A family of threefolds with unusual features

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① $\{ \chi_{top}(X) \}$, X calabi-yau, Physics.

② A conjecture of Bogomolov

③ Kawamata-Morrison conjecture.

④ A conjecture of Namikawa

⑤ A conjecture of Elkies.
(Physics of String Theory)

Question (1) (From: String & Geometry, ~ Mid 80s)

what integers arise as $\chi_{top}(X)$,

X, $\dim_{\mathbb{C}} X = 3$, compact, Kähler, $c_1(X) = 0$?

[Strominger-Witten, et al '86]

[C. Schoen, 1988]:

A family of such X, proj. algebraic, computes $\chi_{top}(X)$.

§ 1. X : "The Schoens" Antonella Grassi 11.10.2022

• step 0 :

$$\begin{array}{c} B \\ r \downarrow \\ \mathbb{P}^1 \end{array}$$

rational elliptic surface with section

Example:

$\{F, G\}$ general cubics in $\mathbb{P}^2$, $\{\lambda F + \mu G\}$ pencil.

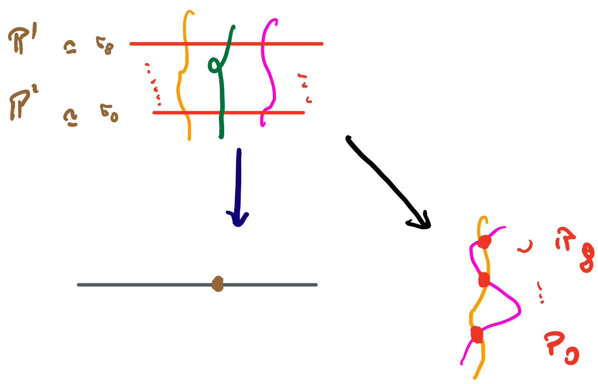
$\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$

$z = [z_0, z_1, z_2] \longmapsto [F(z), G(z)]$

$F \cdot G = \{p_0, \dots, p_8\}$

$$\begin{tikzcd} B = Bl_{\cdot}\mathbb{P}^2 \arrow[d, "r"'] \arrow[dr, "Bl"] & \\ \mathbb{P}^1 & \mathbb{P}^2 \arrow[l, dashed] \end{tikzcd}$$

$[F(z), G(z)] \longleftarrow z$



$r : B \longrightarrow \mathbb{P}^1$ is an elliptic fibration, with section(s)

$\sigma_0, \dots, \sigma_8$ are sections.

Theorem: All rational elliptic surfaces w/ section arise from pencils of plane cubics.

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Example: $F(x,y,z) = z_0^3 + z_1^3 + z_2^3$

$G(x,y,z) = z_0^2 z_1 - 2^{-1/3}\zeta z_0^2 z_2 - 2^{1/3}\zeta^2 z_0 z_2^2 \quad , \quad \zeta^3 = 1$

- Pencil: $\{\lambda F + \mu G\}$ $\qquad F \cdot G = \{F = G = 0\} = \{P_0, \dots P_8\}$

- RES B: $y^2 = x^3 - \frac{1}{4}(27\lambda^6 - 5\lambda^3\mu^3 - 3\mu^6)$

$$\begin{array}{ccc} B & & (x,y,[\lambda,\mu]) \\ r \downarrow & & \downarrow \\ \mathbb{P}^1 & & [\lambda,\mu] \end{array}$$

- Singular fibers: $r^{-1}(q_j)$: $y^2 = x^3$ 

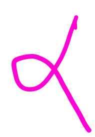
where $q_j = [\lambda_j, \mu_j]$, roots of $(27\lambda^6 - 5\lambda^3\mu^3 - 3\mu^6)$ $\qquad 1 \le j \le 6$

[6 cusps, A. ZANARDINI]

Example: F, G general $\qquad \{\lambda F + \mu G\}$ pencil

- RES B:

$$\begin{array}{ccc} B & & (x,y,[\lambda,\mu]) \\ r \downarrow & & \downarrow \\ \mathbb{P}^1 & & [\lambda,\mu] \end{array} \qquad q_i = [\lambda_i, \mu_i] \qquad 1 \le j \le 12$$

- Singular fibers: $r^{-1}(q_i)$: $y^2 = x^2(x+1)$ 

$\boxed{\text{Def: Discriminant } \stackrel{def}{=} \Sigma(B/\mathbb{P}^1) = \{q_j\}}$

Step 1 The construction

Let $r : B \longrightarrow \mathbb{P}^1$, $r' : B' \longrightarrow \mathbb{P}^1$ RES, with section.

$$X \stackrel{def}{=} B \times_{\mathbb{P}^1} B'$$

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
B \arrow[dr, "r"'] & & B' \arrow[dl, "r'"] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

