Manifolds with odd Euler characteristic and higher orientability

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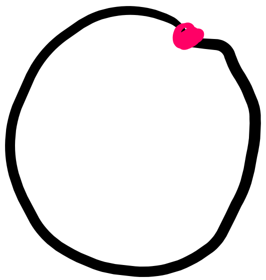
Lisbon TQFT seminar
22/01/'21

Def: The Euler characteristic of a finite CW complex X is

$$\chi(X) := \sum_i (-1)^i k_i = \sum_i (-1)^i b_i = \sum_i (-1)^i \dim H_i(X;\mathbb{F})$$



$$\chi(S^1) = 1 - 1 = 0$$

All manifolds will be smooth, connected, closed

Odd-dimensional manifolds M^n have $\chi = 0$:

Let $\mathbb{F} = \mathbb{Z}/2$:

all manifolds have Poincaré duality for $\mathbb{Z}/2$ homology

$$\dim H_i(M;\mathbb{Z}/2) = \dim H_{n-i}(M;\mathbb{Z}/2)$$

$\leadsto$ cancel out.

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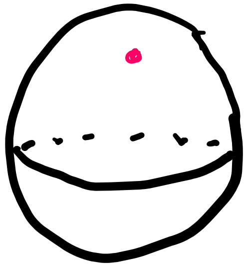
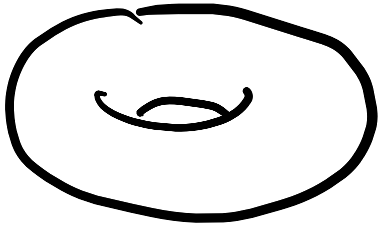
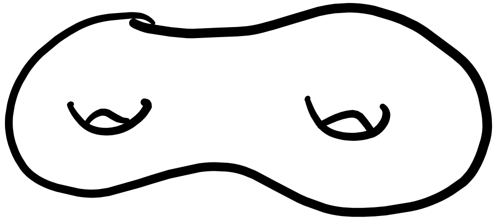
$1 - 0 + 1$



$\chi = 2$

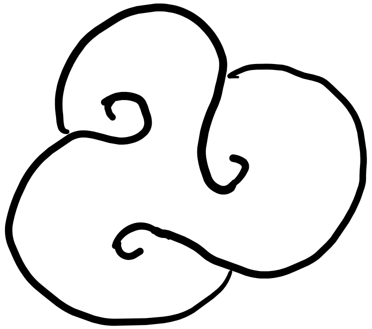


$\chi = 0$



$\chi = -2$

$\dots \quad \chi(\Sigma_g) = 2 - 2g$



$$\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$$

Thm 1: orientable manifolds M^n have even χ unless $n = 4m$.

Proof: even dimensional manifold: parity χ = parity $b_{n/2}$

$n = 4m+2$ by Poincaré duality as M orientable

$$H^{n/2}(M;\mathbb{Q}) \times H^{n/2}(M;\mathbb{Q}) \xrightarrow{\smile} H^n(M;\mathbb{Q}) \cong \mathbb{Q}$$

is a non-degenerate antisymmetric pairing:

symplectic form.

Thm 0: all manifolds M^n have $\chi = 0$ unless $n = 2m$.

Thm 1: orientable manifolds M^n have even χ unless $n = 4m$.

What does it mean to be orientable?

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & TM \arrow[d] & BSO(n) \simeq Gr^+(n,\infty) = \widetilde{BO(n)} \arrow[d] \\ & M^n \arrow[r, "\tau"] \arrow[ur, dashed] & BO(n) \simeq Gr(n,\infty) \end{tikzcd}$$

What does it mean to be orientable?

$$
\begin{tikzcd}
 & & BSpin(n) = \text{2-connected cover } BO(n) \arrow[d] \\
\mathbb{R}^n \arrow[r] & TM \arrow[d] & BSO(n) \simeq Gr^+(n,\infty) = \widetilde{BO(n)} \arrow[d] \\
 & M^n \arrow[r, "\tau"] \arrow[ur, dashed] \arrow[uur, dashed] & BO(n) \simeq Gr(n,\infty)
\end{tikzcd}
$$

Spin = "2-oriented"

Thm 0: all manifolds M^n have $\chi = 0$ unless $n = 2m$.

Thm 1: orientable manifolds M^n have even χ unless $n = 4m$.

$\boxed{\text{Rokhlin '52, Ochanine '81}}$: For spin manifolds of dim $8m+4$, the signature is divisible by 16.

