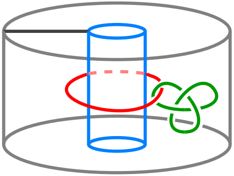
Categorification of Verma modules in low-dimensional topology

Pedro Vaz (Université catholique de Louvain)



$$\begin{tikzcd}
\mathcal{C}_{\smiley} \arrow[r] \arrow[d, two heads, "K_0"'] & \mathcal{M}(\lambda) \arrow[d, two heads, "K_0"] \\
\mathfrak{g} \arrow[r] \arrow[ur] & M(\lambda) = \mathfrak{g} \otimes_{\mathfrak{b}} \mathbb{C}_\lambda
\end{tikzcd}$$

$\mathrm{JWW} \subset \{$Abel Lacabanne, Grégoire Naisse, Ruslan Maksimau, Elia Rizzo$\}$

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Topology ∩ RT ∩ Verma modules

Verma modules are fundamental objects in rep. theory of Lie algebras (e.g. every f.d. irreducible of a Lie algebra is a quotient of a Verma).

Beyond its interest in RT, there were recently found to have applications to topology :

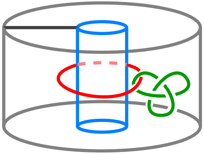
- braid grp reps : Burau, Lawrence–Krammer–Bigelow (Jackson–Kerler '11),
- HOMFLYPT invariants (Naisse–V. '17),
- annular Jones invariants (Iohara–Lehrer–Zhang 18'),

Keywords/minitoc :

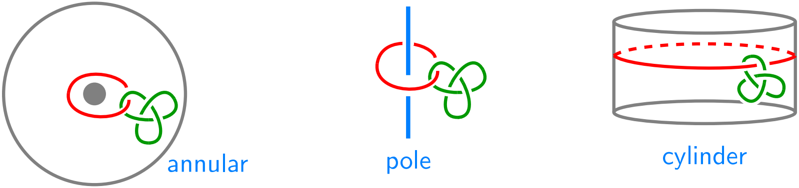
➔ Links in H_g • Jones poly. of type B • blob algebra • Verma modules
➔ Higher Rep.Theory • 2-Verma modules • 2-blob algebra.

This is the first mathematical slide

Suppose you have a link in a solid torus

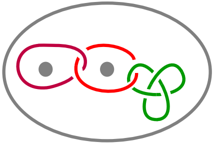


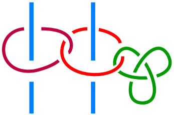
This gives rise to three different kinds of link diagrams :



Extending invariants like Jones's $\mathfrak{sl}_2$-link polynomial from S^3 to the solid torus results in the Jones poly. of type B (Geck–Lambropoulou '97).

This (and similar invs) generalizes to higher genus handlebodies : just think of link diagrams on a disk with g-punctures, or wiggling around g poles (the cylinder is a particular case that only seems to work for $g = 1$).





☞ This is where things get different ! If you want to do braids/tangles you need to use poles (how do you compose tangles in disk with $g > 1$ punctures ?) If you plan to think of more general 3-manifolds it is perhaps useful to use tangles...

Jones ($\mathfrak{sl}_2$) invariant

	disc$\setminus^{\{g \text{ points}\}}$	pole	
	$g \geq 1$	$g = 1$	$g > 1$
WRT	?	Iohara–Lehrer–Zhang ✓ '18	?
Categorification	Asaeda–Przytycki–Sikora ✓ '04	Lacabanne–Naisse–V. ✓ '20	?

☞ ∃ interesting categorification for the cylinder by Ehrig and Tubbenhauer '17.

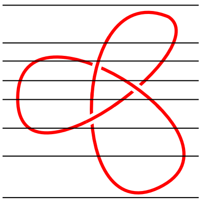
☞ For $\mathrm{WRT}(\mathrm{g} = 1, \mathfrak{g} = \mathfrak{gl}_{\mathrm{k}}, \mathfrak{p} \subseteq \mathfrak{g})$: Lacabanne–V. '20.

Goal of this talk

☞ explain a categorification of the Jones polynomial for links in the solid torus in the pole picture via a categorification of the blob algebra.

ILZ's blob algebra and the pole Jones invariant

The main idea of WRT is to construct quantum link polynomials via a 0+1 TQFT. Consider (quantum) $\mathfrak{sl}_2$ and its 2-dim irrep $V = \mathbb{C}^2(q)$. Since $\mathfrak{sl}_2$ is a Hopf algebra its category of f.d. reps is monoidal. It is even braided...