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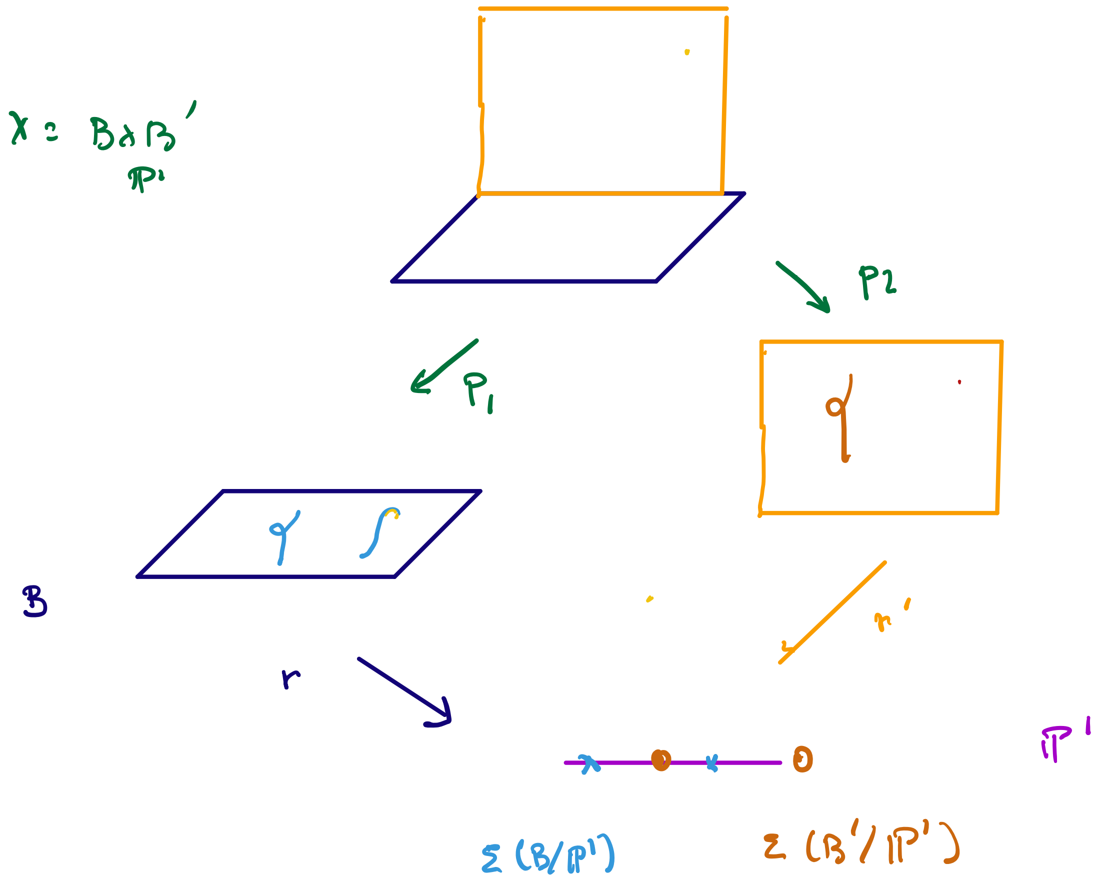

Remark:

(i) $\Sigma(B/\mathbb{P}^1) \cap \Sigma(B'/\mathbb{P}^1) = \emptyset \iff X$ smooth

(ii) $\chi_{top}(X)$ is easily computed.

Proof: Mayer-Vietoris

$r: B \to \mathbb{P}^1$; $r': B' \to \mathbb{P}^1$ family of $\chi_{top}(X)$

(iii) X gen, smooth and $\chi_{top}(X) = 0$

Mostly mild singularities : $c_1(X)$ o.k.

ω_X

$$\boxed{\text{Proposition 1 :} \quad \omega_X \simeq \mathcal{O}_X}$$

Proof:

$$\boxed{\begin{array}{l}\text{Proposition 2 :} \quad X \text{ is simply connected} \\ \qquad h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0\end{array}}$$

Def: X, $\dim_{\mathbb{C}} X = 3$ is a smooth Calabi-Yau 3fold $\Leftrightarrow$ X proj, smooth $\begin{cases} c_1(X) = 0 & (\text{or } K_X \simeq \mathcal{O}_X), \\ h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0 & (h^0(\Omega^i_X) = 0) \end{cases}$

$\rightarrow$ X smooth Schoen is a Calabi-Yau

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[Schoen / Kapusta-Kapusta]

Conditions on $\begin{cases} r: B \to \mathbb{P}^1 \\ r': B' \to \mathbb{P}^1 \end{cases}$ such that, if X is singular.

$\exists\ \pi: \tilde{X} \to X$, $\tilde{X}$ smooth Calabi-Yau

<u>Remark</u>: π is a small resolution.

