Porous medium model in contact with reservoirs

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Summary

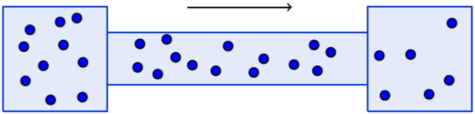
- Motivation
- The porous medium model in contact with reservoirs
- Heuristic of the hydrodynamic limit
- General constants, slow reservoirs and long jumps

Goal

Present microscopic models for the dynamics between particles to obtain the macroscopic laws for the evolution of some quantity of interest in a physical system.

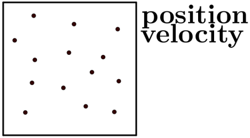
Physical motivation

- Goal: Study the asymptotic behavior of interact particle systems out of equilibrium.



- A channel kept out of equilibrium through the application of a chemical potential. It is predicted that an steady state out-of-equilibrium will be reached, in which the particles flow from one reservoir to another at a constant rate.

Physical motivation



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particles

Statistical Mechanics

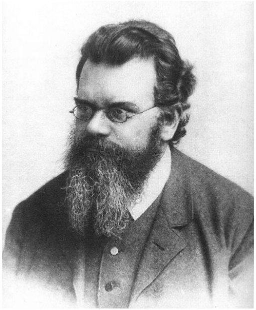
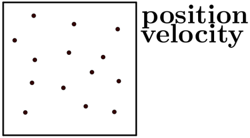


Figure: Ludwig Boltzmann XIX

"Microscopic state behaves randomly"

Physical motivation



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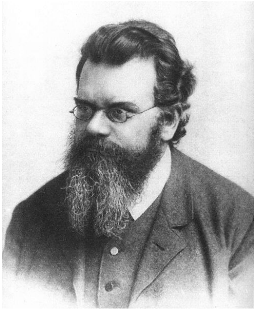

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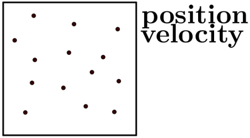

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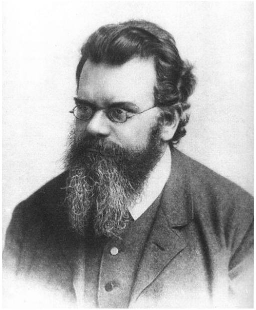


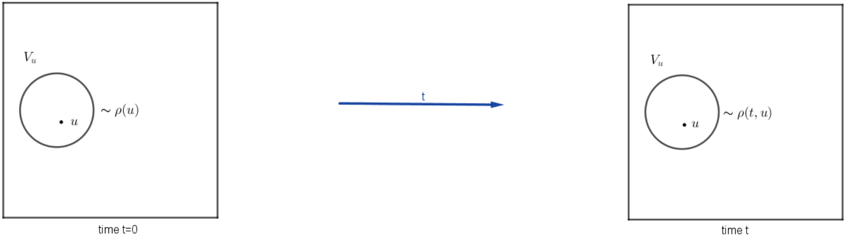
Figure: Ludwig Boltzmann XIX

"Microscopic state behaves randomly"

Frank Ludvig Spitzer 1970

- Frank Ludvig Spitzer (July 24, 1926 – February 1, 1992) was an Austrian mathematician who made fundamental contributions to probability theory especially the theory of interacting particle systems.

Physical motivation



- Find the equilibrium states of the system.
- Characterize these states as a quantity $\rho(\cdot)$ (density, pressure, temperature, energy, etc.)
- Fix a point $u \in V$ and choose a small neighborhood V_u (small for the macroscopic scale but huge from the microscopic point of view). Due to interaction, the system reaches an equilibrium $\rho(u)$.
- Let time evolve and now the equilibrium close to u is given by $\rho(t, u)$. How does $\rho(t, u)$ evolve?

Probability theory and PDE's

Probability $\longleftrightarrow$ PDE'S

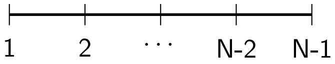
- The temporal evolution of the density of particles in the macroscopic state is described as the weak solution of a PDE.
- A random process gives rise to a process that is deterministic.
- The techniques used here are interesting because they exhibit a different way of demonstrating the existence of weak solutions for PDE's using probabilistic tools.