For $4 | n$:

$H^{n/2}(M;\mathbb{Q}) \times H^{n/2}(M;\mathbb{Q}) \xrightarrow{\smile} H^n(M;\mathbb{Q})$ is a non-degenerate symmetric pairing:

signature = # positive eigenvalues − # negative eigenvalues.

even signature $\Rightarrow$ even dimension $\Rightarrow$ even χ

Thm 0: all manifolds M^n have $\chi = 0$ unless $n = 2m$.

Thm 1: orientable manifolds M^n have even χ unless $n = 4m$.

Thm 2: spin manifolds M^n have even χ unless $n = 8m$.

Thm 0: 0-orientable manifolds M^n have $\chi = 0$ unless $n = 2m$.

Thm 1: 1-orientable manifolds M^n have even χ unless $n = 4m$.

Thm 2: 2-orientable ~~spin~~ manifolds M^n have even χ unless $n = 8m$.

$\rightsquigarrow$ generalise to <u>k-orientable</u>

$\rightsquigarrow$??

Def A manifold is ℓ-parallelizable if τ admits a lift to $BO(n)\langle \ell \rangle$

$$\pi_{i\le \ell} = 0 \qquad \pi_{i>\ell} = \pi_i BO(n)$$

For $BO = \operatorname{colim}_n BO(n)$

$$\pi_i BO = \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z} \pmod 8$$

Q: How do I know if M is ℓ-parallelizable?

$\rightsquigarrow$ check what happens in cohomology

$$\begin{tikzcd}
 & * \arrow[d] \\
 & \vdots \arrow[d] \\
 & B5brane(n) = BO(n)\langle 8 \rangle \arrow[d] \\
 & BString(n) = BO(n)\langle 4 \rangle \arrow[d] \\
 & BSpin(n) = 2\text{-connected cover } BO(n) \arrow[d] \\
TM \arrow[d] & BSO(n) \simeq Gr^+(n,\infty) = \widetilde{BO(n)} \arrow[d] \\
M^n \arrow[r, "\tau"] \arrow[ur, dashed] \arrow[uur, dashed, "?"] & BO(n) \simeq Gr(n,\infty)
\end{tikzcd}$$

Def A manifold is *ℓ-parallelizable* if τ admits a lift to $BO(n)\langle \ell \rangle$ $\rightsquigarrow$ $\pi_{i\leq \ell} = 0$, $\pi_{i>\ell} = \pi_i BO(n)$

Def: *Characteristic classes* of M are the image of $\tau^* : H^*(BO(n);A) \to H^*(M;A)$

Stiefel-Whitney classes: $A = \mathbb{Z}/2$

in $BO = \operatorname{colim}_n BO(n)$, one w_i exists for all i, generated by w_{2^i}

$$\boxed{w_n(M^n) = \text{parity of } \chi(M^n)}$$

M ℓ-parallelizable $\Longrightarrow$ characteristic classes vanish deg $\leq \ell$

$w_1(M) = 0 \iff M$ orientable

$w_{1,2}(M) = 0 \iff M$ spinable

$$\begin{tikzcd}
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\end{tikzcd}$$

However, after spin, characteristic classes are incomplete obstructions to ℓ-parallelizability.

In particular, there exist manifolds w/ all char. classes vanishing that are not parallelizable.

Weaker & easier notion:

Def (H): A manifold M is **k-orientable** if

$$w_i(M) = 0 \quad \text{for} \quad 0 < i < 2^k$$

(or equivalently, $w_i = 0$ for $0 < i \leq 2^{k-1}$)

Def (H): A manifold M is **k-orientable** if

$$w_i(M) = 0 \quad \text{for} \quad 0 < i < 2^k$$

How does this relate to ℓ-parallelizability?

Naively, 2^{k-1}-parallelizable $\Rightarrow$ k-orientable

[Stong '63]: Stiefel-Whitney classes of $BO\langle \ell \rangle$ vanish exponentially with ℓ:

τ admits a lift to the k^{th} non-trivial cover of $BO(n)$ $\Rightarrow$ M is k-orientable

occurring at $\ell = 0, 1, 2, 4 \pmod 8$

k		corresponds to
0-orientable	=	all mfds
1-orientable	=	orientable
2-orientable	=	spinable
3-orientable	$\Leftarrow$	stringable
4-orientable	$\Leftarrow$	5braneable

k-orientable $\Leftarrow$ $\sim 2k$-parallelizable

Thm 0: 0-orientable manifolds M^n have $\chi = 0$ unless $n = 2m$.