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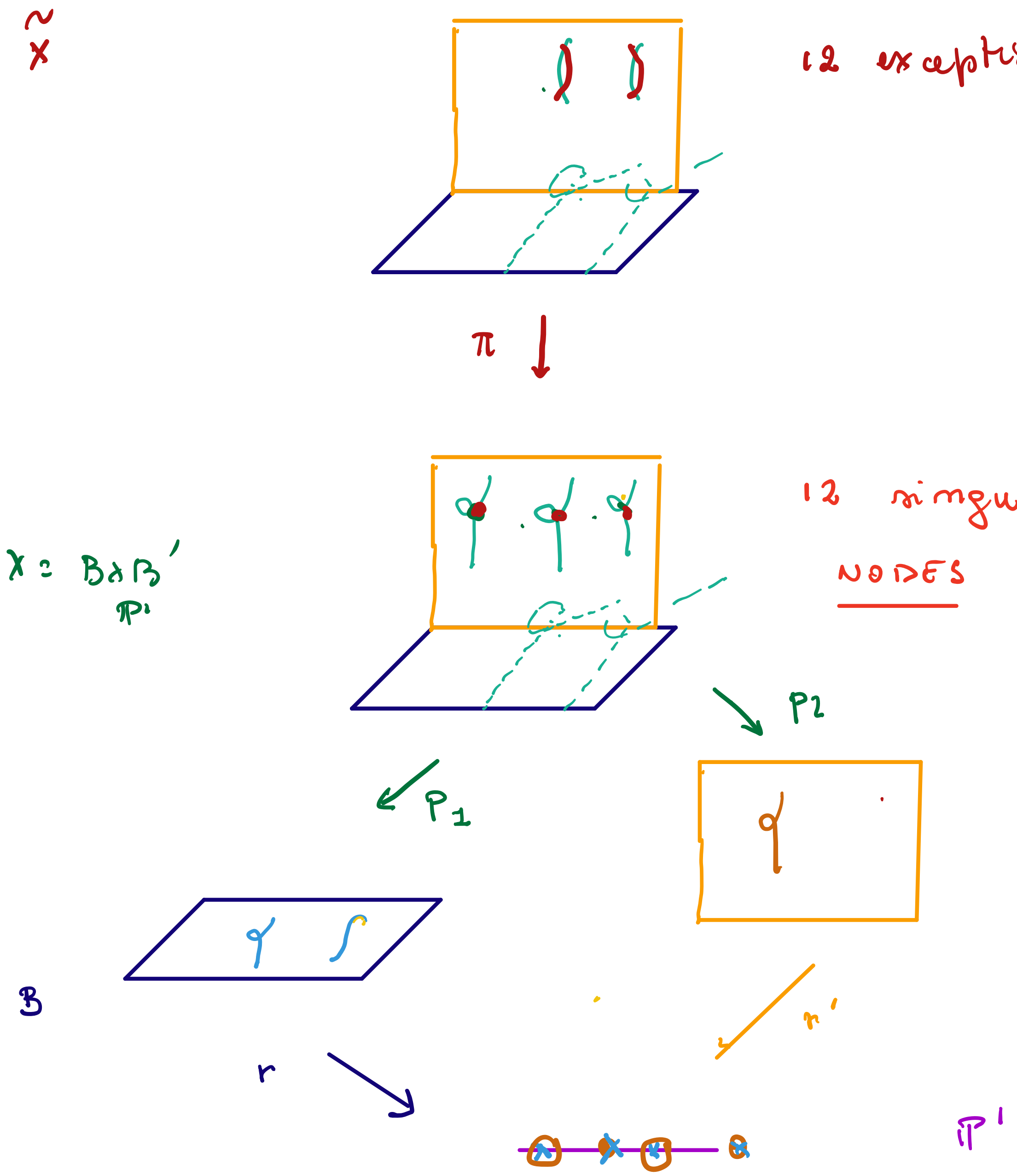
Example



12 exceptional $\mathbb{P}^1 \simeq S^2$

12 singular points

NODES $\quad x^2+y^2=w^2+z^2$

$X = B \times_{\mathbb{P}^1} B'$

$B' = B$

$\Sigma(B/\mathbb{P}^1) = \Sigma(B'/\mathbb{P}^1)$

§ (schoen) . <u>Calculations of $\{\chi_{top}\}$</u>

. some are rigid,

• " unfortunately for the physicists ... ",
many χ_{top} ( )

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Review MSci Net (Beauville)

.... theoretical physicists do not seem to care any longer about these threefolds...
but mathematicians

In fact:

Question ② Bogomolov: X, $K_X \sim \mathcal{O}_X$

Does $\tilde{X} \dashrightarrow \tilde{X}$ birational automorphism extend to a biregular automorphism?