Motivation to study the porous medium model

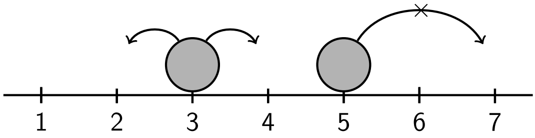
- In 2009 Patrícia Gonçalves, Claudio Landim and Cristina Toninelli deduced the porous medium equation from the porous medium model in $\mathbb{Z}$ without reservoirs.
- Adding two reservoirs on the boundary, i.e, puting an external current in the system, the question that remains is: How can the conditions macroscopically affect the system?

Porous medium model

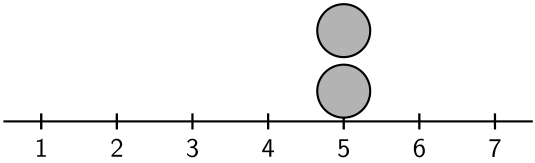
- $\Sigma_N = \{1, 2, \dots, N-1\}$ - the bulk



- Jumps only for the nearest neighbor

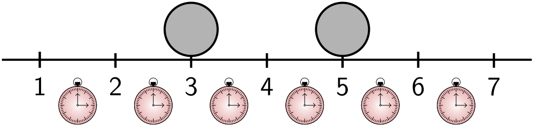


- Only one particle per site

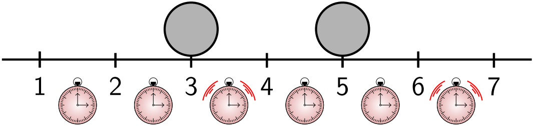


Porous medium model

- Adding a clock on each bond



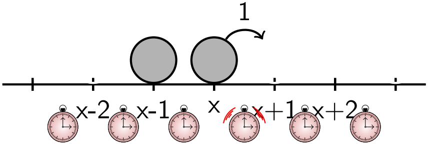
- Two clocks can't ring at the same time



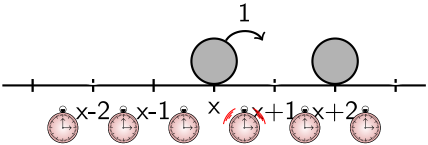
Porous medium model

If the clock rings at the bond $\{x, x+1\}$, the particle can jump from the site x to $x+1$ with a certain rate

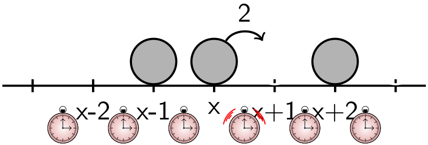
- Case 1:



- Case 2:

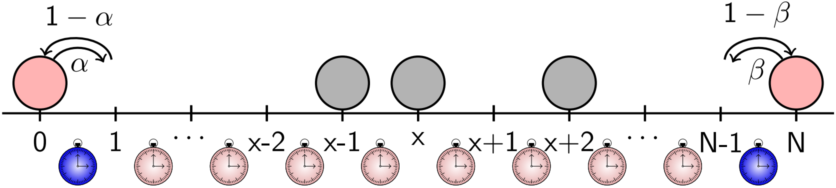


- Case 3:



Porous medium model with reservoirs

- Adding the reservoirs at site $x = 0$ and $x = N$



Infinitesimal generator

- For $N \geq 1$ let $\Sigma_N = \{1, 2, \cdots, N-1\}$.
- We denote the process by $\{\eta_t\}_{t \geq 0}$ which has state space $\Omega_N = \{0, 1\}^{\Sigma_N}$.
- Fix $\alpha, \beta \in (0, 1)$. Given a function $f : \{0, 1\}^{\Sigma_N} \to \mathbb{R}$, the infinitesimal generator of the PMM in contact with reservoirs is given by

$$(L_N f)(\eta) = (L_{N,0} f)(\eta) + (L_{N,b} f)(\eta),$$

where

$$\begin{aligned}
(L_{n,0} f)(\eta) = & \sum_{x=1}^{N-2} [\eta(x)(1 - \eta(x+1)) + \eta(x+1)(1 - \eta(x))][\eta(x-1) \\
& + \eta(x+2)][f(\eta^{x,x+1}) - f(\eta)], \\
(L_{n,b} f)(\eta) = & [\alpha(1 - \eta(1)) + (1 - \alpha)\eta(1)][f(\eta^{1}) - f(\eta)] \\
& + [\beta(1 - \eta(1)) + (1 - \beta)\eta(1)][f(\eta^{N-1}) - f(\eta)]
\end{aligned}$$

