The algebraic structure of groups of area-preserving homeomorphisms

joint work with Dan Cristofaro-Gardiner, Vincent Humilière, Cheuk Yu Mak, Ivan Smith

Sobhan Seyfaddini

CNRS & Sorbonne Univeristé

IST Lisbon
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Introduction

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Conjecture ("Simplicity Conjecture")

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Fathi's question for other surfaces.

Hamiltonian diffeomorphisms

(Σ, ω) compact surface with an area-form ω. (allow $\partial\Sigma \neq \emptyset$)

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$\Sigma = D$: $\mathrm{Ham} =$ area pres diffeos, identity near $\partial\Sigma$.
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Other Σ: $\mathrm{Ham} \lhd \mathrm{Diff}_0(\Sigma, \omega)$.

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- Use Lagrangian Floer homology. Inspired by
  - Ozsvath-Szabo (2003),
  - Mak-Smith (2019),
  - Polterovich-Shelukhin (2021).

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Assume: $\Sigma = D$.

- $\overline{\mathrm{Ham}}(\Sigma) = \mathrm{Homeo}_c(D, \omega)$.
- $\mathrm{Ham}(\Sigma) = \mathrm{Diff}_c(D, \omega)$.

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  - $\mathrm{Cal}(\varphi_H) \leq \|H\|_\infty$.

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If yes, then $\mathrm{Homeo}_c(D,\omega)$ not simple.

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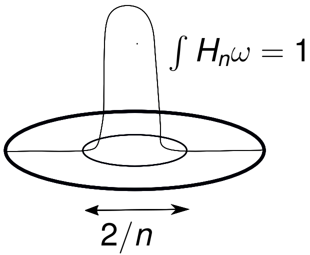
$\mathrm{Cal}(\varphi_{H_n}) = 1$, but $\varphi_{H_n} \xrightarrow{C^0} \mathrm{Id}$.



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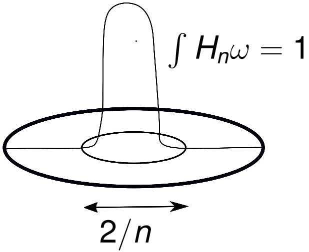
$\mathrm{Cal}(\varphi_{H_n}) = 1$, but $\varphi_{H_n} \xrightarrow{C^0} \mathrm{Id}$.



Not easy to extend Cal to $\mathrm{Homeo}_c(D, \omega)$.

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Outline of the argument

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Let $i \longrightarrow \infty$: RHS $\longrightarrow 0$. We get:

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A few words on the invariant c_d

Assume $\Sigma = S^2$

Construction of c_d: overview.

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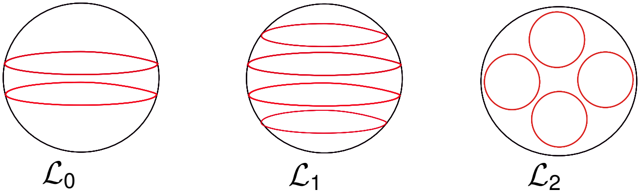
3. Link spectral invariant: corresponding Lagrangian spectral invariant.

Our Lagrangian and its Floer homology.

Links

A link is a finite collection $\mathcal{L} = \{L_1, ..., L_d\}$ of pairwise disjoint circles in Σ.



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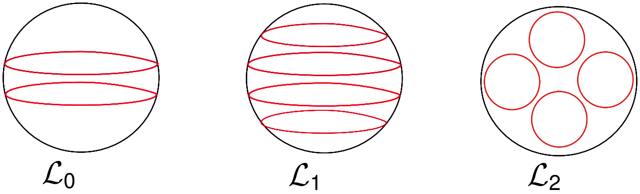

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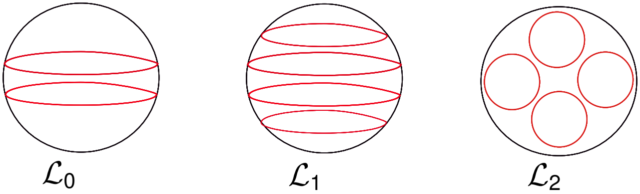
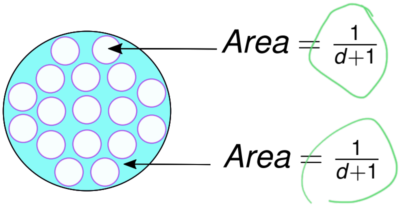


Strategy: Define Lagrangian Floer homology for links.
Previous work:

- Oszvath-Szabo 2000s: Heegaard/knot Floer homology.
- Mak-Smith, 2019: defined Lag Floer for $\mathcal{L}_0$.
- Polterovich-Shelukhin, 2021: spectral invariants for $\mathcal{L}_1$ (any d).

Equidistributed links

$$\mathcal{L} = \{L_1, \ldots, L_d\}$$



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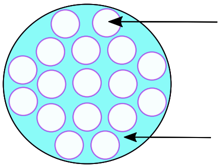

Use $\mathcal{L}$ to build a monotone Lagrangian torus in $\mathbb{C}P^d$.

Symmetric products

$\mathcal{L} = \{L_1, ..., L_d\}$ equidistributed link in (Σ, ω).

