Higher Berry classes for many-body quantum lattice systems

joint work with

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Plan

- Review of "Berry phase" and difficulties with extending to many-body systems
- Local Noether theorem for quantum lattice systems
- Application to "higher Berry phase".

Review of "Berry phase".

Let M be a compact manifold ("parameter space")

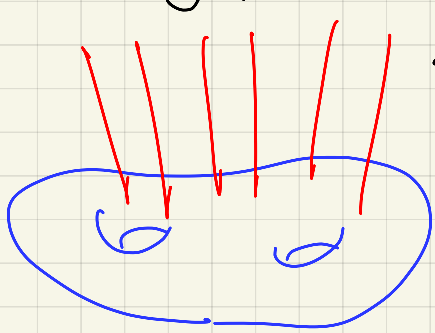
Let V be a fixed f.d. Hilbert space.

Let $H_\mu : V \to V$, $\mu \in M$

be a smooth family of Hamiltonians (self-adjoint operators) s.t.

lowest eigenvalue is "non-degenerate" (eigenspace has dimension 1 $\forall \mu \in M$)

$\mathcal{L}$



M

← line bundle of "ground states"

- $\mathcal{L}$ is a subbundle of a trivial bundle with fiber V.
- There is a canonical connection ∇_B on $\mathcal{L}$ ("Berry connection")

$F = \nabla_B^2 \in \Omega^2(M, i\mathbb{R})$ ("Berry curvature")

Purely imaginary b.c. ∇_B is unitary.

$\left[\frac{F}{2\pi i}\right] \in H^2(M, \mathbb{R})$ is an integral class (lies in the image of $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$)

$\left[\frac{F}{2\pi i}\right]$ depends only on $\mathcal{L}$, not on the Hamiltonian $H(\mu)$.

If $\left[\frac{F}{2\pi i}\right] \neq 0$, the family of ground states is "topologically non-trivial".

Can we generalize this to the case of a many-body quantum system?

Quantum lattice systems

Typical setup: $\mathbb{Z}^d \subset \mathbb{R}^d$

Hilbert space is not specified at the outset.

Instead, specify an algebra of observables,

$$\mathcal{A} = \bigotimes_{p \in \mathbb{Z}^d} \mathcal{A}_p, \qquad \underset{\substack{\uparrow \\ \text{f.d. } C^*\text{-algebra}}}{\mathcal{A}_p} \cong Mat(n_p, \mathbb{C})$$

Define somehow a Hamiltonian on $\mathcal{A}$ which generates the time evolution.

A ground state is now a positive linear function $\omega : \mathcal{A} \to \mathbb{C}$ invariant under evolution.

ω defines the Hilbert space via GNS construction.

<u>Problem:</u> as the Hamiltonian varies, so will ω. The Hilbert spaces for different ω are <u>not naturally isomorphic</u>.

The same problem from a different angle

Suppose the parameters are themselves physical degrees of freedom...

$$\phi^\alpha \rightsquigarrow \phi^\alpha(\vec{x}, t)$$

local coords. on M

$$\vec{x} = (x^1, \dots, x^d)$$

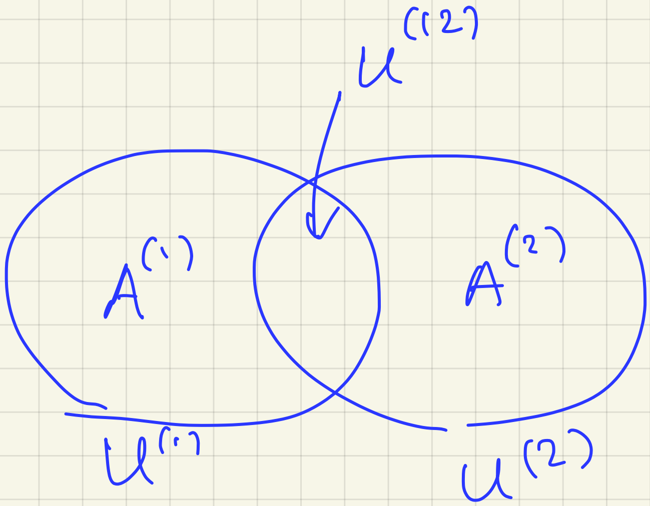
$d = 0$

$$S_{eff} = \int A^B_\alpha(\phi(t)) \frac{d\phi^\alpha}{dt} dt = \int \phi^* A^B \quad (t\dots)$$