Operator-invariant of tangles!

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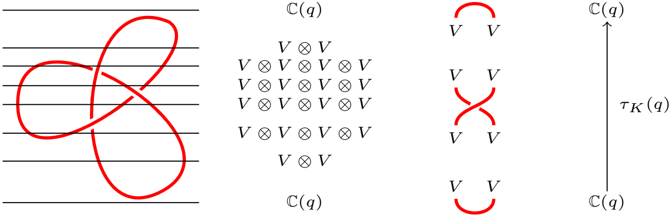

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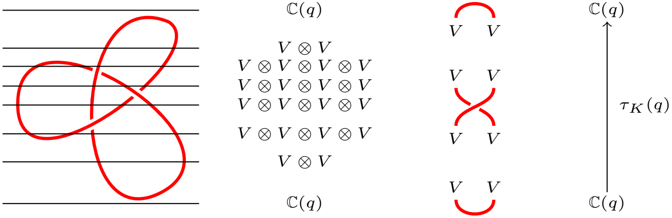


👉 Operator-invariant of tangles !

The Temperlery–Lieb algebra

Pick your favorite natural number d. Then

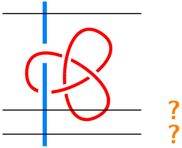
$$\mathrm{End}_{\mathfrak{sl}_2}(V^{\otimes d}) = \mathrm{TL_d}$$

is generated by (d strands)

 and  ($d-1$ of them)

modulo planar isotopies and the local relation $\bigcirc = -(q+q^{-1})$.

If one wants to extend WRT to links in a solid torus, one has do deal with the pole. Note that we had pushed the diagram to the right...



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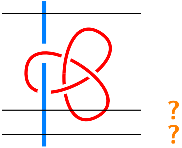
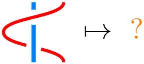




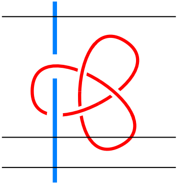
The solution is ... Verma modules!

Verma modules are certain infinite-dimensional representations that have several remarkable properties. For example, every f.d. irrep is a quotient of a Verma module.

$$
\begin{tikzcd}
 & & & & & & & {\color{blue}\mathrm{F}} \\
\cdots \arrow[r, bend right, "{[\lambda;k+1]}"'] & m_{k+1} \arrow[l, bend right, "{[k+2]}"'] \arrow[r, bend right, "{[\lambda;k]}"'] & m_k \arrow[l, bend right, "{[k+1]}"'] \arrow[loop below, "\lambda q^{-2k}"] & \cdots & m_2 \arrow[r, bend right, "{[\lambda;1]}"'] & m_1 \arrow[l, bend right, "{[2]}"'] \arrow[r, bend right, "{[\lambda;0]}"'] & m_0 \arrow[l, bend right, "{[1]}"'] \arrow[loop below, "\lambda"] & {\color{green}\mathrm{K}} \\
 & & & & & & & {\color{red}\mathrm{E}}
\end{tikzcd}
$$

$$[\lambda;k] = \frac{\lambda q^{-k} - \lambda^{-1}q^{k}}{q-q^{-1}}, \qquad [n] = \frac{q^n - q^{-n}}{q-q^{-1}}, \qquad [\mathrm{E},\mathrm{F}] = \frac{\mathrm{K}-\mathrm{K}^{-1}}{q-q^{-1}}$$

This is the universal Verma module for $\mathfrak{sl}_2$, irred. over the field $\Bbbk(\lambda, q)$.



$M(\lambda) \otimes V \otimes V$

$M(\lambda)$

$$\begin{tikzcd} M(\lambda) \\ M(\lambda) \arrow[u, "\zeta_K(q,\lambda)"'] \end{tikzcd}$$

The blob algebra (Martin–Saleur '94)

Iohara–Lehrer–Zhang '18 : $\mathrm{End}_{\mathfrak{sl}_2}(M(\lambda) \otimes V^d) = \mathcal{B}_d(\lambda, q)$

☞ Note that the Verma appears at the left (we have pushed our link diagram to the right).

⚠ The double braiding  is not a composite of two crossings.