Counterexample (Schoen) $\exists\ \tilde{X}$, Schoen /

$\exists$: $\tilde{X} \dashrightarrow \tilde{X}$ birational automorphism which.

does not extend to
$$\begin{tikzcd} \tilde{X} \arrow[r, leftrightarrow, "\times"] \arrow[d] & \tilde{X} \arrow[d] \\ X \arrow[r, leftrightarrow] & X \end{tikzcd}$$
biregular.

§ Physics of Mirror symmetry

Mirror symmetry: for X, Y Calabi-Yau

kähler moduli, X ⟷ cx moduli, Y

eg, $\to$ $h^{1,1}(X) = h^{2,1}(Y)$ (mid 80's)

Question ③ (Morrison '92) X CY

Does $\overline{Nef}(X) \subseteq H^2(X,\mathbb{Q})$ have a fundamental domain which is a rational polyhedral cone for Aut(X)?

Theorem (AG - Morrison, '92)

The conjecture is nontrivially satisfied for X Schoen, general.

Def: (kähler cone)

$Nef(X) = \{ A\ \mathbb{Q}\text{-Divisors on } X \ / \ A\cdot\Gamma \geq 0 \ , \Gamma \in \overline{NE}(X) \}$

$\overline{NE}(X) = \overline{\{ \Gamma, \text{ effective curves} \}}$

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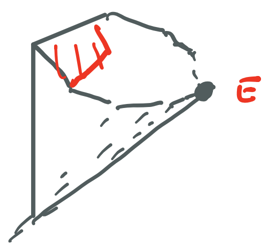
Remarks

(i) Most X, CY in the physics literature $\mathrm{Nef}(X)$ is rationally polyhedral.

(ii) ③ $\rightsquigarrow$ Morrison-Kawamata conjecture in more generality.

Proof: (Idea of)

- Addition law in the fibers gives infinitely many edges of $\overline{NE(X)}$, $\overline{\mathrm{Nef}(X)}$



Not to scale!

$\dim H^2(X,\mathbb{Z}) = 19$

- Addition law in the fibers gives fundamental domain and action of $\mathrm{Aut}(X)$.

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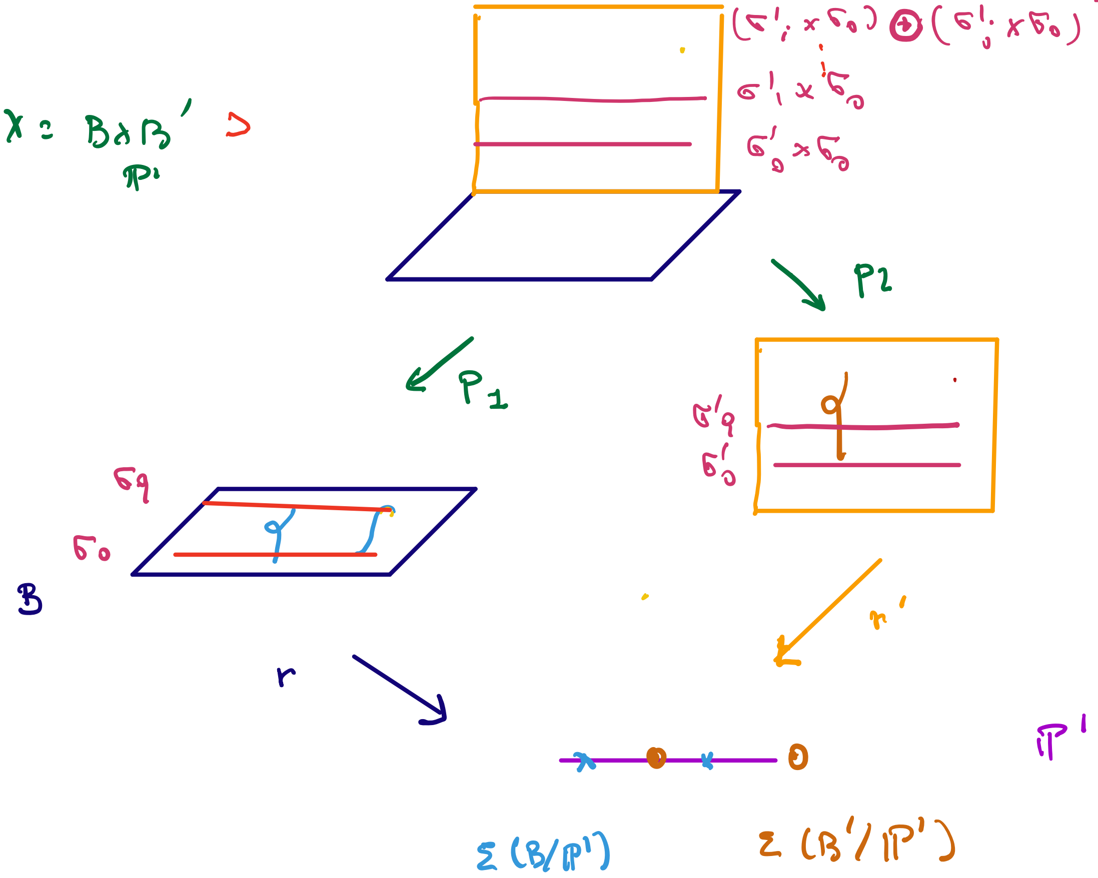
$X = B \times_{\mathbb{P}^1} B' \supset$