Here $A^B = A^B_\alpha \, d\phi^\alpha$ is the "connection 1-form".

$$D_B = d + A^B$$

A^B is defined only locally.



$$A^{(1)} - A^{(2)} = df^{(12)}$$

$$f^{(12)} : U^{(12)} \to i \cdot \mathbb{R}/2\pi\mathbb{Z}$$

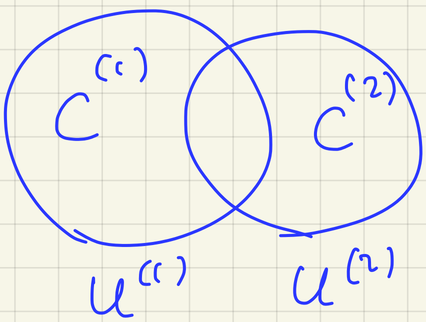
$F^B = dA^B$ is an honest 2-form

$d > 0$

$$S_{eff} = \int \phi^* C + \ldots$$

$$C = \frac{1}{(d+1)!} C_{\alpha_1 \ldots \alpha_{d+1}} d\phi^{\alpha_1} \ldots d\phi^{\alpha_{d+1}}$$

is a $(d+1)$-form (locally)



$$C^{(1)} - C^{(2)} = d\lambda^{(12)}$$

$$\lambda^{(12)} \in \Omega^d(U^{(12)}, i\mathbb{R})$$

$$\lambda^{(12)} + \lambda^{(23)} + \lambda^{(13)} = d\rho^{(123)}$$

$$\rho^{(123)} \in \Omega^{d-1}(U^{(123)}, i\mathbb{R})$$

$$\vdots$$

$\mathcal{F} = dC \in \Omega^{d+2}(M, i\mathbb{R})$ is an honest $(d+2)$-form

$(C, \lambda, \rho, \ldots)$ is a Beilinson-Deligne cocycle

($(d+1)$-form gauge field)

Alternative description:

Cheeger-Simons differential character

Summary

- For $d > 0$ expect a "higher Berry phase" described by a $(d+1)$-form gauge field.
- $\left[\frac{\mathcal{F}}{2\pi i} \right] \in H^{d+2}(M, \mathbb{R})$ should be possible to define without knowing the details of the Hamiltonian
- $\left[\frac{\mathcal{F}}{2\pi i} \right]$ is possibly quantized. (perhaps under some additional conditions)

"Philosophy".

- To understand lattice systems, it is natural to use Quantum Statistical Mechanics.
- Mathematical apparatus: Operator algebras (analysis)
- This is not as hard as it seems (some nice algebra emerges)

Currents in QFT

Symmetry $\Longrightarrow$ current j^μ

- "global"
- acts locally

- conserved: $\partial_\mu j^\mu = 0$
- local

Ambiguity: $j^\mu \mapsto j^\mu + \partial_\rho h^{\mu\rho}$, $h^{\mu\rho} = -h^{\rho\mu}$

Equivalently: $j^0 = \rho \mapsto \rho + \partial_i h^{0i}$, $i = 1, 2, 3$

$(Q = \int \rho \, d^3x$ is unchanged$)$

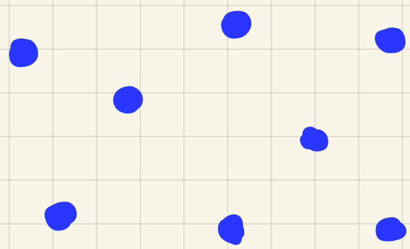
$j^k \mapsto j^k - \partial_0 h^{0k} + \partial_i h^{ki}$.

additional ambiguity
(Net current through a surface not affected)

All these ambiguities are physically harmless*

* I think.

Currents and conserved quantities on a lattice.



$\Lambda \subset \mathbb{R}^d$

(e.g. $\Lambda = \mathbb{Z}^d$)

algebra of observables: $\mathcal{A} = \overline{\bigotimes_{p \in \Lambda} \mathcal{A}_p}$

$\mathcal{A}_p = \mathrm{Mat}(n_p, \mathbb{C})$.