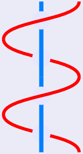
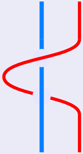
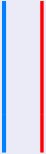
The blob algebra

is defined by generators (one blue and d red strands)

 and 

together with the identity, modulo planar isotopies and the local relations  $= -(q+q^{-1})$, and

 $= -(\lambda q + \lambda^{-1} q^{-1})$

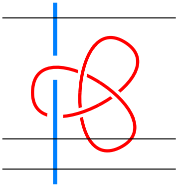
q^{-1}  $= \ (\lambda q + \lambda^{-1} q^{-1})$  $- \ q$ 

It sounds like a plan!

Recall that we want to see ...

... a categorification of the Jones polynomial for links in the solid torus in the pole picture via a categorification of the blob algebra.

- The main tool is a categorification of a $\otimes$ of a Verma module and several irreps of $\dim = 2$, which is realized as derived category of a certain DGA.
- The commuting 2-actions of $\mathfrak{sl}_2$ and of the blob algebra are then realized via DG-functors.



$\mathcal{C}(M(\lambda) \otimes V \otimes V)$

$\mathcal{C}(M(\lambda))$

$$\begin{array}{c} \mathcal{C}(M(\lambda)) \\ \Big\uparrow \mathcal{F}(\zeta_K(q,\lambda)) \\ \mathcal{C}(M(\lambda)) \end{array}$$

\begin{Advertising Higher Representation Theory}

Actions on categories ? HRT !

👍 Actions of groups, algebras ..., on **categories** rather than vector spaces.

• Categorical actions of Lie algebras : first developed by Chuang and Rouquier (2004) to solve a conjecture on modular rep. theory of the symmetric group called the Broué conjecture (parallel ideas by Frenkel, Khovanov and Stroppel based on earlier work of Khovanov and cols).

• This was boosted by the categorification of quantum groups by Lauda, Khovanov–Lauda and Rouquier. Converged to what is called nowadays Higher Representation Theory.

• Usual basic structures of rep. theory (v. spaces and linear maps) get replaced by category theory analogs (categories and functors) $\Rightarrow$ richer (higher) structure invisible to traditional rep.theory.

Higher Representation Theory

Broué Conjecture

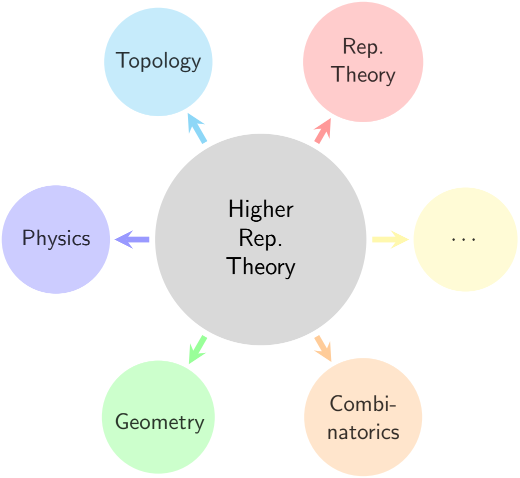
Khovanov Homology

Kahzdan-Luzstig conjectures

5-Branes

HF theory

...



\end{Advertising Higher Representation Theory}

cyclotomic KLRW algebras

Khovanov–Lauda–Rouquier–Webster '08-'10

Categorifications of tensor products of f.d. irreps are given through cyclotomic KLRW algebras.

These are **algebraic categorifications** : We work with (certain) categories of modules over certain $\Bbbk$-algebras where $\mathfrak{g}$ acts via (certain) endofunctors.

☞ The approach consists of :

- replacing weight spaces by categories,

and

- defining functors E and F that
  - move between the weight spaces and
  - satisfy the $\mathfrak{sl}_2$-relations.

Khovanov–Lauda, Rouquier & Webster's catg's

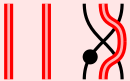
Fix a field $\Bbbk$.

The following exists equally for $\mathfrak{g}$ of symmetrizable type and with the red strands labelled by dominant integral $\mathfrak{g}$-weights.