f

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§ Deformations and Resolution (Geometric Transitions)

(Y. Namikawa - M. Rossi)

Question (4) (Namikawa) X, Calabi-Yau /

$\tilde{X} \longrightarrow X$ small resolution

Does $\exists\ \tilde{Y}$, C.Y. $\tilde{Y} \longrightarrow Y$, Y, NODES $x^2+y^2=z^2+w^2$

and flat deformation. $\tilde{X} \rightsquigarrow \tilde{Y}$?

Theorem (Namikawa // Rossi) $\exists$ counterexamples:

$\{X_i\}$ family of NEW schemes

Remark: Conifold transitions, Physics.

Recall:

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Example: $F(x,y,z) = z_0^3 + z_1^3 + z_2^3$

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- Pencil: $\{\lambda F + \mu G\}$

- $$\boxed{B: \quad y^2 = x^3 - \frac{1}{4}(27\lambda^6 - 5\lambda^3\mu^3 - 3\mu^6)}$$

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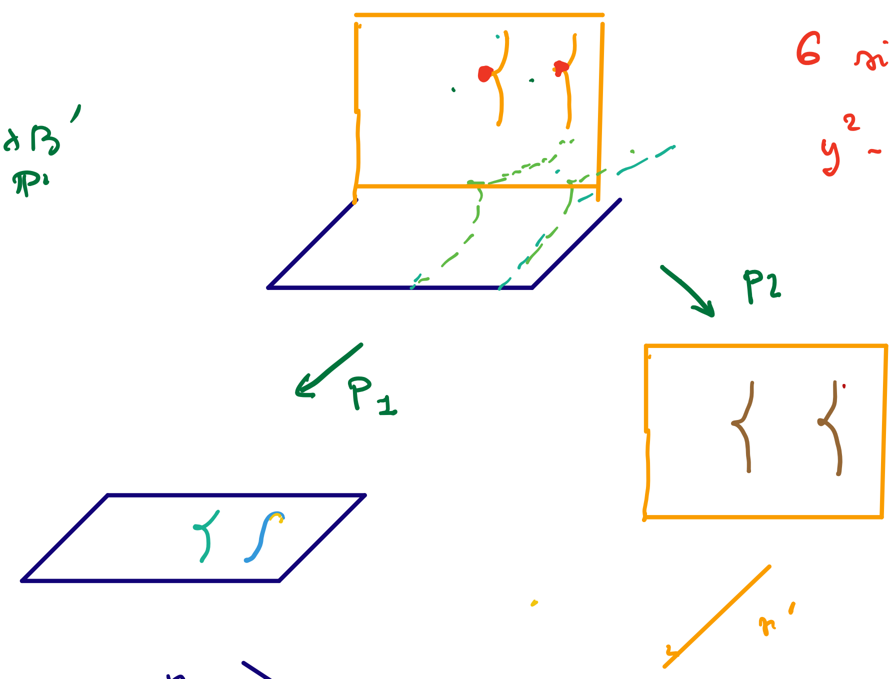
- Singular fibers : $r^{-1}(q_i)$: $y^2 = x^3$ $\{$

where $q_j = [\lambda_j, \mu_j]$, roots of $(27\lambda^6 - 5\lambda^3\mu^3 - 3\mu^6)$ $\boxed{1 \le j \le 6}$

$X = B \times_{\mathbb{P}^1} B$, 6 singularities. NOT NODAL

Cartoon:

$X = B \times_{\mathbb{P}^1} B'$



6 singular point:

$y^2 - x^3 = w^2 - z^3$

$B' = B$

B

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$\mathbb{P}^1$

$$\Sigma(B/\mathbb{P}^1) = \Sigma(B'/\mathbb{P}^1)$$

$$\boxed{\exists \{X_i\}_{0 \le i \le 5} \quad \pi_i : X_i \to X \quad \text{small resolutions.}}$$

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§ Physics of F-Theory.

<u>Def</u>: $X \xrightarrow{p} B$ elliptic fibration

$E \mapsto p \in \mathcal{U} \subset B$ dense , $E \simeq T^2$ elliptic curve.