Hamiltonian:

$A \mapsto$ "$[H, A]$" $= \delta_H(A)$

derivation of $\mathcal{A}$

$\delta(AB) = \delta(A) \cdot B + A \cdot \delta(B)$

$A, B \in \mathcal{A}$.

- δ_H is NOT bounded
- δ_H is not defined everywhere on $\mathcal{A}$.
- Unbounded derivations do not form a Lie algebra.

Physically relevant derivations:

$$\delta_\Phi : A \mapsto \sum_{\Gamma \in \mathcal{P}(\Lambda)} [\Phi(\Gamma), A] \quad , \quad \Phi(\Gamma) \in \mathcal{A}_\Gamma$$

Properties of $\Phi(\Gamma)$

- $\Phi(\Gamma)^* = -\Phi(\Gamma)$.
- $\Phi(\Gamma) \to 0$ as $\operatorname{diam}(\Gamma) \to \infty$
- $\operatorname{Tr}(\Phi(\Gamma)) = 0$.

N.B.

- $\Phi(\Gamma)$ are not uniquely defined by δ_Φ.
- no notion of "density of energy".

Alternative:

$$\delta_F : A \mapsto \sum_{p \in \Lambda} [F_p, A] \quad , \quad F_p \in \mathcal{A}.$$

- $F_p^* = -F_p$.
- F_p is approximately localized near p.
- $\operatorname{Tr}(F_p) = 0$

Q1 How do we describe the ambiguity in F_p for a given δ_F ?

Conservation equation:

$$\delta_H F_j = - \sum_{k \in \Lambda} J^F_{kj}$$

J^F_{kj} ← "current from j to k"

↑ conserved quantity at j

- $J^F_{kj} = - \left(J^F_{kj}\right)^*$
- $J^F_{kj} = - J^F_{jk}$
- $J^F_{kj} \to 0$ as $|j-k| \to \infty$
- J^F_{kj} is approximately localized near j, k.

Q2 Does such a J^F exist for any "Hamiltonian" F ?

Q3 How does one describe the ambiguity in J^F for a given F_p, $p \in \Lambda$?

Finite-range (UL) chains

$$a : \underbrace{\Lambda \times \Lambda \times \ldots \times \Lambda}_{q+1 \text{ times}} \to \mathcal{A}$$

q-chain

- skew-symmetric $\qquad q \geq 0$
- traceless
- $a_{j_0 \ldots j_q}$ localized on a ball of radius R centered at j_k, $k \in \{0, \ldots, q\}$
- bounded

Let C_q^{UL} be the space of q-chains.

$$(\partial a)_{j_1 \ldots j_q} = \sum_{j_0} a_{j_0 j_1 \ldots j_q}.$$

$\partial : C_q^{UL} \to C_{q-1}^{UL}$ is a differential: $\partial^2 = 0$.

Let J_{jk} be a 1-chain.

$$(\partial J^F)_k = \sum_j J_{jk} .$$

Conservation equation takes the form

$$\delta_H(F) = -\partial J^F.$$

ambiguity?

$$J \mapsto J + \partial M , \quad M \in C_2^{UL}.$$

No other ambiguities f.r.

Theorem

Homology of $(C_\bullet^{UL}, \partial)$ is trivial for $q > 0$.

$$H_0(C_\bullet^{UL}) \cong \text{finite-range derivations}$$

Def. UL Noether complex is

$$\cdots \xrightarrow{\partial} C_2^{UL} \xrightarrow{\partial} C_1^{UL} \xrightarrow{\partial} C_0^{UL} \xrightarrow{\delta} C_{-1}^{UL}$$

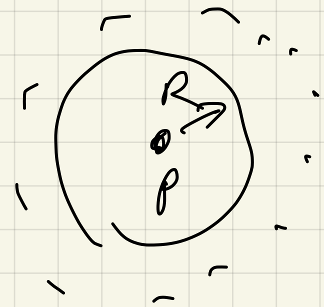
C_{-1}^{UL} = "nice" derivations

UL Noether complex has trivial homology.

Rapidly decaying (UAL) chains

Uniformly Almost Local chains.

$a : \underbrace{\Lambda \times \dots \times \Lambda}_{q+1 \text{ times}} \to \mathcal{A}$



- skew-symmetric
- traceless
- $a_{j_0 \dots j_q}$ approximately localized near any of $j_0, \dots, j_q$.

