Multiple zeta values in deformation quantization

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"Phase space" = manifold/variety X equipped with a **Poisson bracket**

$$\{-,-\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

Axioms:

1. $\{f, g\} = -\{g, f\}$
2. $\{f, gh\} = \{f, g\}h + g\{f, h\}$
3. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

Example (1d particle): $X = \mathbb{R}^2_{(x,p)} = T^*\mathbb{R}$, functions $\mathcal{O}_X = \mathbb{C}[x, p]$

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} \qquad \{x, p\} = 1$$

Quantization

Example (1d particle): replace $x \rightsquigarrow \hat{x} = x\cdot$ and $p \rightsquigarrow \hat{p} = -\hbar\partial_x$

$$A_\hbar = \frac{\mathbb{C}\langle\hat{x},\hat{p}\rangle}{\hat{x}\hat{p}-\hat{p}\hat{x}=\hbar} \cong (\mathbb{C}[x,p],\star_\hbar)$$

using Moyal–Groenewold star product:

$$f\star_\hbar g = \sum_{n=0}^{\infty}\frac{\hbar^n}{2^n n!}\sum_{i=0}^{n}(-1)^i\frac{\partial^n f}{\partial x^i\partial p^{n-i}}\frac{\partial^n g}{\partial x^{n-i}\partial p^i} = fg + \tfrac{1}{2}\hbar\{f,g\} + O(\hbar^2)$$

Definition (Bayen–Flato–Fronsdal–Lichnerowicz–Sternheimer 1978)

Let $(X,\{-,-\})$ be a Poisson manifold. A **deformation quantization** of $(X,\{-,-\})$ is a family of associative products $\star_\hbar$ on $\mathcal{O}_X$ such that

$$f\star_\hbar g = fg + \tfrac{1}{2}\hbar\{f,g\} + O(\hbar^2)$$

Today: only formal deformations

$$\star : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X[[\hbar]]$$

$$f \star g = fg + \hbar \underbrace{B_1(f,g)}_{=\frac{1}{2}\{f,g\}} + \hbar^2 B_2(f,g) + \cdots$$

Basic question: Does a quantization of $(X, \{-,-\})$ always exist?

Answer when $(\{x_i, x_j\})_{i,j}$ **nondegenerate (symplectic):** yes – Berezin, Deligne, Fedosov, Kirillov, Kostant, Schlichenmaier, Souriau, Toeplitz, ...

Kontsevich formality theorem

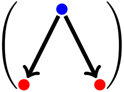
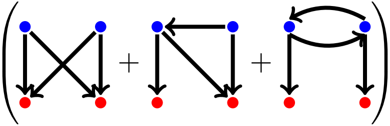
Theorem (Kontsevich 1997)

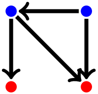
Every smooth Poisson manifold X has a "canonical" quantization. In fact there is an equivalence

$$\frac{\{\textit{Poisson brackets on } X\}}{\sim} \cong \frac{\{\textit{noncommutative deformations of } \mathcal{O}_X\}}{\sim}$$

Precise statement is stronger: $\wedge^\bullet T_X \cong CC^*(\mathcal{O}_X)$ as dg Lie algebras

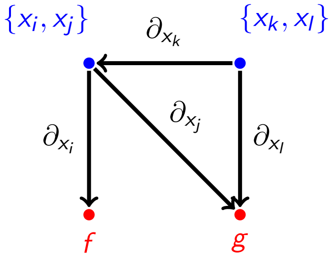
Explicit formula when $X = \mathbb{R}^n$:

$f \star g = fg + \hbar$  $+ \hbar^2$  $+ \cdots$

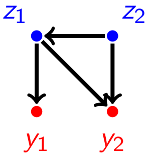
 $= \begin{pmatrix}\text{complicated} \\ \text{integral}\end{pmatrix} \cdot \begin{pmatrix}\text{derivatives of} \\ f, g \text{ and } \{-,-\}\end{pmatrix}$

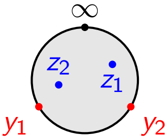
Cattaneo–Felder: it's the Feynman expansion for a disk correlator in a 2d TFT (Ikeda/Schaller–Strobl's Poisson sigma model)