Vafa '96 : Evidence of F-Theory

"F-theory" on $X \xrightarrow{p} B$ $\longleftrightarrow$ IIB on B, with 7-branes

constructed as dual

X CY, p elliptic.

$SL(2,\mathbb{Z})$ symmetry of E $\longleftrightarrow$ $SL(2,\mathbb{Z})$ symmetry of IIB

$E \sim T^2 = \mathbb{C}/\Lambda$

- $\exists$ "many" CY with elliptic fibrations.
  (A.I. searches)

- Mirror symmetry: other dualities.

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<u>Questions</u> : Compute the "spectrum" in physics

$\updownarrow$?

geometry of $X \longrightarrow B$

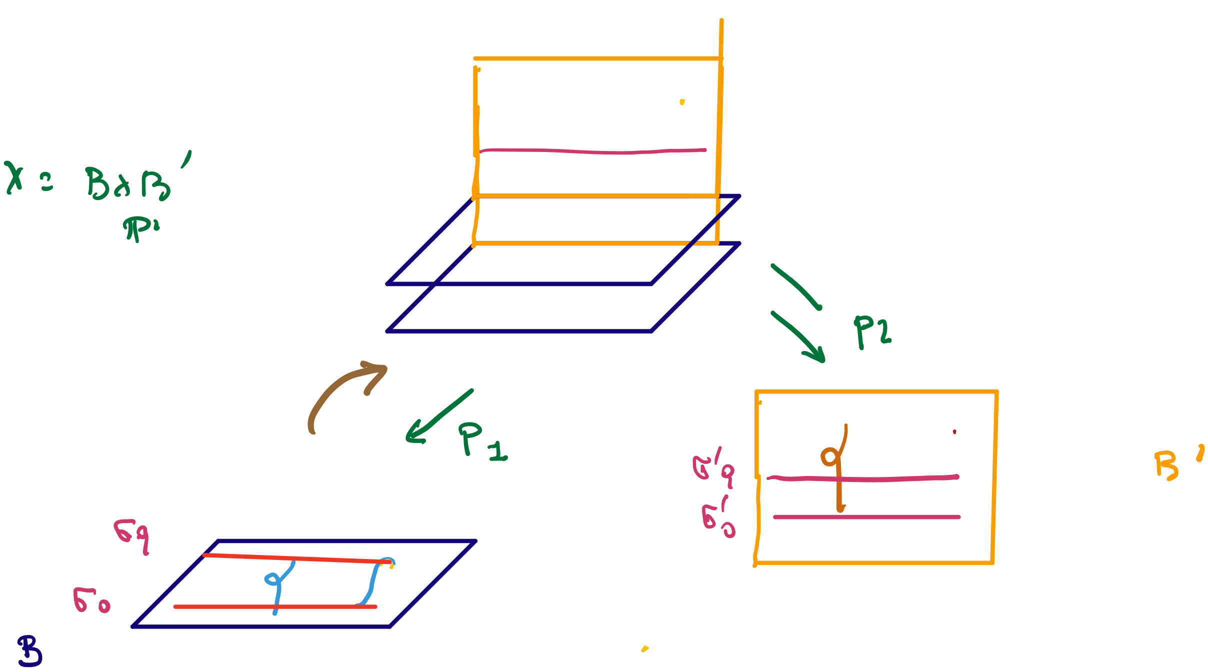
Objects of interest for the spectrum.

- $MW(X/B)$: Mordell-Weil group of sections. of $X \xrightarrow{} B$ σ

- Enumerative invariants

<u>Def</u>/ Theorem: $MW(X/B)$ is abelian, finite generated.

$X = B \times_{\mathbb{P}^1} B'$



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(via dualities)

Physics prediction : $X \to B$ elliptic C-Y 3fold :

$\mathrm{rk}\, MW(X) \leq 20$ if $B \neq \mathbb{P}^2$

$\mathrm{rk}\, MW(X) \leq 24$ if $B = \mathbb{P}^2$

Remark : (i) · $\dim X = 2$ (K3) Physics : $\mathrm{rk}\, MW \leq 18$

· (Cox, Kloosterman, Kuwata) :

$\mathrm{rk}\, MW \leq 18$, all values occur.

What is top $\mathrm{rk}\, MW(X)$, $\dim X = 3$?