$\partial : C_q^{UAL} \to C_{q-1}^{UAL}$, $\partial^2 = 0$.

UAL derivations:

$a : A \mapsto \sum_j [a_j, A]$. $A \in \mathcal{A}_{al}$

dense subalgebra in $\mathcal{A}$

Theorem

Homology of $\dots \to C_1^{UAL} \to C_0^{UAL} \to C_{-1}^{UAL}$ is trivial

UAL derivations

Important technical point:

one can set up the definitions so that $\mathcal{A}_{al}$, C_q^{UAL} are "nice" topological vector spaces (Frèchet spaces) $\forall q \geq -1$ and ∂_q is a continuous map $\forall q \geq 0$

Applications

- any "local" symmetry of a "local" Hamiltonian gives a "local" current

- currents are determined up to exact 1-chains.

- conserved quantity determines its density up to exact 0-chains

Example: energy current.

$$J^E_{jk} = [H_k, H_j].$$ (Kitaev, "Anyons")

more generally, for any two 0-chains ("densities") F_j, G_j can define a 1-chain:

$$[F, G]_{jk} = [F_j, G_k] - [F_k, G_j].$$

How does this binary operation fit into our story?

There is a bracket of degree $+1$

on $C_\bullet^{UL}$ & $C_\bullet^{UAL}$

$$[a, b]_{i_0 \cdots i_{p+q+1}} = \frac{1}{p!\,q!} [a_{i_0 \cdots i_p}, b_{i_{p+1} \cdots i_{p+q+1}}]$$

a: p-chain, b: q-chain

$\pm$ permutations

$(C_\bullet^{UL, UAL}, \partial, [\cdot, \cdot])$ is a (1-shifted) DG Lie algebra (DGLA)

- graded Jacobi for $[\,,\,]$
- graded Leibniz for $([\,,\,], \partial)$

$C_{-1} \times C_{-1} \to C_{-1}$ - Lie bracket of derivations

$C_0 \times C_0 \to C_1$ - symmetric operation on 0-chains

C.f. in QFT:

skew-symmetric in a, b

$$[J^a(\vec{x}, 0), J^b(\vec{y}, 0)] = f^{abc} J^c(\vec{x})\, \delta^3(\vec{x} - \vec{y})$$

symmetric in a, b → + Schwinger terms

QFT - lattice dictionary

QFT in $D+1$ space-time dimensions	Lattice models in D spatial dimensions
D-form	0-chain
$(D-1)$-form = current	1-chain
p-form	$(D-p)$-chain
de Rham d	∂
?	$[\ ,\ ]$
Submanifold of dimension p	$(D-p)$-cochain (to be discussed)

What is this good for?

Not sure ... but there is an analogous structure which is VERY useful.

Let $\omega : \mathcal{A} \to \mathbb{C}$ be a state.

<u>Def</u>. An observable A does not excite ω if $\langle [A, B] \rangle = 0$ $\forall B \in \mathcal{A}$.

This <u>seems</u> rare.

<u>Def</u>. A derivation δ_f does not excite ω if $\langle \delta_f(B) \rangle = 0$ $\forall B \in \mathcal{A}$

<u>Example</u>: Derivations associated to unbroken symmetries

<u>Def</u>. A q-chain a does not excite ω if $\langle [a_{j_0 \dots j_q}, B] \rangle = 0$ $\forall B \in \mathcal{A}$

Also <u>seems</u> rare.

Let C_q^ω be the space of chains which do not excite ω, $q \geq -1$

Th. Let ω is a gapped ground state of a Hamiltonian $H = \sum_j H_j$ arising from a 0-chain H_j, then $(C_\bullet^\omega, \partial)$ has trivial homology.

Application #1

$\delta_H : A \mapsto \sum_j [H_j, A]$ does not excite the ground state ω of H

If H is gapped, then $\exists\, \tilde{H}_j$ s.t.