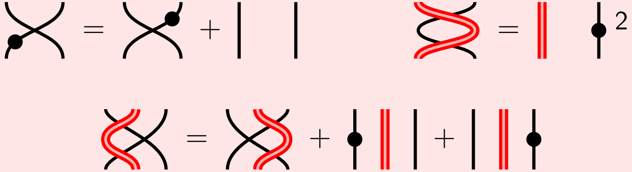
Definition

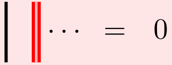
The KLRW algebra T_{d+1} is the graded, associative, unital $\Bbbk$-algebra generated by isotopy classes of braid-like diagrams

- Strands can either be black or red
  - there are $d+1$ red strands, which cannot intersect each other.
  - black strands can cross red strands and each other and they can carry dots.
- Multiplication is concatenation.

For example, 

Generators are required to satisfy local relations. For example :





Cyclotomic

The cyclotomic condition makes T_{d+1} f.d. Without it we have an affine algebra : call it T_{d+1}^{aff}.

Write $T_{d+1}(\nu)$ for the subalgebra of all diagrams having ν black strands. We have $\bigoplus_{\nu \geq 0} T_{d+1}(\nu) = T_{d+1}$.

Categorical $\mathfrak{sl}_2$-action

Define

$$\mathrm{F}^{d+1}(\nu)\colon T_{d+1}(\nu)\text{-mod}_{\mathrm{g}} \to T_{d+1}(\nu+1)\text{-mod}_{\mathrm{g}}$$

as the functor of *induction* for the map that adds a black strand at the right of a diagram from $T_{d+1}(\nu)$, and let $\mathrm{E}^{d+1}(\nu)$ be its *right adjoint* (/shift).

These functors have very nice properties...

Theorem *(Webster '10) :*

▷ The functors E^{d+1} and F^{d+1} are *biadjoint* and

▷ the composites $\mathrm{E}^{d+1}\mathrm{F}^{d+1}(\nu)$ and $\mathrm{F}^{d+1}\mathrm{E}^{d+1}(\nu)$ satisfy a *direct sum decomposition* lifting the commutator relation.

$$\mathrm{E}^{d+1}\mathrm{F}^{d+1}(\nu) \simeq \mathrm{F}^{d+1}\mathrm{E}^{d+1}(\nu) \oplus_{[d+1-2\nu]} \mathrm{Id}(\nu) \qquad \text{if } d+1 \geq \nu,$$

$$\mathrm{F}^{d+1}\mathrm{E}^{d+1}(\nu) \simeq \mathrm{E}^{d+1}\mathrm{F}^{d+1}(\nu) \oplus_{[2\nu-d-1]} \mathrm{Id}(\nu) \qquad \text{if } d+1 \leq \nu.$$

▷ Moreover, $K_0(T_{d+1}) \cong V^{\otimes(d+1)}$ (as $\mathfrak{sl}_2$)-modules)

Categorification of tensor products with a Verma

The idea is to see T_{d+1} as a dg-algebra with zero differential and "integrate" the cyclotomic condition

$$= 0$$

into a *dg-algebra* $(\mathcal{T}_{(1,d)}, \partial)$, together with a quasi-isomorphism

$$(\mathcal{T}_{(1,d)}, \partial) \simeq (T_{d+1}, 0).$$

- To construct such an algebra we note that T^{aff}_{d+1} acts on T_{d+1} (the first is ∞-dim while the second is f.d.).
- Writing a free resolution of T_{d+1} as a module over T^{aff}_{d+1} one gets a DGA $(\mathcal{T}_{(1,d)}, \partial)$ whose homology is T_{d+1} (this is nontrivial).

Will it work ?

Basically, we intend to categorify the rational fraction $\frac{\lambda q^{-k}-\lambda^{-1}q^{k}}{q-q^{-1}}$.

- We know that a categorification of multiplication by $[n]$ is $\mathbb{Q}[X]/X^n$ (secretly this is $H(G_1(n))$) via grading shifts of some id. functor $\oplus_{[n]}\mathrm{Id}$

- But $\mathbb{Q}[X]/X^n$ is a module over $\mathbb{Q}[X]$ (secretly this is $H(G_1(\infty))$) for which

$$\mathbb{Q}[X]/X^n \longleftarrow \mathbb{Q}[X] \xleftarrow{X^n} \mathbb{Q}[X]$$

gives as a free resolution (grading shifts involved !).

- We can write this as a DGA $(\mathbb{Q}[X,\omega]/\omega^2, \partial)$ with $\partial X = 0$, $\partial\omega = X^n$, which has homology $\mathbb{Q}[X]/X^n$.

- Tensoring M with $\mathbb{Q}[X,\omega]/\omega^2$ gives ...