Empirical measure

- For each configuration $\eta \in \{0,1\}^{\Sigma_N}$ we define the **empirical measure** $\pi^N(\eta, du)$ on $[0,1]$ as

$$\pi^N(\eta, du) = \frac{1}{N} \sum_{x \in \Sigma_N} \eta(x) \delta_{\frac{x}{N}}(du).$$

- If $H : [0,1] \to \mathbb{R}$, then **the integral of H with respect to the empirical measure** π_t^N will be given by

$$\langle \pi_t^N, H \rangle := \int H(u) \pi_t^N(\eta, du) = \frac{1}{N} \sum_{x=1}^{N-1} H_x \eta_t(x),$$

where $H_x = H(\frac{x}{N})$.

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Hydrodynamic limit

Definition

A sequence of probability measures $\{\mu_N\}_{N\geq 1}$ in $\{0,1\}^{\Sigma_N}$ is said to be associated to a density profile $\rho_0 : [0,1] \to [0,1]$ if, for all $\gamma > 0$ and for any continuous function $H : [0,1] \to \mathbb{R}$ the following limit holds:

$$\lim_{N\to\infty} \mu_N \left(\eta \in \{0,1\}^{\Sigma_N} : \left| \frac{1}{N} \sum_{x=1}^{N-1} H\left(\frac{x}{N}\right) \eta(x) - \int_{[0,1]} H(u)\rho_0(u)du \right| > \gamma \right) = 0.$$

$$\pi_0^N \xrightarrow{\mu_N} \pi_0(du) = \rho_0(u)du$$

$$\pi_t^N \xrightarrow{\mathbb{P}_{\mu_N}} \pi_t(du) = \rho_t(u)du$$

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Hydrodynamic limit

Theorem

- $\rho_0 : [0,1] \to [0,1]$ measurable function
- $\{\mu_N\}_{N\geq 1}$ associated to a profile

For each $t \in [0, T]$, $\gamma > 0$ and for all functions $H \in C[0,1]$, we have that

$$\lim_{N\to\infty} \mathbb{P}_{\mu_N}\left(\eta_\cdot : \left|\frac{1}{N}\sum_{x=1}^{N-1} H\left(\frac{x}{N}\right)\eta_t(x) - \int_{[0,1]} H(u)\rho(t,u)du\right| > \gamma\right) = 0,$$

holds, where $\rho(t,\cdot)$ is the unique weak solution of the porous medium equation.

The weak solution

Let $\rho_0 : [0,1] \to [0,1]$ be a measurable function. We say that $\rho : [0,T] \times [0,1] \to [0,1]$ is a weak solution of the porous medium equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t \rho_t(u) = \Delta\,(\rho_t(u))^2, & (t,u) \in [0,T] \times (0,1), \\ \rho_t(0) = \alpha, \quad \rho_t(1) = \beta, & t \in [0,T], \\ \rho_0(\cdot) = \rho_0(\cdot), \end{cases}$$

if the following conditions hold:

1. $\rho \in L^2(0,T;\mathcal{H}^1)$;
2. ρ satisfies the weak formulation:

$$\int_0^1 \rho_t(u) H_t(u)\,du - \int_0^1 \rho_0(u) H_0(u)\,du - \int_0^t \int_0^1 \rho_s(u) \partial_s H_s(u)\,du\,ds$$
$$- \int_0^t \int_0^1 (\rho_s(u))^2 \Delta H_s(u)\,du\,ds - \int_0^t \beta^2 \partial_u H_s(1) - \alpha^2 \partial_u H_s(0)\,ds = 0,$$

for all $t \in [0,T]$ and any function $H \in C_0^{1,2}([0,T] \times [0,1])$.

Comparing the micro and macro

- Weak formulation
- Dynkin's formula

Heuristic of the hydrodynamic limit

- Analyse the time evolution of the empirical measure associated to the process $\{\eta_t\}_{t\geq 0}$.
- We define the empirical measure process as $\pi_t^N(\eta, du) = \pi^N(\eta_{tN^2}, du)$.
- If $H \in C_0^{1,2}([0,T] \times [0,1])$, then the integral of H with respect to the empirical measure π_t^N will be given by

$$\langle \pi_t^N, H \rangle := \frac{1}{N} \sum_{x=1}^{N-1} H_x \eta_{tN^2}(x),$$

where $H_x = H(\frac{x}{N})$.

