The geometric interpretation of the Peter-Weyl theorem

João P. Nunes
(Dep. Mathematics, IST, Lisbon)

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Joint work with:

T.Baier, J.Hilgert, O.Kaya and J.Mourão

Let K be a compact Lie group, let $\hat{K}$ be the set of equivalence classes of irreps of K, labeled by highest weight λ and let dx be the Haar measure.

Peter-Weyl Theorem: One has an orthogonal decomposition

$$L^2(K, dx) = \widehat{\bigoplus}_{\lambda \in \hat{K}} V_\lambda^{\oplus \dim V_\lambda}.$$

Let $f \in L^2(K, dx)$. Then, there exist endomorphisms $A_\lambda \in End(V_\lambda), \lambda \in \hat{K}$, such that

$$f(x) = \sum_{\lambda \in \hat{K}} \mathrm{tr}(\pi_\lambda(x) A_\lambda).$$

On the other hand, the Borel-Weil-Bott theorem realizes irreps of K as spaces of holomorphic sections of line bundles over coadjoint orbits

$$V_\lambda \cong H^0(\mathcal{O}_\lambda, L_\lambda).$$

We will describe how Hamiltonian flows in imaginary time and geometric quantization relate the 2 results by assigning geometric cycles in T^*K to each summand in the Peter-Weyl decomposition.

Plan of the talk

- Geodesics on the space of Kähler metrics
- Geometric quantization
- Toric Manifolds and T^*K
- The Kirwin-Wu polarization on T^*K

Geodesics on the space of Kähler metrics

Recall that a Kähler manifold (M, ω, J, γ) is a symplectic manifold (M, ω) with compatible integrable complex structure J. The two structures define a Riemannian metric (M, γ). Locally, on a sufficiently small open set $U \subset M$, the Kähler form can be written in terms of a (non-unique) Kähler potential

$$\omega = i\partial\bar{\partial}\kappa, \quad \kappa \in \mathbb{C}^{\infty}(U, \mathbb{R}).$$

If M is compact, from the $\partial\bar{\partial}$-lemma, the space of Kähler forms in the class $[\omega] \in H^{1,1}(M)$ is described by

$$\mathcal{H} = \{\phi \in C^{\infty}(M) : \omega_{\phi} = \omega + i\partial\bar{\partial}\phi > 0\},$$

(two Kähler forms in $[\omega]$ differ by a *global* Kähler potential), so that the space of Kähler metrics in the class $[\omega]$ is given by $\mathcal{H}/\mathbb{R}$.

The Mabuchi metric on $\mathcal{H}$ is

$$||\delta\phi||_{\phi}^{2} = \int_{M} (\delta\phi)^{2} d\mu_{\phi}, \;\; d\mu_{\phi} = \frac{1}{n!}\omega_{\phi}^{n}.$$

The expression for the curvature of $\mathcal{H}$, as well as other arguments, show that (Donaldson, Semmes), *morally*,

$$\mathcal{H} \cong Ham_{\mathbb{C}}(M,\omega)/Ham(M,\omega),$$

an infinite-dimensional non-compact symmetric space for the "group" of complexified symplectomorphisms of (M,ω). (This group does not really exist but this is a useful analogy.) This led Donaldson to suggest that geodesics on $\mathcal{H}$ should be generated by "complexified" Hamiltonian flows. This can be made concrete and explicit, and can applied to interesting examples.

Geodesics on $\mathcal{H}$ are described by the non-linear equation

$$\ddot{\phi} = \frac{1}{2}||\nabla\dot{\phi}||^2_{\phi}.$$

The (very hard to obtain) analytical and geometrical properties of these geodesics play an important role in recent developments in Kähler geometry, namely on the relation between stability and the existence of constant scalar curvature Kähler metrics in the Yau-Tian-Donaldson program.

For instance, the K-energy is an important functional on $\mathcal{H}$, which is convex along geodesics and whose critical points give constant scalar curvature metrics.

Let (M, ω, J, γ) be a compact Kähler manifold and suppose that all the structures (symplectic form ω, complex structure J, Riemannian metric γ) are real analytic.

If X is a real analytic vector field, its time t flow, $\varphi_t : M \to M$, will be real analytic in t. Power series (in one variable) have a radius of convergence in the *complex plane*. This is defined on small open sets on M (Gröbner). Since M is compact there exists some $T > 0$ such that we can analytically continue to complex time τ for $|\tau| < T$.

Let z^j be local holomorphic coordinates on M and consider their (complex) time τ flow

$$z_\tau^j = e^{\tau X} z^j = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X^k (z^j).$$

Whenever it is well defined, the operator $\exp(\tau X)$ acts as an automorphism of the algebra of (real analytic) functions:

$$e^{\tau X}(fg) = e^{\tau X}(f) e^{\tau X}(g).$$

Therefore, on overlapping holomorphic coordinate charts the operator $\exp(\tau X)$ preserves the (holomorphic) coordinate transformations defining M as a complex manifold.