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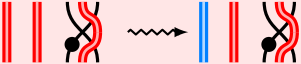
• Tensoring M with $\mathbb{Q}[X,\omega]/\omega^2$ gives ... $\frac{\lambda q^{-k} - \lambda^{-1} q^k}{q - q^{-1}}[M]$ (☺ hooray !) in (an appropr. defined) K_0.

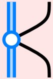
One can give a presentation of $(\mathcal{T}_{(1,d)}, \partial)$

dg-enhancement of cyclotomic KLRW algebras

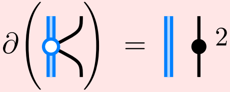
1. Replace the first red strand in the diagrams from T_{d+1} by a blue strand (secretelly labeled λ).

   For example, 

2. Black strands can be pinned to the blue strand, which we depict as

   ☜ New generator ! (homological degree 1)

3. 

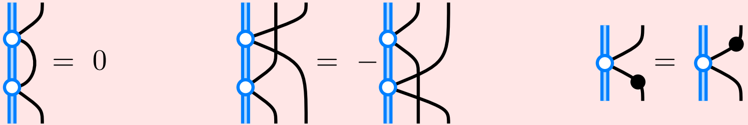
   $\partial(\text{all other generators}) = 0$

$+$ gr.Leibniz rule w.r.t. the hom. grading.

☞ When no diff. : new λ-grading.

Categorification of tensor products with a Verma

4. The generators are required to satisfy the local relations of T_{d+1} and



The (super)algebra $\mathcal{T}_{(1,d)}$ is free, acts faithfully on a supercommutative ring. For $\mathfrak{g} = \mathfrak{sl}_r$ it is isomorphic to a (higher level) Hecke algebra of type A (Rizzo : this generalizes Maksimau–Stroppel '18 and Maksimau–V. '19).

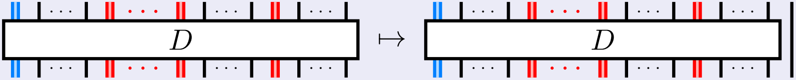
Theorem (Lacabanne–Naisse–V. '20)

The dg-algebra $(\mathcal{T}_{(1,d)}, \partial)$ is formal with

$$H\big(\mathcal{T}_{(1,d)}, \partial\big) \cong T_{d+1}.$$

Now just **forget** *there is a differential on* $\mathcal{T}_{(1,d)}$.

To define an $\mathfrak{sl}_2$-categorical action we use the map that adds a vertical black strand at the right of a diagram from $\mathcal{T}_{(1,d)}$:



Writing $\mathcal{T}_{(1,d)} = \bigoplus_{\nu\geq 0} \mathcal{T}_{(1,d)}(\nu)$, this gives rise to a functor of *induction*

$$\mathrm{F}^{(1,d)}(\nu)\colon \mathcal{T}_{(1,d)}(\nu)\text{-mod}_{\mathrm{g}} \to \mathcal{T}_{(1,d)}(\nu+1)\text{-mod}_{\mathrm{g}}$$

between (suitable !) categories of modules.

We also set $\mathrm{E}^{(1,d)}(\nu)$ as its *right adjoint* (/shift). They are not biadjoint.

Theorem (Lacabanne–Naisse–V. '20)

These functors fit in a SES

$$0 \to \mathrm{E}^{(1,d)}\mathrm{F}^{(1,d)}(\nu) \longrightarrow \mathrm{F}^{(1,d)}\mathrm{E}^{(1,d)}(\nu) \longrightarrow \oplus_{[\lambda,2\nu]}\mathrm{Id}(\nu) \to 0,$$

Here, for $N \in \mathbb{Z}$, $[\lambda_i, N] = \dfrac{\lambda_i q^{-N} - \lambda_i^{-1} q^N}{q - q^{-1}}$.

In the sequel it is better to work in the derived category.

Let $\mathcal{D}_{dg}(\mathcal{T}_{(1,d)}, 0)$ be the derived dg-category of dg-modules over $(\mathcal{T}_{(1,d)}, 0)$.

☞ The previous theorem can be restated as a quasi-isomorphism

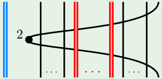
$$\mathrm{Cone}\big(\mathrm{E}^{(1,\mathrm{d})}\mathrm{F}^{(1,\mathrm{d})}(\nu) \longrightarrow \mathrm{F}^{(1,\mathrm{d})}\mathrm{E}^{(1,\mathrm{d})}(\nu)\big) \xrightarrow{\cong} \oplus_{[\lambda,2\nu]}\mathrm{Id}(\nu),$$

of dg-functors.