Given $\{-,-\}$ in coordinates $x_1, \ldots, x_n$ on $\mathbb{R}^n$, want to compute $f \star g$

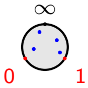


$$\begin{pmatrix} \text{derivatives of} \\ f, g \text{ and } \{-,-\} \end{pmatrix} := \sum_{i,j,k,l} (\partial_{x_i} f) \cdot \left(\partial_{x_j}\partial_{x_l} g\right) \cdot \left(\partial_{x_k}\{x_i, x_j\}\right) \cdot \{x_k, x_l\}$$

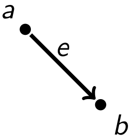


$\rightsquigarrow$ $\mathfrak{C}_{n,\,m} = \left\{ \right.$ 

$\left. \right\} \Big/$ holomorphic iso.

e.g. $\mathfrak{C}_{n,2} \cong \left\{ \right.$ 

$\left. \right\} \cong \mathbb{H}^n \setminus \{z_i = z_j\}_{i \neq j}$



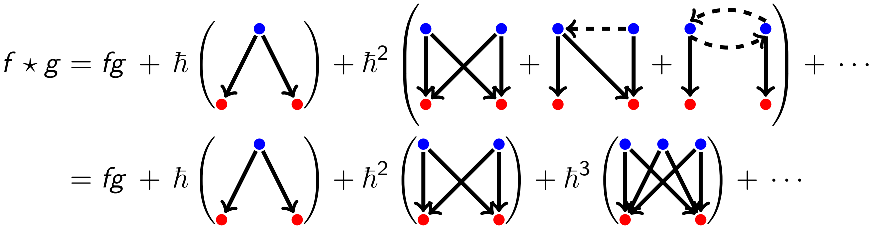
$$\rightsquigarrow \qquad 2\alpha_e := \frac{d \log(a, \overline{a}; b, \infty)}{2\mathrm{i}\pi} + \frac{d \log(a, \overline{a}; \overline{b}, \infty)}{2\mathrm{i}\pi}$$

$$\omega := \alpha_{e_1} \wedge \cdots \wedge \alpha_{e_N} \in 2^{-N}\mathbb{Z} \left\langle \frac{d \log f}{2\mathrm{i}\pi} \,\middle|\, f \text{ a cross ratio} \right\rangle \subset \Omega^\bullet(\mathfrak{C}_{n,m})$$

$$\begin{pmatrix}\text{complicated} \\ \text{integral}\end{pmatrix} := \int_{\mathfrak{C}_{n,m}} \omega \in \mathbb{R}$$

Recovering Groenewold–Moyal

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial p} \qquad \begin{pmatrix} \{x, x\} & \{x, p\} \\ \{p, x\} & \{p, p\} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



$$= \sum_{n=0}^{\infty} \frac{\hbar^n}{2^n n!} \sum_{i=0}^{n} (-1)^i \frac{\partial^n f}{\partial x^i \partial p^{n-i}} \frac{\partial^n g}{\partial x^{n-i} \partial p^i}$$

$X = \mathbb{R}^n$ with coordinates x_i, linear bracket

$$\{x_i, x_j\} = \sum c_{ij}^k x_k \qquad \leftrightarrow \qquad \text{Lie algebra } \mathfrak{g}$$

Similar analysis:

- Series truncates for $f, g \in \mathbb{C}[x_i]$
- Can compute

$$x_i \star x_j - x_j \star x_i = \hbar \sum c_{ij}^k x_k$$

- Conclude

$$(\mathbb{C}[x_i], \star_\hbar) \cong \frac{\mathbb{C}\langle x_i \rangle}{x_i x_j - x_j x_i = \hbar \sum c_{ij}^k x_k} =: U(\mathfrak{g}, \hbar)$$

$$\{X, P\} = XP$$

$$X \star P = q(\hbar) XP \qquad P \star X = q(-\hbar) XP$$

Our software:

$$q(\hbar) = 1 + \frac{\hbar}{2} + \frac{\hbar^2}{24} - \frac{\hbar^3}{48} - \frac{\hbar^4}{1440} + \frac{\hbar^5}{480} + \left(\frac{251\zeta(3)^2}{2048\pi^6} - \frac{17}{184320} \right) \hbar^6 + \cdots$$