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$\mathrm{Cal}: \mathrm{Homeo} \longrightarrow \mathbb{R}$

$(\phi \in \mathrm{Homeo} \to \mathrm{Cal}(\phi)) < +\infty$

find $\theta \in \mathrm{Homeo}_c(D, \omega)$ w/ $\mathrm{Cal}(\theta) = +\infty$

$H : D \to \mathbb{R}$

$H(r, \theta) = f(r)$

$\int H \omega = +\infty$

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2. S_d acts on Σ^d: $\sigma \in S_d$, $\sigma \cdot (x_1, \ldots, x_d) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(d)})$.

$c_d : \mathrm{Diff}_c(\mathbb{D}, \omega) \to \mathbb{R}$

1. c_d is a q.m. w/ defect $\frac{1}{d} \to 0$

$$|c_d(\varphi\psi) - c_d(\varphi) - c_d(\psi)| \leq \frac{1}{d} \xrightarrow{d \to +\infty} 0$$

$$\mu(\varphi) := \lim_{d \to \infty} c_d(\varphi)$$

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Lagrangian: $L_1 \times \ldots \times L_d$ in the quotient.

$$\mathrm{Sym}^d(\mathcal{L}) \subset \mathrm{Sym}^d(\Sigma) \setminus \Delta.$$

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- Key idea: Cobordism between Maslov 2 J-discs on $\mathrm{Sym}^d(\mathcal{L})$ and the Clifford torus.

Thank You!

Bonus:

- Lagrangian spectral invariants.
- Calabi property.
- The disc potential.

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Lagrangian spectral invariants:

$$\ell : C^\infty([0,1] \times M) \to \mathbb{R}.$$

$\ell(H) :=$ "the action level at which $[L]$ appears in Lag Floer homology."

The action functional

$\Omega(L) := \{x : [0,1] \longrightarrow M : x(0), x(1) \in L, [x] = 0 \in \pi_1(M, L)\}$.
$H \in C^\infty([0,1] \times M)$. The action functional $\mathcal{A}_H : \Omega(L) \longrightarrow \mathbb{R}$

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- Define

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The spectral invariant c_d

$HF(\mathrm{Sym}(\mathcal{L})) = H_*(\mathrm{Sym}(\mathcal{L})) \implies$ can define spectral invariants:

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$$\mathrm{Sym}^d(H) : \mathrm{Sym}^{\mathrm{d}}(\Sigma) \to \mathbb{R}$$

$$[(x_1, \ldots, x_d)] \mapsto H_t(x_1) + \ldots + H_t(x_d).$$

Define

$$c_d(H) := \frac{1}{d}\ell(\mathrm{Sym}^d(H)).$$

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General fact: $L \subset (M, \omega)$ Lagrangian, with associated spectral invariant

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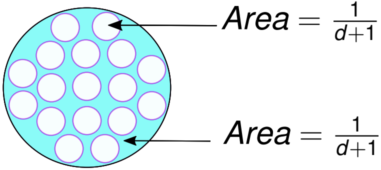
Translation to our setting: $\mathcal{L} = \{L_1, ..., L_d\} \subset (\Sigma, \omega)$

$$\frac{1}{d} \sum_{i=1}^{d} \int_0^1 \min_{L_i} H_t \, dt \ \leq c_d(H) \leq \frac{1}{d} \sum_{i=1}^{d} \int_0^1 \max_{L_i} H_t \, dt.$$

$$\mathcal{L} = \{L_1, \dots, L_d\}$$



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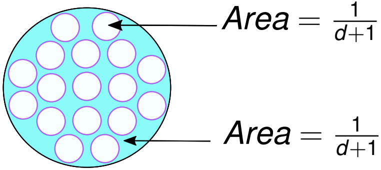

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$\frac{1}{d} \sum_{i=1}^{d} \min_{L_i} H_t \approx$ Riemann sum for $\int H_t \, \omega$

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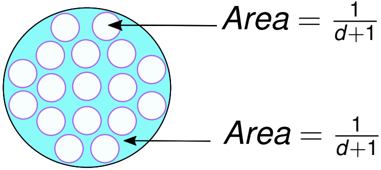

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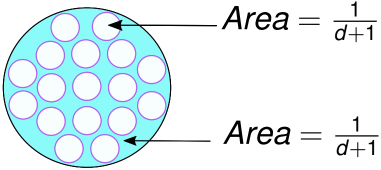

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Hence,

$$\int_0^1 \int H_t \, \omega \, dt \leq \lim_{d \longrightarrow \infty} c_d(H) \leq \int_0^1 \int H_t \, \omega \, dt.$$

The disc potential.

Discs with boundary on $\mathrm{Sym}^d(\mathcal{L})$

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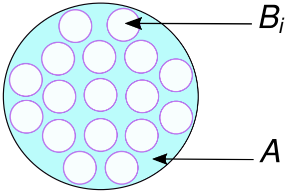

Correspond to the discs $B_1, \ldots, B_d$ and the region A.

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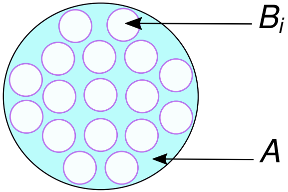

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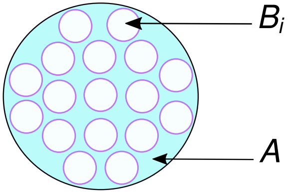

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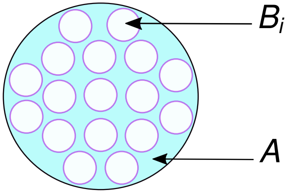

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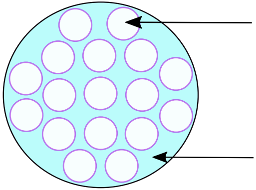

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