Bringing the differential ∂ into the picture we can define analogous of the functors $\mathrm{F}^{(1,d)}$ and $\mathrm{E}^{(1,d)}$ on $\mathcal{D}_{dg}(\mathcal{T}_{(1,d)}, \partial)$.

The previous q.i. descends to a q.i. of mapping cones $(\alpha = d + 1 - 2\nu)$

$$\mathrm{Cone}\big(\mathrm{E}_\partial^{(1,\mathrm{d})}\mathrm{F}_\partial^{(\mathrm{d}+1)}(\nu) \longrightarrow \mathrm{F}_\partial^{(1,\mathrm{d})}\mathrm{E}_\partial^{(1,\mathrm{d})}(\nu)\big) \xrightarrow{\cong}$$



$$\mathrm{Cone}\Big(\bigoplus_{\mathrm{p}\geq 0} \mathrm{q}^{1+2\mathrm{p}+\alpha}\mathrm{Id}(\nu) \xrightarrow{\qquad\qquad} \bigoplus_{\mathrm{p}\geq 0} \mathrm{q}^{1+2\mathrm{p}-\alpha}\mathrm{Id}(\nu)\Big) \cong \bigoplus_{[\alpha]} \mathrm{Id}(\nu)$$

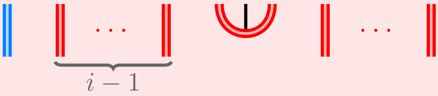
Theorem (Lacabanne–Naisse–V. '20)

There are isomophisms of $\mathfrak{sl}_2$-modules

$$\mathbf{K}_0^\Delta(\mathcal{T}^{(1,d)}, 0) \cong M(\lambda) \otimes V^{\otimes d},$$
$$\mathbf{K}_0^\Delta(\mathcal{T}^{(1,d)}, \partial) \cong V \otimes V^{\otimes d} \cong V^{\otimes(d+1)}.$$

A categorical blob action

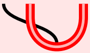
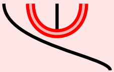
Following Webster we define the cup bimodule B_i for $1 < i \leq d-2$ as the $(\mathcal{T}^{(1,d-2)}, \mathcal{T}^{1,d})$-bimodule generated by the diagram



with additional black strands crossing the diagram.

The generator is placed in $\deg_{h,q,\lambda}\Big($  $\Big) = (0,0,0)$.

The diagrams are taken up to regular isotopy, and subjected to the same local relations as $\mathcal{T}^{1,d}$ together with the extra local relations

 $= 0$,  $= 0$,  $= -$  , etc...

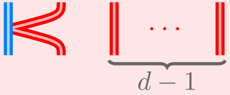
The cap bimodule $\overline{B}_i$ is defined similarly, by taking the mirror along the horizontal axis of B_i. However, $\deg_{h,q,\lambda}\left(\right.$  $\left.\right) = (-1,-1,0)$.

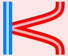
Set $\mathcal{T} = \bigoplus_{d\geq 0} \mathcal{T}_{(1,d)}$. One define the coevaluation and evaluation dg-functors as

$$\mathsf{B}_i := B_i \otimes^{\mathrm{L}}_{\mathcal{T}} - : \mathcal{D}_{dg}(\mathcal{T}^{(1,d-2)}, 0) \to \mathcal{D}_{dg}(\mathcal{T}^{(1,d)}, 0),$$

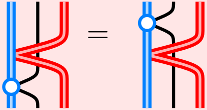
$$\overline{\mathsf{B}}_i := \overline{B}_i \otimes^{\mathrm{L}}_{\mathcal{T}} - : \mathcal{D}_{dg}(\mathcal{T}^{(1,d)}, 0) \to \mathcal{D}_{dg}(\mathcal{T}^{(1,d-2)}, 0).$$

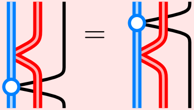
The double braiding bimodule X is the $(\mathcal{T}^{(1,d)}, \mathcal{T}^{(1,d)})$-bimodule generated by the diagram



$\deg_{(h,q,\lambda)}\Big($  $\Big) = (0, 0, -1)$

with local relations





We define the double braiding functor as

$$\Xi := X \otimes^{\mathrm{L}}_{\mathcal{T}} - : \mathcal{D}_{dg}(\mathcal{T}^{(1,d)}, 0) \to \mathcal{D}_{dg}(\mathcal{T}^{(1,d)}, 0).$$

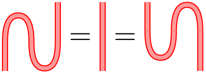
⚠ In order to prove the (categorical) relations of $\mathcal{B}_d(\lambda, q)$ one needs resolutions of the bimodules involved, and this takes us to the world of A_∞-bimodules.