(ii) $\dim X = 3$

$\exists$ examples (many) $\mathrm{rk}\, MW \leq 9$

$$\boxed{\text{Question: Others !?}}$$

Def : Elkies' "Excellent family"

$$W_{NDE}: \; y^2 = x^3 + (p_4 \zeta^4 + p_{10} \zeta)x + \zeta^9 + p_6 \zeta^3 + p_{18}$$

(x, y, z) $(6, 9, 2)$

p_j invariant form of deg j

for ST_{33} Shephard-Todd unitary reflection in $\mathbb{P}^4$

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Theorem (Elkies): Take

$P_j|_{\mathbb{P}^2}, \mathbb{P}^2$ general, C quadratic in $\mathbb{P}^2$

$\Rightarrow$ $\mathrm{rk}\, MW(W_{NDE}/\mathbb{P}^2) = 10$

Question 5: (i) Is W_{NDE} Calabi-Yau ?

(ii) The spectrum (invariants)

Problem: to answer.

Analyze the singularities of W_{NDE}

Theorem: (G-Weigand):

$\exists\, X_i \to B$, Calabi-Yau, Schoen, B: RES

(i) $\mathrm{rk}\, MW(X_i/B) = 10$

(ii) Compute the spectrum of F-theory

$\updownarrow$

dictionary (+) Compute invariants, in particular

<u>relative genus 0</u> Gopakumar-Vafa,

which are birational invariants.

Dictionary : Geometry $\leftrightarrow$ F-theory

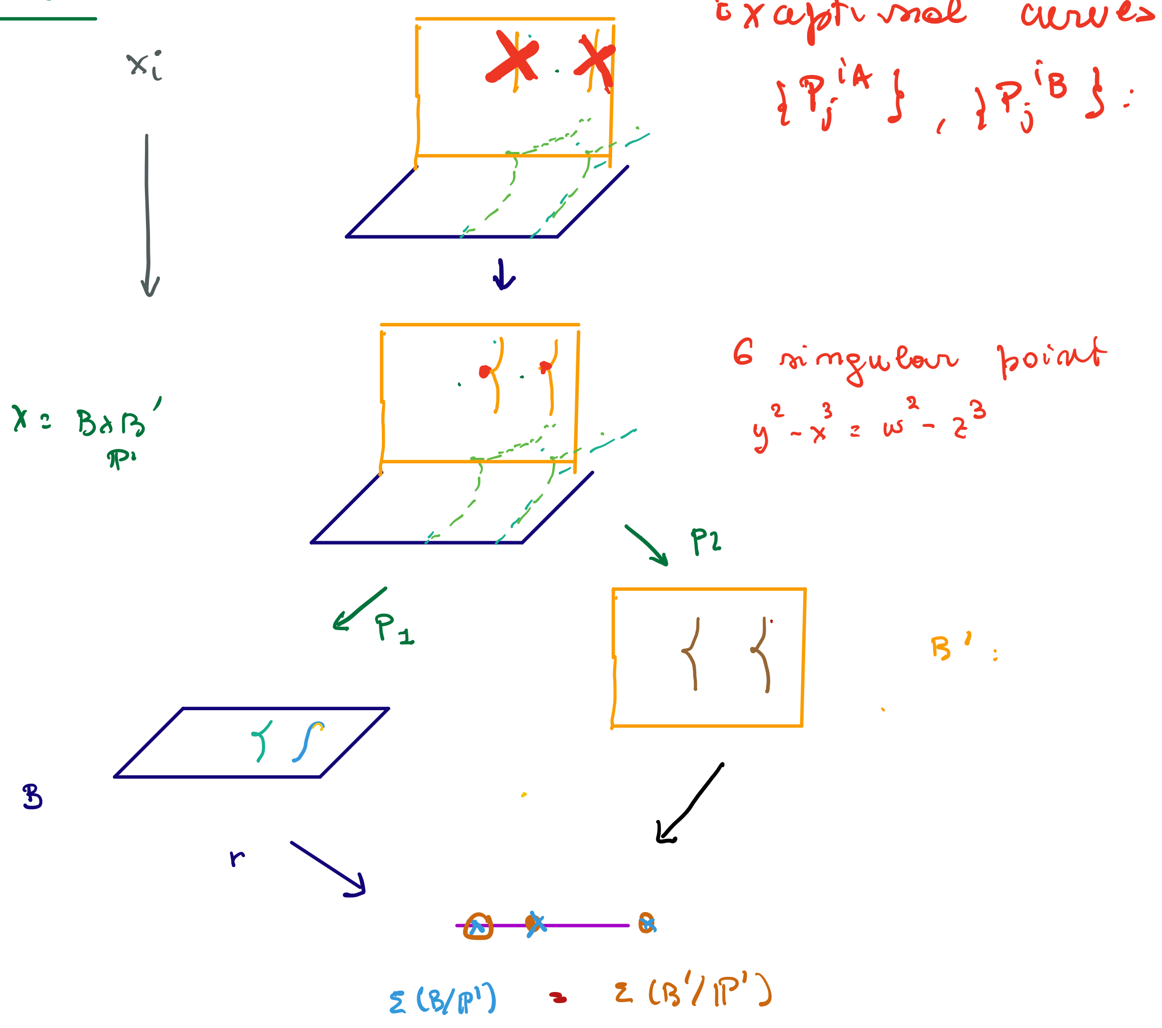
in particular:

- $MW(X_i/B)$ $\Leftrightarrow$ gauge group $\bigoplus_{a=1}^{10} U(1)_a$
- rk $MW(X_i/B)$ $\leftrightarrow$ V, vector multiplets
- Genus 0 G-V invariants $\leftrightarrow$ H_{ch}, charged hypermultiplets.
- Shioda-map (height pairings) $\leftrightarrow$ $U(1)_a$ charges

Anomaly cancellations hold.