Thm 1: 1-orientable manifolds M^n have even χ unless $n = 4m$.

Thm 2: 2-orientable ~~spin~~ manifolds M^n have even χ unless $n = 8m$.

Thm (H.): k-orientable manifolds M^n have even χ unless $n = 2^{k+1}m$

Moreover, if $4 \mid n$ but $2^{k+1} \nmid n$, then $8 \mid \sigma(M)$.

k		corresponds to	dimensions with odd χ possible
0-orientable	$=$	all mfds	$2m$
1-orientable	$=$	orientable	$4m$
2-orientable	$=$	spinable	$8m$
3-orientable	$\Leftarrow$	stringable	$16m$
4-orientable	$\Leftarrow$	5braneable	$32m$
k-orientable	$\Leftarrow$	$\sim 2k$-parallelizable	$2^{k+1}m$

k	corresponds to	dimensions with odd χ possible	k-orientable manifolds w/ odd χ	
0-orientable	= all mfds	$2m$	$\mathbb{RP}^{2m}$	$\chi = 1$
1-orientable	= orientable	$4m$	$\mathbb{CP}^{2m}$	$\chi = 2m+1$
2-orientable	= spinable	$8m$	$\mathbb{HP}^{2m}$	$\chi = 2m+1$
3-orientable	$\Leftarrow$ stringable	$16m$	$(\mathbb{OP}^2)^m$	$\chi = 3^m$
4-orientable	$\Leftarrow$ 5braneable	$32m$		
k-orientable	$\Leftarrow$ $\sim 2k$-parallelizable	$2^{k+1}m$		

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3-orientable	$\Leftarrow$ stringable	$16m$	$(\mathbb{OP}^2)^m$	$\chi = 3^m$
4-orientable	$\Leftarrow$ 5braneable	$32m$	$\mathcal{X}^{32m}$?	
k-orientable	$\Leftarrow$ ~2k-parallelizable	$2^{k+1}m$	$\mathcal{X}^{2^{k+1}m}$?	

Open question: Does there exist a 4-orientable manifold with odd Euler characteristic χ^{32m}?

4-orientable $= w_{1-8}(M)=0 \Leftarrow M$ string $/BO\langle 8\rangle \not\Leftarrow M$ 8-connected

In particular, is there an 8-connected manifold w/ odd χ?

No 4-orientable Poincaré complex w/ odd χ $\Longrightarrow$ Hopf invariant 1:

no division algebras beyond $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

$=$ no $\mathbb{X}P^2$: $\cdot\, a^2$, $\cdot\, a$, $\cdot\, 0$ for $|a|>8$

$$H^*(\mathbb{X}P^2; \mathbb{Z}) = \frac{\mathbb{Z}\langle a\rangle}{a^3}$$

Let's try to find a manifold of dim $32m$ with odd χ and show it's 4-orientable.

First attempt: $\mathbb{Q}\mathbb{P}^2$'s.

Def: A rational projective plane $\mathbb{Q}\mathbb{P}^2$ is a smooth, closed, simply conn. manifold with $H^*(M;\mathbb{Q}) = \dfrac{\mathbb{Q}\langle a\rangle}{a^3}$

$a^2 \bullet$

$a \bullet$ $\chi = 3$

$0 \bullet$

[Kennard, Su '17]:

$\mathbb{Q}\mathbb{P}^2$'s exist in dimensions $4, 8, 16, 32, 128$ and no more up to 2^{13} except for <u>possibly</u> in $544, 1024, 2048, 4160, 4352$

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But None of these are spin except for in dims 8 & 16 $(\mathbb{H}\mathbb{P}^2, \mathbb{O}\mathbb{P}^2)$

$\rightsquigarrow$ others are 1-orientable, not 2-orientable. $\Longrightarrow$ no candidates X^{32m}

<u>New attempt</u>: Symmetric spaces: quotients of Lie groups by sub-Lie groups

$\rightsquigarrow$ interesting manifolds

$$\mathbb{R}P^n = Gr(1, n+1) = \frac{O(n+1)}{O(n)\,O(1)}$$

$$\mathbb{C}P^n = \frac{U(n+1)}{U(n)\,U(1)}$$

$$\mathbb{H}P^n = \frac{Sp(n+1)}{Sp(n)\,Sp(1)}$$

$$\mathbb{O}P^2 = F_4 / Spin(9)$$

Cartan gave a complete classification of simply connected, compact symmetric spaces that are quotients of simple Lie groups.