Nevertheless: algebra determined by

$$X \star P = \frac{q(\hbar)}{q(-\hbar)} P \star X = e^{\hbar} P \star X$$

Morally: $X = e^x$ and $P = e^p$ where $\{x, p\} = 1$.

$$\zeta(s) = \sum_{k \geq 1} \frac{1}{k^s}$$

Theorem (Euler 1735): $\zeta(2m) = (-1)^{m+1} \frac{B_{2m}(2\pi)^{2m}}{2(2m)!} \in \mathbb{Q}\pi^{2m}$

Open Question: Is $\zeta(2m+1) \in \mathbb{Q}(\pi)$?

Conjecture: $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

Theorem (Apéry 1978): $\zeta(3) \notin \mathbb{Q}$

Theorem ((Ball–)Rivoal 2000): Infinitely many $\zeta(3), \zeta(5), \zeta(7), \ldots \notin \mathbb{Q}$

Theorem (Zudilin 2000): At least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11) \notin \mathbb{Q}$

Definition

A **normalized multiple zeta value (MZV)** of weight n is a number of the form

$$\widetilde{\zeta}(n_1, \ldots, n_d) = \frac{1}{(2i\pi)^n} \sum_{k_1 > k_2 > \cdots > k_d \geq 1} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_d^{n_d}} \in \begin{cases} \mathbb{R} & n \text{ even} \\ i\mathbb{R} & n \text{ odd} \end{cases}$$

where $n_1 \geq 2$ and $n_1 + \cdots + n_d = n$.

Additional "honourary" normalized MZVs:

- $1 \in \mathbb{R}$ has weight 0
- $\frac{1}{2} = \frac{i\pi}{2i\pi} \in \mathbb{R}$ has weight 1

$$\widetilde{\mathcal{Z}} := \mathbb{Z} \cdot \{\text{normalized MZVs}\} \subset \mathbb{C}$$

Weight filtration:

$$\begin{array}{ccccccccc} \widetilde{\mathcal{Z}}_0 & \subset & \widetilde{\mathcal{Z}}_1 & \subset & \widetilde{\mathcal{Z}}_2 & \subset & \cdots & \subset & \widetilde{\mathcal{Z}} \\ \| & & \| & & \| & & & & \\ \mathbb{Z} & \subset & \mathbb{Z} \cdot \frac{1}{2} & \subset & \mathbb{Z} \cdot \underbrace{\frac{\zeta(2)}{(2\mathrm{i}\pi)^2}}_{=\frac{-1}{24}} & \subset & \cdots & & \end{array}$$

Shuffle product:

$$\widetilde{\mathcal{Z}}_m \widetilde{\mathcal{Z}}_n \subset \widetilde{\mathcal{Z}}_{m+n}$$

e.g.

$$\widetilde{\zeta}(m)\widetilde{\zeta}(n) = \widetilde{\zeta}(m, n) + \widetilde{\zeta}(n, m) + \widetilde{\zeta}(n + m)$$

For unnormalized MZVs:

- $\mathbb{Q}$-dimension of weight spaces conjectured by Zagier
  - Proven to be an upper bound (Terasoma, Deligne–Goncharov)
- $\mathbb{Q}$-basis conjectured by Hoffman: $\zeta(2s$ and $3s)$
  - Proven to generate (Brown)

For normalized MZVs:

$\mathbb{Z}$-module generators of $\widetilde{\mathcal{Z}}_n$

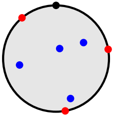
n	0	1	2	3	4	5	6
real	1	$\frac{1}{2}$	$\frac{1}{24}$	$\frac{1}{48}$	$\frac{1}{5760}$	$\frac{1}{11520}$	$\frac{1}{2903040}$ $\frac{\zeta(3)^2}{128\pi^6}$
imaginary				$\frac{i\zeta(3)}{8\pi^3}$	$\frac{i\zeta(3)}{16\pi^3}$	$\frac{i\zeta(3)}{192\pi^3}$ $\frac{i\zeta(5)}{64\pi^5}$	$\frac{i\zeta(3)}{384\pi^3}$ $\frac{i\zeta(5)}{128\pi^5}$

- **Quantum groups:** coefficients of Drinfel'd associator
- **Knot theory:** coefficients of Kontsevich integral
- **Homotopical algebra:** formality of the operad E_2
- **Algebraic geometry:** periods integrals on moduli space $\mathcal{M}_{0,N}$ (Brown 2006, conj. by Goncharov–Manin)
- **Physics:** values of certain Feynman integrals
- ...