Proposition (Lacabanne–Naisse-V. '20)

1. The functor $\Xi : \mathcal{D}_{dg}(T^{\lambda,r}, 0) \to \mathcal{D}_{dg}(T^{\lambda,r}, 0)$ is an autoequivalence, with inverse given by $\Xi^{-1} := \mathrm{RHOM}_T(X, -)$.
2. There are natural isomorphisms of functors $\mathrm{E} \circ \Xi \cong \Xi \circ \mathrm{E}$, $\mathrm{E} \circ \mathsf{B}_i \cong \mathsf{B}_i \circ \mathrm{E}$ and $\mathrm{E} \circ \overline{\mathsf{B}}_i \cong \overline{\mathsf{B}}_i \circ \mathrm{E}$ (similarly for F in the place of E).
3. There are quasi-isomorphisms of A_∞-bimodules

$$\mathcal{T}^{(1,d)} \xrightarrow{\simeq} \bar{B}_{i\pm 1} \otimes^{\mathrm{L}}_{\mathcal{T}} B_i,$$

$$q(\mathcal{T}^{(1,d)})[1] \oplus q^{-1}(\mathcal{T}^{(1,d)})[-1] \xrightarrow{\simeq} \bar{B}_i \otimes^{\mathrm{L}}_{\mathcal{T}} B_i.$$

The q.i. above correspond to  and  $= -q - q^{-1}$

Theorem (Lacabanne–Naisse–V. '20)

There is a quasi-isomorphism

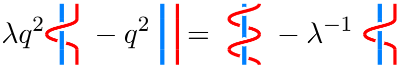
$$\mathrm{Cone}\Big(\lambda q^2 \Xi[1] \to q^2\,\mathrm{Id}[1]\Big)[1] \xrightarrow{\simeq} \mathrm{Cone}\Big(\Xi \circ \Xi \to \lambda^{-1}\Xi\Big)$$

of dg-functors, and a quasi-isomorphism

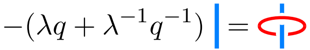
$$\lambda q(\mathcal{T}^{(1,d)})[1] \oplus \lambda^{-1}q^{-1}(\mathcal{T}^{(1,d)})[-1] \xrightarrow{\simeq} \bar{B}_1 \otimes^{\mathrm{L}}_{\mathcal{T}} X \otimes^{\mathrm{L}}_{\mathcal{T}} B_1,$$

of A_∞-bimodules.

These correspond to the last two defining relations of $\mathcal{B}_d(\lambda, q)$:



and



Final remarks

• A link diagram with a pole gives a functor from $\mathcal{D}_{dg}(\mathcal{T}^{(1,0)}, 0)$ to itself, categorifying the Jones invariant.

At the time beeing we cannot tell this is equivalent to APS...

1. give a htpy construction (calculational-friendly) of the 2-blob (imitating Mackaay–Webster seems too technical at the moment)...
2. other $\mathfrak{g}$'s, other V's
3. several poles
4. $\dots$

• A different application (Naisse–V. '17) of categorification of parabolic Verma modules for $\mathfrak{gl}_{2n}$ allows a HRT construction of Khovanov–Rozansky HOMFLYPT homology (the connection between HOMFLYPT polynomial and Verma modules was not known before).

Thanks for your attention!

Geometric constructions?

Higher structure?

$\mathbb{Z}$-h.w. 2-Vermas? (2-$\mathcal{O}$?)

2-Ariki-Koike?
2-row quotients (generalized blob)?

2-Vermas

Topology

HOMFLYPT HKhR w/ Naisse

Annular HKh w/ Lacabanne, Naisse

…

2-blob algebra
w/ Lacabanne, Naisse

Hecke algebras / S_n-reps w/ Maksimau, Rizzo

2-representation theory?
à la Mazorchuck–Miemietz–Mackaay et al.