$$G/K$$

7 series

12 exceptional ones

Label	G	K	Dimension
AI	$\mathrm{SU}(n)$	$\mathrm{SO}(n)$	$(n-1)(n+2)/2$
AII	$\mathrm{SU}(2n)$	$\mathrm{Sp}(n)$	$(n-1)(2n+1)$
AIII	$\mathrm{SU}(p+q)$	$\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$	$2pq$
BDI	$\mathrm{SO}(p+q)$	$\mathrm{SO}(p)\times\mathrm{SO}(q)$	pq
DIII	$\mathrm{SO}(2n)$	$\mathrm{U}(n)$	$n(n-1)$
CI	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	$n(n+1)$
CII	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p)\times\mathrm{Sp}(q)$	$4pq$
EI	E_6	$\mathrm{Sp}(4)/\{\pm I\}$	42
EII	E_6	$\mathrm{SU}(6)\cdot\mathrm{SU}(2)$	40
EIII	E_6	$\mathrm{SO}(10)\cdot\mathrm{SO}(2)$	32
EIV	E_6	F_4	26
EV	E_7	$\mathrm{SU}(8)/\{\pm I\}$	70
EVI	E_7	$\mathrm{SO}(12)\cdot\mathrm{SU}(2)$	64
EVII	E_7	$E_6\cdot\mathrm{SO}(2)$	54
EVIII	E_8	$\mathrm{Spin}(16)/\{\pm\mathrm{vol}\}$	128
EIX	E_8	$E_7\cdot\mathrm{SU}(2)$	112
FI	F_4	$\mathrm{Sp}(3)\cdot\mathrm{SU}(2)$	28
FII	F_4	$\mathrm{Spin}(9)$	16
G	G_2	$\mathrm{SO}(4)$	8

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3 Rosenfeld planes : odd χ

EIII	EVI	EVIII
$(\mathbb{O}\otimes\mathbb{C})\mathbb{P}^2$	$(\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2$	$(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2$
E_6	E_7	E_8
32	64	128
$\mathbb{Z}$-cohom. known	$\mathbb{Z}/2$-cohom known [Nakagawa '01]	?
2-orientable not 3-orientable [Ishitoya '91]		

Thm (H.):

$(\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2$ is 3-orientable

3 Rosenfeld planes: odd χ

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$\mathbb{Z}$-cohom. known	$\mathbb{Z}/2$-cohom known [Nakagawa '01]	?
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Thm (H.):

$(\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2$ is 3-orientable and possibly 4-orientable.

depends on 2 unknown coefficients in Sq^8 on a generator of the cohomology.

Status: Still open whether $\mathcal{X}^{32m}$ exists.

3 Rosenfeld planes : odd χ

EIII	EVI	EVIII
$(\mathbb{O}\otimes\mathbb{C})\mathbb{P}^2$	$(\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2$	$(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2$
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$\mathbb{Z}$-cohom. known	$\mathbb{Z}/2$-cohom known [Nakagawa '01]	?
2-orientable not 3-orientable [Ishitoya '91]		

The proof of:

Thm: k-orientable manifolds M^n have even χ unless $n = 2^{k+1} m$.

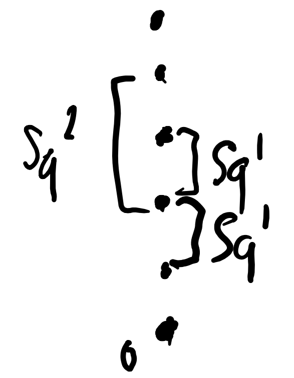
Moreover, if $4 \mid n$ but $2^{k+1} \nmid n$, then $8 \mid \sigma(M)$.

First, recap of some classical theory.