Theorem (Brown 2011, building on Deligne–Goncharov, Levine, Voevodsky, Zagier, . . .)

All periods of unramified mixed Tate motives lie in $\mathbb{Q}\widetilde{\mathcal{Z}}[\frac{1}{2i\pi}]$.

Main result

$$\mathfrak{C}_{n,m} = \Big\{$$  $$\Big\} \Big/ \text{holomorphic iso.}$$

$$\mathcal{A}^{\bullet}(\mathfrak{C}_{n,m}) := \mathbb{Z}\left\langle \frac{d\log f}{2\mathrm{i}\pi} \;\middle|\; f \text{ a cross ratio} \right\rangle \subset \Omega^{\bullet}(\mathfrak{C}_{n,m})$$

Theorem (Banks–Panzer–P.)

Suppose that $\omega \in \mathcal{A}^{dim}(\mathfrak{C}_{n,m})$ *is absolutely integrable. Then*

$$\int_{\mathfrak{C}_{n,m}} \omega \in \begin{cases} \widetilde{\mathcal{Z}}_{n+m-2} & m > 0 \\ \widetilde{\mathcal{Z}}_{n-1} & m = 0 \end{cases}$$

Corollary (case $m = 2$)

Coefficients at order $\hbar^n$ *in Kontsevich's star product lie in* $4^{-n}\widetilde{\mathcal{Z}}_n \cap \mathbb{R}$

$$\widetilde{\zeta}(n_1,\ldots,n_d) = \frac{1}{(2\mathrm{i}\pi)^n} \sum_{k_1>k_2>\cdots>k_d\geq 1} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_d^{n_d}} \qquad = L_{n_1,\ldots,n_d}(1)$$

in terms of **multiple polylogarithm**

$$L_{n_1,\ldots,n_d}(z) := \frac{1}{(2\mathrm{i}\pi)^n} \sum_{k_1>k_2>\cdots>k_d\geq 1} \frac{z^{k_1}}{k_1^{n_1} k_2^{n_2} \cdots k_d^{n_d}}$$

e.g.

$$L_1(z) = \sum_{k\geq 1} \frac{z^k}{k} = \frac{\log(1-z)}{2\mathrm{i}\pi} \qquad L_2(z) = \text{dilogarithm}$$

Alternate notation:

$$n_1,\ldots,n_d \qquad \leftrightarrow \qquad s_1\cdots s_n = \underbrace{00\cdots01}_{n_1}\underbrace{00\cdots01}_{n_2}\cdots\underbrace{00\cdots01}_{n_d}$$

Check:

$$dL_{s_1\cdots s_n} = (-1)^{s_1} \frac{L_{s_2\cdots s_n} dz}{2\mathrm{i}\pi(z-s_1)}$$

Rewrite

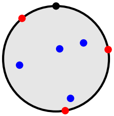
$$\widetilde{\zeta}(n_1, \ldots, n_d) = L_{s_1 \cdots s_n}(1) \qquad dL_{s_1 \cdots s_n} = (-1)^{s_1} \frac{L_{s_2 \cdots s_n} dz}{2\mathrm{i}\pi(z - s_1)}$$

and therefore (Kontsevich, Le–Murakami)

$$\widetilde{\zeta}(n_1, \ldots, n_d) = (-1)^d \underbrace{\int_0^1 \frac{dt_1}{2\mathrm{i}\pi(t_1 - s_1)} \int_0^{t_1} \frac{dt_2}{2\mathrm{i}\pi(t_2 - s_2)} \cdots \int_0^{t_{n-1}} \frac{dt_n}{2\mathrm{i}\pi(t_n - s_n)}}_{\displaystyle \int_0^1 s_1 \cdots s_n}$$

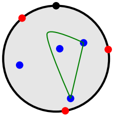
Chen iterated integral

NB: diverges if $s_1 = 1$ or $s_n = 0$, so “regularize”: $\log(\epsilon) = 0$

$\mathfrak{C}_{n,m} = \{$  $\} \big/$ holomorphic iso.