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Heuristic of the hydrodinamic limit

Dynkin's formula

- $\{\eta_t\}_{t\geq 0}$ a Markov process with countable state space and with generator L.
- $F:\mathbb{R}^+\times E\to\mathbb{R}$ be a bounded function such that
  - $\forall \eta\in E, F(\cdot,\eta)\in C^2(\mathbb{R}^+)$,
  - there exists a finite constant C, such that $\sup_{(s,\eta)}|\partial_s^j F(s,\eta)|\leq C$, for $j=1,2$.

For $t\geq 0$, let

$$M_t^F = F(t,\eta_t) - F(0,\eta_0) - \int_0^t (\partial_s + L)F(s,\eta_s)ds.$$

Them $\{M_t^F\}_{t\geq 0}$ is a martingale wrt $\mathcal{F}_t=\sigma(\eta_s; s\leq t)$.

Heuristic of the hydrodynamic equation

- Fix $H \in C_0^{1,2}([0,T] \times [0,1])$.
- By Dynkin's formula, taking the function $F(t, \eta_t) = \langle \pi_t^N, H \rangle = \frac{1}{N} \sum_{x=1}^{N-1} H_x \eta_{tN^2}(x)$, we have that

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t (\partial_s + N^2 L_N) \langle \pi_s^N, H \rangle ds,$$

  is a martingale with respect to the natural filtration $\mathcal{F}_t := \sigma(\eta_s : s \leq t)$.
- Since $F(s, \cdot)$ is time independent, then $\partial_s F(s, \cdot) = 0$. So,

$$M_t^N(H) = \langle \pi_t^N, H \rangle - \langle \pi_0^N, H \rangle - \int_0^t N^2 L_N \langle \pi_s^N, H \rangle ds$$

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Heuristic of the hydrodynamic equation

$$N^2 L_N \langle \pi_t^N, H \rangle = \frac{1}{N} \sum_{x=1}^{N-1} \Delta_N H_x \varphi_x(\eta) + c_1(t,H) + c_{N-1}(t,H),$$

where

$$\mathrm{c}_1(t,H) = \nabla_N^+ H_0 \big(\alpha - \eta(1) + \alpha\eta(1) + \eta(1)\eta(2) - \alpha\eta(2)\big)$$

$$\begin{aligned} \mathrm{c}_{N-1}(t,H) = & -\nabla_N^- H_N \big(\beta - \eta(N-1) + \eta(N-2)\eta(N-1) \\ & + \eta(N-1)\beta - \eta(N-2)\beta\big), \end{aligned}$$

and

$$\varphi_x(\eta) = \eta(x-1)\eta(x) + \eta(x)\eta(x+1) - \eta(x-1)\eta(x+1).$$

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and

$$\varphi_x(\eta) = \eta(x-1)\eta(x) + \eta(x)\eta(x+1) - \eta(x-1)\eta(x+1).$$

Heuristic of the hydrodynamic equation

Definition

The **discrete laplacian** of H in $\frac{x}{N}$ with $x \in \Sigma_N$ is given by

$$\Delta_N H_x = N^2\{H_{x-1} - 2H_x + H_{x+1}\}.$$

We also define the **discrete gradients** as

$$\begin{aligned} \nabla_N^+ H_x &= N(H_{x+1} - H_x), \\ \nabla_N^- H_x &= -N(H_{x-1} - H_x). \end{aligned}$$

Heuristic of the hydrodynamic equation

Hypothesis

- μ_N be a measure in $\{0,1\}^{\Sigma_N}$.
- $\eta(1) \longleftrightarrow \alpha$ with a certain error e_1, $\mathbb{E}_{\mu_N}[e_1] \xrightarrow{N\to\infty} 0$.
- $\eta(N-1) \longleftrightarrow \alpha$ with a certain error e_{N-1}, $\mathbb{E}_{\mu_N}[e_{N-1}] \xrightarrow{N\to\infty} 0$.