Steenrod squares: cohomology operations of $\mathbb{Z}/2$-cohom

$$Sq^i : H^k(X;\mathbb{Z}/2) \longrightarrow H^{k+i}(X;\mathbb{Z}/2)$$

$$Sq^{|x|}(x) = x \cup x \,, \quad Sq^0(x) = x \,, \quad Sq^{i>|x|}(x) = 0$$



Form Steenrod algebra,

obey Adem relations:

$$a<2b: \qquad Sq^a \circ Sq^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-1}{a-2c} Sq^{a+b-c} \circ Sq^c$$

Wu classes: For a manifold M^n, squares with target H^n can be represented by cupping with a class.

$$Sq^i : H^{n-i}(M;\mathbb{Z}/2) \longrightarrow H^n(M;\mathbb{Z}/2) \qquad i \leq \frac{n}{2}$$

$$x \longmapsto Sq^i(x) = v_i \cup x \qquad v_i \in H^i(M;\mathbb{Z}/2)$$

Wu relation:

$$w = Sq(v)$$

$w_1 = Sq^0 v_1 = v_1$

$w_2 = v_2 + Sq^1 v_1$

$\vdots$

- For M^{2n}, $w_{2n} = Sq^n v_n = v_n^2$
- $w_i = 0 \quad 0<i<2^k \iff v_i = 0 \quad 0<i<2^k$

The proof of:

Thm: k-orientable manifolds M^n have even χ unless $n = 2^{k+1} m$

Moreover, if $4 \mid n$ but $2^{k+1} \nmid n$, then $8 \mid \sigma(M)$.

$\Uparrow$

orientable mfds dim $4m+2$ have even χ

spin mfds dim $8m+4$ have even χ

$\vdots$

Thm': Let $M^{2n'}$ be a k-orientable manifold of dim $\underline{2n' = 2^{k+1} m + 2^k}$.

Then $\underline{v_{n'} = 0}$,

$v_n = 0 \Rightarrow w_{2n} = v_n^2 = 0 = \text{parity } \chi(M)$

& for $x \in H^n(M^{2n}; \mathbb{Z}/2)$, $x \cup x = Sq^n(x) = v_n \cup x = 0$

For $4 \mid 2n'$,

intersection form is

<u>even</u> symmetric bilin form

$\Rightarrow 8 \mid \sigma$

Thm': Let M^{2n} be a k-orientable manifold of dim $2n = 2^{k+1}m + 2^k$.

Then $v_n = 0$.

Proof: We show $Sq^n : H^n(M;\mathbb{Z}/2) \longrightarrow H^{2n}(M;\mathbb{Z}/2)$ vanishes by:

Claim: There is a relation in the Steenrod algebra:

$$Sq^n = Sq^{2^k m + 2^{k-1}} = \sum_{i=1}^{2^{k-1}} Sq^i \circ \underbrace{\eta^{n-i}}_{\text{sum of composites of squares}}$$

e.g.:

$$Sq^{4m+2} = Sq^2 Sq^{4m} + Sq^1 Sq^{4m} Sq^1$$

$$Sq^{4m+2}\left[\,\vdots\; z\,\right] + \ldots$$

Claim $\Rightarrow$ Thm':

M k-orientable $\Leftrightarrow$ $w_i = 0$ for $0 < i < 2^k$

$\Leftrightarrow$ $v_i = 0$ for $0 < i < 2^k$

$\Leftrightarrow$ Sq^i w/ target top cohom vanish $0 < i < 2^k$

$\Rightarrow$ Sq^n " " " " vanishes

$\Rightarrow$ $v_n = 0$

Claim: $$Sq^n = Sq^{2^k m + 2^{k-1}} = \sum_{i=1}^{2^{k-1}} Sq^i \circ \underbrace{\eta^{n-i}}_{\text{sum \& composites of squares.}}$$

Proof:

Adem relns: $$Sq^a \circ Sq^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-1}{a-2c} Sq^{a+b-c} \circ Sq^c \qquad (a<2b)$$

Apply to
$a = 2^{k-1}$
$b = 2^k m$

$$\underbrace{\binom{2^k m - 1}{2^{k-1}}}_{\text{non-zero mod 2}} Sq^{2^k m + 2^{k-1}} = \underbrace{Sq^{2^{k-1}} \circ Sq^{2^k m}}_{=} + \sum_{c=1}^{2^{k-2}} \binom{2^k m - c - 1}{2^{k-1} - 2c} \underbrace{Sq^{n-c}}_{=} \circ Sq^c$$

Reapply

Apply to
$a = 2^{k-1} - c$
$b = 2^k m$

$$\underbrace{\binom{2^k m - 1}{2^{k-1} - c}}_{\text{non-zero mod 2}} Sq^{n-c} = \underbrace{Sq^{2^{k-1}-c} \circ Sq^{2^k m}}_{=} + \sum_{c'=1}^{\lfloor 2^{k-2} - \frac{c}{2} \rfloor} \binom{2^k m - c' - 1}{2^{k-1} - c - 2c'} \underbrace{Sq^{n-c-c'}}_{=} \circ Sq^{c'}$$

$\Longrightarrow$ iterated application yields formula. $\square$

Thm:

$(\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2$ is 3-orientable and possibly 4-orientable.