- $\tilde{H}_j$ is a 0-chain
- $\delta_H = \delta_{\tilde{H}}$ (i.e. H & $\tilde{H}$ generate the same dynamics)
- $\langle [\tilde{H}_j, B] \rangle = 0 \quad \forall\, B \in \mathcal{A}$

i.e. $|0\rangle$ is the eigenstate of each $\tilde{H}_j$. ← due to Kitaev

Application #2 : Higher Berry class

Def. A family of states $\omega_\mu : \mathcal{A} \to \mathbb{C}$, $\mu \in M$
is called smooth

if $\exists\ G \in \Omega^1(M, \mathfrak{D}_{al})$

$\mathfrak{D}_{al}$: space of UAL derivations

such that (M, ω) is "covariantly constant"

w.r. to $D = d + G$.

$$d \langle A \rangle_\mu = \langle G(A) \rangle_\mu \quad \forall A \in \mathcal{A}$$

Or, if $A : M \to \mathcal{A}$ is a smooth function:

$$d \langle A \rangle = \langle dA + G(A) \rangle = \langle DA \rangle$$

note:

$$0 = d^2 \langle A \rangle = \langle D^2(A) \rangle = \langle F(A) \rangle$$

where $F = dG + \frac{1}{2}[G, G]$.

Thus $F \in \Omega^2(M, \mathfrak{D}^{\omega}_{al})$

also, $DF = 0$.

Th. Let $H : M \to \mathfrak{D}_\ell$ is a smooth family of "Hamiltonians" with unique ground states ω_μ, $\mu \in M$ such that $\langle A \rangle_\mu \in C^\infty(M, \mathbb{C})$ $\forall A \in \mathcal{A}$, then such a G exists.

(follows from a theorem of Y. Ogata and A. Moon).

This provides a justification for studying smooth families of states.

We will further assume that for some $\mu_0 \in M$ the state ω_{μ_0} is a unique ground state of a gapped UL "Hamiltonian". Then

$$\dots \to \Omega^p(M, C_1^\omega) \xrightarrow{\partial} \Omega^p(M, C_0^\omega) \xrightarrow{\partial} \Omega^p(M, \mathfrak{D}_{al}^\omega)$$

has trivial homology $\forall p \geq 0$.

Now let's try to define the Berry curvature $\mathcal{F} \in \Omega^{d+2}(M, i\mathbb{R})$

<u>$d=0$</u>

$\mathcal{F} = \langle F \rangle$

(makes sense b.c. $\mathfrak{D}_{al} = \mathfrak{D} = \left\{ \begin{array}{l} \text{traceless} \\ \text{anti-self-adjoint} \\ \text{elements of } \mathcal{A} \end{array} \right\}$

Check that it is closed:

$d\mathcal{F} = \langle DF \rangle = 0.$

<u>$d>0$</u>

$F = dG + \frac{1}{2}[G, G] \in \Omega^2(M, \mathfrak{D}^{\omega}_{al})$

But $\langle F \rangle$ now does not make sense.

$\mathfrak{D}^{\omega}_{al}$ now consists of formal sums

$B = \sum_{j \in \Lambda} B_j$, $B_j \in \mathcal{A}_{al}$ is "approximately localized at j".

$\sum_j \langle B_j \rangle$ is divergent ...

Maurer-Cartan element

Recall that given a DG Lie algebra $(\mathfrak{g}^\bullet, \mathbf{d}, [\ ,\ ])$, an MC element is a degree-1 element G satisfying

$$\mathbf{d}G + \frac{1}{2}[G,G] = 0.$$

$G \in \Omega^1(M, \mathcal{D}_{al})$ is a degree-1 element in

$$\mathfrak{g}^\bullet = \bigoplus_{p,q} \Omega^p(M, T_q)$$

where $T_\bullet$ is the complex

$$\begin{array}{lccccccc} & \cdots \xrightarrow{\partial} & C_1^{(\omega)} & \xrightarrow{\partial} & C_0^{\upsilon} & \xrightarrow{\partial} & \mathcal{D}_{al} \\ \text{deg:} & & -2 & & -1 & & 0 \end{array}$$

$\mathbf{d} = d + \partial$

$[\ ,\ ]$ is a combination of $[\ ,\ ]$ and wedge product of forms.