Proof: Take X_i as in Namikawa-Rossi

Cartoon:



(i) o $\sigma_0, \ldots, \sigma_q$ of RES give 8 sections, ind.

- $X_i \to X$ small res. gives 8 other ind.

(ii) careful analysis.

Relative Gopakumar Vafa invariants:

$$m_{\{0,P^i_j A\}} = 1, \quad m_{\{0,P^i_j B\}} = 1, \quad ,, \quad m_{\{0,P^i_j A+B\}} = 1,$$

$$\boxed{\textbf{Question}: \text{ Is this a } W_{NAE} \text{ ?}}$$

Theorem:

$$\exists \quad X_i \longrightarrow \mathbb{P}^2, \qquad \mathrm{rk}\ MW(X_i/\mathbb{P}_2) = 10$$

X_i : birational to:

$$W_i : y^2 = x^3 + \beta_{18}, \qquad \deg \beta_{18} \text{ reducible.}$$

Proof: sketch:

$$\begin{tikzcd} X_i \arrow[d] \arrow[dr] & \\ B \arrow[r] & \mathbb{P}^2 \end{tikzcd}$$

Blow up in the first example of R.E.S.

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- Fix a section of

$$\begin{tikzcd} X_i \arrow[d] \\ B \arrow[u, bend left] \arrow[r] & \mathbb{P}^2 \end{tikzcd}$$

- Build a Relative log canonical model: W_i

$$\begin{tikzcd} X_i \arrow[r, dashed] \arrow[d] & W_i \arrow[dl] \\ \mathbb{P}^2 \end{tikzcd}$$

- :

$$\beta_{18} = \prod_i (\text{equations of } 6 \text{ cuspidal cubics})^2$$

as in example 1.

Corollary: W_i has

- $\mathbb{Q}$-factorial canonical (not terminal) sing. over $\{P_0, \dots P_8\} \in \mathbb{P}^2$
- Non $\mathbb{Q}$-factorial terminal sing over the cusps in the pencils in $\mathbb{P}^2$
- The singular locus of the discriminant has. 15 points.

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Example: $F(x,y,z) = z_0^3 + z_1^3 + z_2^3$

$$G(x,y,z) = z_0^2 z_1 - 2^{-1/3}\zeta z_0^2 z_2 - 2^{1/3}\zeta^2 z_0 z_2^2 \quad , \quad \zeta^3 = 1$$

$\cdot \{P_0, \dots P_8\} = \{F = G = 0\}$

$$B: \quad y^2 = x^3 - \frac{1}{4}(27\lambda^6 - 5\lambda^3\mu^3 - 3\mu^6)$$

$$\begin{tikzcd} B \arrow[r] \arrow[d, "r"'] & \mathbb{P}^2 \arrow[dl, dashed] \\ \mathbb{P}^1 & \end{tikzcd}$$

- 6 singular fibers : $r^{-1}(q_j)$: $y^2 = x^3$ $\{$

where $q_j = [\lambda_j, \mu_j]$, roots of $(27\lambda^6 - 5\lambda^3\mu^3 - 3\mu^6)$ $\quad 1 \le j \le 6$

Corollary: If p_{18} is irreducible,

and $p_4 = p_{10} = p_6 = 0$

w_i is not a W_{NDE}.

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<u>Def</u> : Elkies' "Excellent family"

$$: y^2 = x^3 + (p_4 \zeta^4 + p_{10} \zeta) x + \zeta^9 + p_6 \zeta^3 + p_{18}$$

(x, y, z) $(6, 9, 2)$

p_j invariant form of deg j
for ST_{33} Shephard-Todd unitary reflection

<u>Remark</u>: p_4 : is the equation of the Burkhardt quartic, record quartic, 45 nodes.

irreducible p_{18} : equation of dual of the Burkhardt
not normal. (Freitag-Salvati-Manni)
45 components of codimension one.

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Questions:

- Is $\beta_{18} = \beta'_{18}|_{\mathbb{P}^2}$, β'_{18}-ST_{33} invariant?

- What about general W_{NDE}?

- Compute all Gopakumar-Vafa for $X_i \to B$