Theorem ([Nak01])

$$H^*((\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2,\mathbb{Z}/2)=\mathbb{Z}/2\,[y_2,y_3,y_{12},y_{16},y_{20}]\,/\,J$$

where J is an ideal generated by twelve homogeneous relations:

$$J=\left(\begin{array}{l} y_3^3,\, y_{16}y_2+y_{12}y_3^2+y_2^6y_3^2,\, y_{16}y_3,\, y_{12}^2y_2+y_{12}y_2^4y_3^2+y_{20}y_3^2,\\ y_{12}^2y_3,\, y_{12}y_{16}+y_2^{14}+y_{12}y_2^5y_3^2+y_2^{11}y_3^2,\, y_{12}^3+y_{16}y_{20}+y_2^5y_{20}y_3^2,\\ y_{12}^2y_{16}+y_{20}^2+y_{12}y_2^{11}y_3^2,\, y_{12}^2y_{20}+y_{12}y_2^{13}y_3^2+y_{12}y_2^3y_{20}y_3^2,\\ y_{12}y_{16}^2+y_{12}y_2^{13}y_3^2,\, y_{16}^3+y_{12}y_{16}y_{20}+y_{12}y_2^5y_{20}y_3^2,\, y_{16}^2y_{20}+y_2^{13}y_{20}y_3^2 \end{array}\right)$$

$$Sq^1y_2 = y_3.$$

$$Sq^2(y_3)=y_2y_3.$$

$$\begin{aligned} Sq^2(y_{12}) &= y_2^7+y_2y_{12}+y_2^4y_3^2,\\ Sq^4(y_{12}) &= y_2^8+y_2^2y_{12}+\alpha' y_2^5y_3^2,\\ Sq^8(y_{12}) &= y_{20}+y_2^4y_{12}+\alpha'' y_2^7y_3^2+\beta'' y_2y_3^2y_{12}, \end{aligned}$$

for some coefficients α', α'', $\beta''\in\mathbb{Z}/2$,

$$\begin{aligned} Sq^2(y_{16}) &= 0,\\ Sq^4(y_{16}) &= y_2^7y_3^2,\\ Sq^8(y_{16}) &= y_{12}^2+\gamma'' y_2^9y_3^2+\delta'' y_2^3y_3^2y_{12}, \end{aligned}$$

for some coefficients γ'', $\delta''\in\mathbb{Z}/2$, and

$$\begin{aligned} Sq^2(y_{20}) &= y_2^{11}+y_2y_{20}+\mu y_2^8y_3^2+\nu y_2^2y_3^2y_{12},\\ Sq^4(y_{20}) &= y_{12}^2+y_2^6y_{12}+\mu' y_2^9y_3^2+\nu' y_2^3y_3^2y_{12},\\ Sq^8(y_{20}) &= y_{12}y_{16}+y_2^8y_{12}+\lambda'' y_2^{11}y_3^2+\mu'' y_2^5y_3^2y_{12}+\nu'' y_2y_3^2y_{20}, \end{aligned}$$

for some coefficients μ, ν, μ', ν', λ'', μ'', $\nu''\in\mathbb{Z}/2$.