Choose $s_0, s_1, \ldots, s_{n+1} \in \{z_i, \bar{z}_i, y_i\}$, define "disk polylog" (multivalued!)

$$L_{s_0;s_1\cdots s_n;s_{n+1}} : \mathfrak{C}_{n,m} \to \mathbb{C}$$



$$\mapsto \int_{s_0}^{s_{n+1}} s_1 \cdots s_n$$

regularizing divergences via Deligne's tangential base points.

These functions and their differentials generate a locally constant subsheaf

$$\mathcal{A}^\bullet(\mathfrak{C}_{n,m}) \subset \mathcal{U}^\bullet_{\mathfrak{C}_{n,m}} \subset \Omega^\bullet_{\mathfrak{C}_{n,m}}$$

with monodromy unipotent for the weight filtration.

$$\text{Constants:} \quad \widetilde{\mathcal{Z}} \subset \mathcal{U}^0_{\mathfrak{C}_{n,m}}$$

Theorem (BPP "de Rham theorem for disk polylogs")

$\mathcal{U}^\bullet$ is a resolution of the constant sheaf $\widetilde{\mathcal{Z}}$ by acyclic local systems. Hence

$$H^\bullet(\mathcal{U}^\bullet(\mathfrak{C}_{n,m}), d) \cong H^\bullet(\mathfrak{C}_{n,m}; \widetilde{\mathcal{Z}}).$$

Sketch of proof.

Induction on n, m via $f : \mathfrak{C}_{n,m} \to \mathfrak{C}_{n-k,m-j}$.

Resolution: $\mathbb{Z}$-linear lift of Brown's Poincaré lemma via fibrewise KZ equation $dL = L' \cdot dz/(z-s)$.

Acyclic: have $\mathfrak{C}_{n,m} = K(PureBraids_n, 1)$, show group cohomology of the monodromy representation vanishes (i.e. $R^{>0} f_* \mathcal{U}^\bullet_{\mathfrak{C}_{n,m}} = 0$). □

Corollary

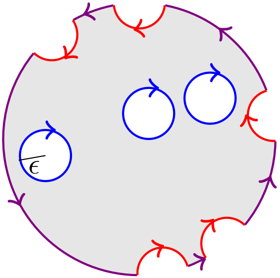
Every volume form in $\mathcal{U}^\bullet(\mathfrak{C}_{n,m})$ has a primitive in $\mathcal{U}^\bullet(\mathfrak{C}_{n,m})$.

Theorem (BPP "Fubini theorem for disk polylogs")

Given $f : \mathfrak{C}_{n,m} \to \mathfrak{C}_{n-k,m-j}$ *and integrable* $\omega \in \mathcal{U}^\bullet(\mathfrak{C}_{n,m})$*, have*

$$\int_{\mathfrak{C}_{n,m}} \omega = \int_{\mathfrak{C}_{n-k,m-j}} \left(\int_{fibres} \omega \right) \qquad \int_{fibres} \omega \in \mathcal{U}^\bullet(\mathfrak{C}_{n-k,m-j})$$

and weight drops by k*. Main theorem:* $\omega \in \mathcal{A}^\bullet(\mathfrak{C}_{n,m})$ *and* $f : \mathfrak{C}_{n,m} \to pt$*.*



$$\int_{disk} \frac{L dz \wedge d\overline{z}}{(z-s)(\overline{z}-z)}$$
$$= \lim_{\epsilon \to 0} \int_{\partial_\epsilon disk} \tilde{L} \frac{dz}{z-s}$$
$$= \sum Res + \int_{outer\ \ cycle}$$

estimates + unipotent monodromy
$\rightsquigarrow$ weight drop

Conjecture (BPP)

Coefficients of the star product at $\hbar^n$ generate $\widetilde{\mathcal{Z}}_n$.

Strategy: operadic motivic lift (in progress with Dupont and Panzer)

Convergence of power series? Motivic Galois action?

Thank you!