Theorem: (Mourão-N '15) There exists $T > 0$ such that for $|\tau| < T$ there exists a global complex structure J_τ on M, defined locally by the coordinates z_τ^j, and a unique biholomorphism

$$\varphi_\tau : (M, J_\tau) \to (M, J).$$

We get two equivalent Kähler structures (ie nothing new)

$$(M, \omega, J, \gamma) \cong (M, \varphi_\tau^* \omega, J_\tau, \varphi_\tau^* \gamma_\tau).$$

When X is an *Hamiltonian vector field*, $X = X_h, h \in C^\omega(M)$, however, J_τ is still compatible with the original symplectic form ω and we get a *new Kähler structure*. (Note that $\{z_\tau^i, z_\tau^j\}_{PB} = 0$ since X_h is Hamiltonian.)

Theorem: (Mourão-N '15) For $|\tau| < T$,

i) (M, ω, J_τ) is a Kähler manifold (w/ a new Riemannian metric γ_τ)

ii) There is a reasonably explicit formula for the local Kähler potential κ_τ.

Theorem: (Mourão-N '15) The family of Kähler metrics γ_τ, $\tau = is, s \in \mathbb{R}$, is a geodesic with respect to the Mabuchi metric.

If M is not compact the Mabuchi metric is not defined but the geodesic equation still makes sense and, sometimes, the above results still hold even if M is not compact and for $T = +\infty$.

Geometric Quantization

Let (M, ω) be a symplectic manifold, dim $M = 2n$.

Geometric quantization is a rich framework where one can study mathematical issues related to the problem of quantization.

Assume that there exists $L \to M$ with hermitian structure h and compatible connection ∇, with curvature $F_\nabla = -i\omega$. One calls (M, L, h, ∇) the *prequantization data.*

The prequantum Hilbert space $\Gamma_{L^2}(M, L)$ is too big for irreduciblity. Choose a *polarization*

$$P \subset TM \otimes \mathbb{C}$$

that is, a Lagrangian $(rank(P) = n, \omega_{|_P} = 0)$, involutive distribution in $TM \otimes \mathbb{C}$.

The *Hilbert space of quantum states* is then (a completion of, if necessary)

$$\mathcal{H}_P = \{s \in \Gamma_{L^2}(M, L) : \nabla_{\bar{P}} s = 0\}.$$

There are two main cases:

I) (M, ω, J, γ) is Kähler and $P = T^{1,0}(M)$, so that $P \cap \bar{P} = 0$. Then, $\mathcal{H} = H^0(M, L)$ (possibly with conditions on growth at infinity).

II) $P = \bar{P}$ is a real polarization. Sections of L covariantly constant along P must be supported on leaves of P where ∇ has trivial holonomy. These are the *Bohr-Sommerfeld leaves* (BS leaves).

If M is compact, in good cases, some open dense subset of M will be fibered by Arnold-Liouville tori. Of these, only a finite number will be BS. Thus, sections which are covariantly constant along P will be *distributional* in nature.

Need to study weak solutions of the equations of covariant constancy.

Mixed polarizations have been less studied and will play the leading role in this talk.

Ideology

A major, and geometrically very rich, problem in geometric quantization is the dependence of $\mathcal{H}_P$ on the choice of P. In some cases, if $\mathcal{J}$ is a nice space of Kähler complex structures for (X, ω) one gets a Hilbert bundle

$$\mathcal{H} \to \mathcal{J},$$

with fiber $\mathcal{H}_P$ over $P \in \mathcal{J}$. Of course, one would like to find a natural unitary (projectively) flat connection on $\mathcal{H}$ provinding for a unitary identification of quantizations for different polarizations through parallel transport.

If $\mathcal{J}$ has some (partial) compactification $\bar{\mathcal{J}}$, where $\partial\bar{\mathcal{J}}$ includes some mixed and real polarizations, one would like to have continuous interpolation between quantum states for holomorphic and real polarizations on the boundary.

Sometimes the holomorphic quantum states can be generated by a concrete analytical gadget applied to quantum states in a real polarization $P_0 \in \partial\bar{\mathcal{J}}$. The unitarity (or lack of) of this operator then decides the equivalence of quantizations in different polarizations $P, P' \in \bar{\mathcal{J}}$.

This gadget, a *generalized coherent state transform*, is intimately related to complexified Hamiltonian symplectomorphisms and to the (Mabuchi) geodesics they generate.

This program can be applied very concretely to rich families of examples which include complex Lie groups, complex tori and classical theta functions, non-abelian theta functions on the moduli space of holomorphic vector bundles over an elliptic curve and symplectic toric manifolds.

Today, I will mention symplectic toric manifolds for context and the main focus will be the case of cotangent bundles of Lie groups.