Then

$$
\begin{aligned}
M_t^N(H) = {} & \frac{1}{N}\sum_{x=1}^{N-1} H_x\eta(x) - \frac{1}{N}\sum_{x=1}^{N-1} H_x\eta_0(x) \\
& - \int_0^t \frac{1}{N}\sum_{x=1}^{N-1} \Delta_N H_x \varphi_x(\eta)ds \\
& + \int_0^t \nabla_N^+ H_0 \alpha^2 \, ds + e_1 - \int_0^t \nabla_N^- H_N \beta^2 \, ds + e_{N-1}.
\end{aligned}
$$

Heuristic of the hydrodynamic equation

Hypothesis

- μ_N be a measure in $\{0,1\}^{\Sigma_N}$.
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Then

$$\begin{aligned} M_t^N(H) = {} & \frac{1}{N}\sum_{x=1}^{N-1} H_x \eta(x) - \frac{1}{N}\sum_{x=1}^{N-1} H_x \eta_0(x) \\ & - \int_0^t \frac{1}{N}\sum_{x=1}^{N-1} \Delta_N H_x \varphi_x(\eta) ds \\ & + \int_0^t \nabla_N^+ H_0 \alpha^2 \, ds + e_1 - \int_0^t \nabla_N^- H_N \beta^2 \, ds + e_{N-1}. \end{aligned}$$

Heuristic of the hydrodynamic equation

- Since $M_0^N(H) = 0$ and the expectation of a martingale with respect to any measure μ_N is constant, then $\mathbb{E}_{\mu_N}[M_t^N(H)] = \mathbb{E}_{\mu_N}[M_0^N(H)] = 0$,

$$
\begin{aligned}
0 = & \frac{1}{N}\sum_{x=1}^{N-1} H_x\big(\mathbb{E}_{\mu_N}[\eta(x)] - \mathbb{E}_{\mu_N}[\eta_0(x)]\big) \\
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\end{aligned}
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Heuristic of the hydrodynamic equation

- Since $M_0^N(H) = 0$ and the expectation of a martingale with respect to any measure μ_N is constant, then $\mathbb{E}_{\mu_N}[M_t^N(H)] = \mathbb{E}_{\mu_N}[M_0^N(H)] = 0$,

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Heuristic of the hydrodynamic equation

- Let ρ_t be a density profile which is the solution of the partial differential equation that we are looking for.
- Let $\rho_t^N(x) = \mathbb{E}_{\mu_N}[\eta_{tN^2}(x)]$.
- Suppose that $\rho_t^N(x) \sim \rho_t(\frac{x}{N})$.
- Assuming that the expectation of the product is the product of the expectation and $\rho_t(\frac{x\pm 1}{N}) \sim \rho_t(\frac{x}{N})$, for each $x \in \Sigma_N$ we can write

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PMM with slow reservoirs

- In the hydrodynamic scenario we obtain that the time evolution of the spatial density of particles, in the diffusive scaling, is given by the weak solution of the porous medium equation, with boundary conditions that depend on θ.
- The behavior of the system is strongly affected and new boundary conditions may be derived at the macroscopic level.

- $\theta < 1$ Dirichlet type, that is,

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho^2(t,u)\,, & \text{for } t>0\,,\ u\in(0,1)\,,\\ \rho(t,0)=\alpha\,, & \text{for } t>0\,,\\ \rho(t,1)=\beta\,, & \text{for } t>0\,,\\ \rho(0,u)=\rho_0(u)\,, & u\in[0,1]\,. \end{cases}$$

- $\theta = 1$ Robin type, that is,

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho^2(t,u)\,, & \text{for } t>0\,,\ u\in(0,1)\,,\\ \partial_u\rho^2(t,0)=m(\rho(t,0)-\alpha)\,, & \text{for } t>0\,,\\ \partial_u\rho^2(t,1)=m(\beta-\rho(t,1))\,, & \text{for } t>0\,,\\ \rho(0,u)=\rho_0(u)\,, & u\in[0,1]\,. \end{cases}$$

- $\theta > 1$ Neumann type, that is

$$\begin{cases} \partial_t \rho(t,u) = \partial_u^2 \rho^2(t,u)\,, & \text{for } t>0\,,\ u\in(0,1)\,,\\ \partial_u\rho(t,0)=0\,, & \text{for } t>0\,,\\ \partial_u\rho(t,1)=0\,, & \text{for } t>0\,,\\ \rho(0,u)=\rho_0(u)\,, & u\in[0,1]\,. \end{cases}$$

Long jumps

- The presence of long jumps, have a drastic effect on the macroscopic behavior and critical exponents of the system.
- Fractional Laplacian.
- Fractional porous medium equation.

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