$*$	b_i	generators
33	2	$y_2^9\, y_3\, y_{12},\ y_2^5\, y_3\, y_{20}$
34	5	$y_2^{11}\, y_{12},\ y_2\, y_{12}\, y_{20},\ y_2^3\, y_{12}\, y_{16} + y_2^4\, y_3^2\, y_{20},\ y_2^3\, y_{12}\, y_{16},\ y_2^7\, y_{20}$
35	3	$y_2^{10}\, y_3\, y_{12},\ y_3\, y_{12}\, y_{20},\ y_2^6\, y_3\, y_{20}$
36	6	$y_2^5\, y_3^2\, y_{20},\ y_2^4\, y_{12}\, y_{16} + y_2^5\, y_3^2\, y_{20},\ y_2^{12}\, y_{12},\ y_2^8\, y_{20},\ y_{16}\, y_{20},\ y_2^2\, y_{12}\, y_{20}$
37	3	$y_2^{11}\, y_3\, y_{12},\ y_2\, y_3\, y_{12}\, y_{20},\ y_2^7\, y_3\, y_{20}$
38	6	$y_3^2\, y_{12}\, y_{20} + y_2\, y_{16}\, y_{20},\ y_2^5\, y_{12}\, y_{16},\ y_2^{13}\, y_{12},\ y_2^3\, y_{12}\, y_{20},\ y_2\, y_{16}\, y_{20},\ y_2^9\, y_{20}$
39	3	$y_2^2\, y_3\, y_{12}\, y_{20},\ y_2^8\, y_3\, y_{20},\ y_2^{12}\, y_3\, y_{12}$
40	6	$y_2\, y_3^2\, y_{12}\, y_{20},\ y_2^{10}\, y_{20},\ y_2^2\, y_{16}\, y_{20},\ y_{12}^2\, y_{16},\ y_{12}^2\, y_{16} + y_{20}^2,\ y_2^4\, y_{12}\, y_{20}$
41	3	$y_2^9\, y_3\, y_{20},\ y_2^{13}\, y_3\, y_{12},\ y_2^3\, y_3\, y_{12}\, y_{20}$
42	5	$y_2^2\, y_3^2\, y_{12}\, y_{20} + y_2^3\, y_{16}\, y_{20} + y_2\, y_{20}^2,\ y_2^5\, y_{12}\, y_{20},$ $y_2^2\, y_3^2\, y_{12}\, y_{20} + y_2^3\, y_{16}\, y_{20},\ y_2^2\, y_3^2\, y_{12}\, y_{20},\ y_2^{11}\, y_{20}$
43	2	$y_2^4\, y_3\, y_{12}\, y_{20},\ y_2^{10}\, y_3\, y_{20}$
44	5	$y_2^6\, y_{12}\, y_{20},\ y_2^4\, y_{16}\, y_{20},\ y_{12}^2\, y_{20} + y_2^2\, y_{20}^2, y_2^{12}\, y_{20},\ y_{12}^2\, y_{20} + y_2^4\, y_{16}\, y_{20}$
45	2	$y_2^5\, y_3\, y_{12}\, y_{20},\ y_2^{11}\, y_3\, y_{20}$
46	4	$y_2\, y_{12}^2\, y_{20},\ y_2\, y_{12}^2\, y_{20} + y_2^5\, y_{16}\, y_{20},\ y_2^7\, y_{12}\, y_{20},\ y_2^{13}\, y_{20}$
47	2	$y_2^6\, y_3\, y_{12}\, y_{20},\ y_2^{12}\, y_3\, y_{20}$
48	4	$y_{12}\, y_{16}\, y_{20},\ y_2^6\, y_{16}\, y_{20},\ y_2^2\, y_{12}^2\, y_{20} + y_{12}\, y_{16}\, y_{20},\ y_2^8\, y_{12}\, y_{20}$

$*$	b_i	generators
49	2	$y_2^7\, y_3\, y_{12}\, y_{20},\ y_2^{13}\, y_3\, y_{20}$
50	3	$y_2\, y_{12}\, y_{16}\, y_{20},\ y_2^7\, y_{16}\, y_{20} + y_2\, y_{12}\, y_{16}\, y_{20},\ y_2^9\, y_{12}\, y_{20}$
51	1	$y_2^8\, y_3\, y_{12}\, y_{20}$
52	3	$y_2^2\, y_{12}\, y_{16}\, y_{20} + y_{12}\, y_{20}^2,\ y_2^2\, y_{12}\, y_{16}\, y_{20},\ y_2^{10}\, y_{12}\, y_{20}$
53	1	$y_2^9\, y_3\, y_{12}\, y_{20}$
54	2	$y_2^{11}\, y_{12}\, y_{20},\ y_2^3\, y_{12}\, y_{16}\, y_{20}$
55	1	$y_2^{10}\, y_3\, y_{12}\, y_{20}$
56	2	$y_2^{12}\, y_{12}\, y_{20},\ y_2^4\, y_{12}\, y_{16}\, y_{20}$
57	1	$y_2^{11}\, y_3\, y_{12}\, y_{20}$
58	2	$y_2^{13}\, y_{12}\, y_{20},\ y_2^5\, y_{12}\, y_{16}\, y_{20}$
59	1	$y_2^{12}\, y_3\, y_{12}\, y_{20}$
60	1	y_{20}^3
61	1	$y_2^{13}\, y_3\, y_{12}\, y_{20}$
62	1	$y_2\, y_{20}^3$
63	0	
64	1	$y_2^2\, y_{20}^3$

Check 4-orientable:

Do v_1, v_2, v_4, v_8 vanish?

