , **smooth $\infty$-groups**.

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**Relevance:** Dirac operators on $LM$, TMF, $M_5$-brane theory, . . .

[Witten, Killingback; Stolz, Teichner; Waldorf; ABGHR; Sämann, Jurco; . . . ]

- Generalise from presheaves of groupoids to **presheaves of $\infty$-groupoids**: $\mathbf{H}_\infty := \mathcal{F}\mathrm{un}(N\mathcal{C}\mathrm{art}^{\mathrm{op}}, \mathbf{S})$; call these **smooth spaces**.

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**Example:** For $X \in \mathbf{S}_*$ a pointed space, $\Omega X$ is a group object in $\mathbf{S}$.

# Groups extensions and functors

- One can define group actions, principal $\infty$-bundles, and group extensions in any $\infty$-topos $\mathbf{H}$ [Nikolaus, Schreiber, Stevenson; SB].

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- Essentially, a **principal $\widehat{G}$-bundle** is an effective epimorphism $\pi\colon P \to X$ and a principal $\widehat{G}$-action on $P$ that preserves $\pi$ (up to coherent homotopy). E.g. for $X \in \mathbf{S}_*$, $P_0X \to X$ is $\Omega X$-bundle.

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**Theorem [SB]**

Let $\mathrm{L}\colon \mathbf{H} \to \mathbf{H}'$ be a functor of $\infty$-topoi preserving realisations of simplicial objects and finite products. Then, $\mathrm{L}$ preserves groups, principal $\infty$-bundles, and group extensions.

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**Definition**

We call the functor $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ the **smooth singular complex**.

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**Theorem [SB]**

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## Definition—using above theorems

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Let $H$ be as above, acting on itself via left multiplication. Let $\mathcal{G} \in \mathcal{G}rb(H)$ represent a generator of $\mathrm{H}^3(H;\mathbb{Z}) \cong \mathbb{Z}$. Then,

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- Induces $\mathrm{H}^k(H;\mathsf{U}(1)^H) \cong \mathrm{H}^k(H;\mathsf{U}(1))$, under which $\mathrm{Sym}(\mathcal{G}) \to H$ is represented by the same Čech cocycle as $\mathcal{G}$.

Thank you for your attention!