G is <u>not</u> an MC element b.c.

$$F = dG + \frac{1}{2}[G, G] \neq 0.$$

But can use G as a "seed":

$$\mathbf{G} = G + \sum_{p=2}^{\infty} g^p, \qquad g^p \in \Omega^p(M, T_{p-1})$$

Can solve the MC equation recursively:

$$F + \partial g^{(2)} = 0 \quad ✓$$

$$D g^{(2)} - \partial g^{(3)} = 0 \quad ✓$$

$$D g^{(3)} + \frac{1}{2}\{g^{(2)}, g^{(2)}\} + \partial g^{(4)} = 0 \quad ✓$$

$$\vdots$$

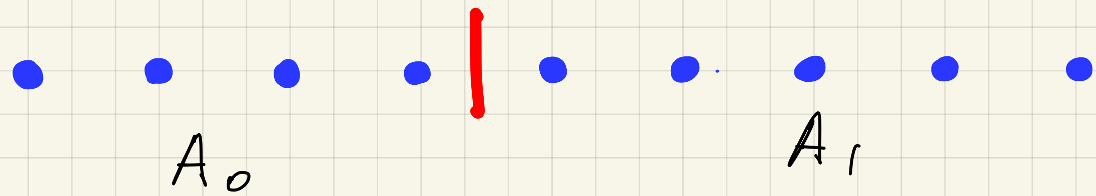
What do we do now?

Need to extract an observable out of all these form-valued chains...

Integrating chains

$d=1$

How to integrate a 1-chain (= current):



$$\int_{A_0A_1} h = \sum_{\substack{p\in A_0\\ q\in A_1}} h_{pq}$$ = "flux of $h^{(1)}$ through a section"

Main property

$$\int_{A_0A_1} \partial h' = 0$$ for a 2-chain h'.

Consider $$\mathcal{F}^{(3)} = \int_{A_0A_1} \langle g^{(3)} \rangle \in \Omega^3(M, i\mathbb{R})$$

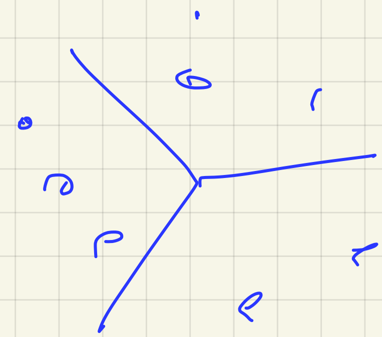
$$d\mathcal{F}^{(3)} = \int_{A_0A_1} \langle D g^{(3)} \rangle = -\int_{A_0A_1} \langle \partial g^{(4)} \rangle = 0$$

Can also check that $[\mathcal{F}^{(3)}] \in H^3(M, i\mathbb{R})$ does not depend on the section.

General d : Integrating a d-chain

- Choose $d+1$ conical regions $A_0, \dots, A_d \subset \mathbb{R}^d$ s.t. $\bigcup_i A_i = \mathbb{R}^d$

- $$\int_{A_0 \cdots A_d} h = \sum_{\substack{i_0 \in A_0 \\ \vdots \\ i_d \in A_d}} h_{i_0 \dots i_d} .$$



Hence we let

$$\mathcal{F}^{(d+2)} = \int_{A_0 \dots A_d} \langle g^{(d+2)} \rangle$$

$$d\mathcal{F}^{(d+2)} = \int_{A_0 \dots A_d} \langle D g^{(d+2)} \rangle = \pm \underbrace{\int_{A_0 \dots A_d} \langle \partial g^{(d+3)} \rangle}_{= 0}$$

<u>Th</u>. $[\mathcal{F}^{(d+2)}] \in H^{d+2}(M, i\mathbb{R})$ does not depend on the choice of $A_0 \dots A_d$, or the MC element G

$\Rightarrow$ it is a topological invariant of the family.

Concluding remarks

- $[\mathcal{F}^{(d+2)}]$ is not expected to be quantized, in general.
- But can be shown to be quantized for $d=1$
- $\mathcal{F}^{(d+2)}$ depends on various choices
  $\Rightarrow$ not physical.
- Suppose (M,ω) is a G-equivariant family (G = a compact Lie group)
  Can attach to it an element of $H_G^{(d+2)}(M, i\mathbb{R})$
- This is interesting even for $M=\{*\}$:
  get topological invariants of G-invariant states taking values in G-invariant polynomials on the Lie algebra of G
  $\Rightarrow$ Chern-Simons forms !

References

This work originated with a proposal by A. Kitaev, see his talk at Dan Freed's 60th birthday conference.

Further developed by Leo Spodyneiko & A.K., 2001.03454, 2003.09519

The talk is based on work with Nikita Sopenko (to appear soon)