$\rightsquigarrow Sq^1, Sq^2, Sq^4, Sq^8$ up to H^{64}

Need Cartan formula:

$$Sq^n(x \cup y) = \sum_{i+j=n} Sq^i(x) \cup Sq^j(y)$$

$$Sq^n\left(\prod_{j=1}^k x_j\right) = \sum_{\{p\}} \prod_{j=1}^k Sq^{p_j} x_j$$

Using computer, $v_1 = v_2 = v_4 = 0$, but:

$$\boxed{\begin{aligned} Sq^8(y_2^{12} y_{12} y_{20}) &= (1 + \beta'' + \nu'') y_2 y_{20}^3 \\ Sq^8(y_2^9 y_3^2 y_{12} y_{20}) &= 0. \end{aligned}}$$

$$
\begin{aligned}
Sq^8(y_2^{12}y_{12}y_{20}) &= (1+\beta''+\nu'')y_2y_{20}^3\\
Sq^8(y_2^9y_3^2y_{12}y_{20}) &= 0.
\end{aligned}
$$

Thm:

$(\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2$ is 3-orientable and possibly 4-orientable.

Theorem ([Nak01])

$$H^*((\mathbb{O}\otimes\mathbb{H})\mathbb{P}^2,\mathbb{Z}/2)=\mathbb{Z}/2\,[y_2,y_3,y_{12},y_{16},y_{20}]\,/\,J$$

where J is an ideal generated by twelve homogeneous relations:

$$
J=\left(\begin{array}{l}
y_3^3,\, y_{16}y_2+y_{12}y_3^2+y_2^6y_3^2,\, y_{16}y_3,\, y_{12}^2y_2+y_{12}y_2^4y_3^2+y_{20}y_3^2,\\
y_{12}^2y_3,\, y_{12}y_{16}+y_2^{14}+y_{12}y_2^5y_3^2+y_2^{11}y_3^2,\, y_{12}^3+y_{16}y_{20}+y_2^5y_{20}y_3^2,\\
y_{12}^2y_{16}+y_{20}^2+y_{12}y_2^{11}y_3^2,\, y_{12}^2y_{20}+y_{12}y_2^{13}y_3^2+y_{12}y_2^3y_{20}y_3^2,\\
y_{12}y_{16}^2+y_{12}y_2^{13}y_3^2,\, y_{16}^3+y_{12}y_{16}y_{20}+y_{12}y_2^5y_{20}y_3^2,\, y_{16}^2y_{20}+y_2^{13}y_{20}y_3^2
\end{array}\right)
$$

$$Sq^1y_2 = y_3.$$

$$
\begin{aligned}
Sq^2(y_{12}) &= y_2^7+y_2y_{12}+y_2^4y_3^2,\\
Sq^4(y_{12}) &= y_2^8+y_2^2y_{12}+\alpha' y_2^5y_3^2,\\
Sq^8(y_{12}) &= y_{20}+y_2^4y_{12}+\alpha'' y_2^7y_3^2+\beta'' y_2y_3^2y_{12},
\end{aligned}
$$

for some coefficients α', α'', $\beta''\in\mathbb{Z}/2$,

$$
\begin{aligned}
Sq^2(y_{16}) &= 0,\\
Sq^4(y_{16}) &= y_2^7y_3^2,\\
Sq^8(y_{16}) &= y_{12}^2+\gamma'' y_2^9y_3^2+\delta'' y_2^3y_3^2y_{12},
\end{aligned}
$$

for some coefficients γ'', $\delta''\in\mathbb{Z}/2$, and

$$
\begin{aligned}
Sq^2(y_{20}) &= y_2^{11}+y_2y_{20}+\mu y_2^8y_3^2+\nu y_2^2y_3^2y_{12},\\
Sq^4(y_{20}) &= y_{12}^2+y_2^6y_{12}+\mu' y_2^9y_3^2+\nu' y_2^3y_3^2y_{12},\\
Sq^8(y_{20}) &= y_{12}y_{16}+y_2^8y_{12}+\lambda'' y_2^{11}y_3^2+\mu'' y_2^5y_3^2y_{12}+\nu'' y_2y_3^2y_{20},
\end{aligned}
$$

for some coefficients μ, ν, μ', ν', λ'', μ'', $\nu''\in\mathbb{Z}/2$.