Toric manifolds (no details, just for context)

Let X be a $2n$-dimensional smooth symplectic toric manifold, with moment map μ and moment polytope $P = \mu(X) \subset \mathbb{R}^n$ defined by linear inequalities

$$l_i(x) = \langle x, \nu_i \rangle - \lambda_i \geq 0,\ \lambda_i \in \mathbb{R}, i = 1, \dots, d,$$

where the ν_i's are the inner pointing primitive normal vectors to the facets of P. Assume $\lambda_i \in \mathbb{Z}, i = 1, \dots, n$, so that $[\frac{\omega}{2\pi}] \in H^2(X, \mathbb{Z})$. (We may have to take $[\frac{\omega}{2\pi}]$ half-integer to include the half-form correction.) Recall that X is a Kähler manifold with an holomorphic action of $(\mathbb{C}^*)^n$ and a dense open orbit

$$\check{X} = \mu^{-1}(\check{P}) \cong \check{P} \times T^n \cong (\mathbb{C}^*)^n.$$

The symplectic form on the open orbit, in action-angle coordinates, is simply

$$\omega = \sum_{i=1}^{n} dx^i \wedge d\theta^i,\ (x, \theta) \in \check{P} \times T^n.$$

Toric invariant Kähler structures on X are described by a symplectic potential of the form (Guillemin '94, Abreu '98)

$$g = g_P + \varphi,$$

where $g_P = \sum_{j=1}^{r} \frac{1}{2}\ell_j(x)\log(\ell_j(x))$ and $\varphi \in C^\infty(\check{P})$ is such that the Hessian $H_{g_P+\varphi}$ is positive definite on $\check{P}$ and has singular behaviour along ∂P fixed by g_P. *Note that the Kähler potential is the Legendre transform of g.*

Let h be a smooth strictly convex function on P. Consider the family of toric Kähler structures defined by the symplectic potentials

$$g_s = g_P + sh,\ s > 0.$$

This is a Mabuchi geodesic ray of toric Kähler metrics on X.

Let $\mathcal{P}_{s,h}$ be the corresponding Kähler polarization.

The holomorphic structure on X and P define an holomorphic structure on an appropriate smooth line bundle $L \to X$, a meromorphic section σ_L of L, and a connection $\nabla = d + ixd\theta$ with curvature $-i\omega$. For any toric Kähler structure the quantization of X in the corresponding Kähler polarization is then

$$H^0(X, L) = \langle \sigma^m = w^m \sigma_L, m \in P \cap \mathbb{Z}^n \rangle_{\mathbb{C}},$$

where $w = (w^1, \ldots, w^n)$ gives the holomorphic coordinates.

(If you include the half-form correction then you get the integral points in a corrected polytope P'.)

(X, ω) has a natural real polarization $\mathcal{P}_{\mathbb{R}} = \langle \frac{\partial}{\partial \theta^i}, i = 1, \ldots, n \rangle_{\mathbb{C}}$. It is immediate to check that the BS leaves correspond to the integral points in P.

The generalized coherent state transform associated to the geodesic family of Kähler structures (ω, J_s) is then defined as

$$C_s : H^0(X, J_0, L) \to H^0(X, J_s, L),\ s > 0,$$

where

$$C_s = \exp(s h_{pQ}) \circ \exp(-s\widehat{h}),$$

where h_{pQ} is the Kostant-Souriau prequantum operator for h and $\widehat{h}(\sigma^m) := h(m)\sigma^m$. The prequantum operator $h_{pQ} = i\nabla_{X_h} + h$ is responsible for analytic continuation along the complex time flow $J_0 \to J_s$, while $\widehat{h}$ asymptotically corrects the growth of norms. (There is also a Lie derivative acting on the half-form part that I am not showing for the sake of simplicity.) One obtains,

Theorem (Baier-Florentino-Mourão-N '11, Kirwin-Mourão-N '13, Kirwin-Mourão-N '16)

i) Pointwise in $\check{X}$, $\lim_{s\to+\infty}\mathcal{P}_{s,h} = \mathcal{P}_{\mathbb{R}}$;
ii) The quantization in the real toric polarization is given by distributional sections supported on integral points

$$\mathcal{H}_{\mathcal{P}_{\mathbb{R}}} = \bigoplus_{m\in P\cap\mathbb{Z}^n} \langle \delta^m \rangle_{\mathbb{C}};$$

iii) $\lim_{s\to+\infty} C_s\sigma_s^m = c\cdot\delta^m$, where c is an (m-independent) constant.

Note that the isotypical decomposition for the lift of the T^n-action to L,

$$H^0(X,J_s,L) = \bigoplus_{m\in P\cap\mathbb{Z}^n} \langle \sigma_s^m \rangle_{\mathbb{C}},$$

is related, in the infinite Mabuchi geodesic time limit, to a collection of geometric cycles in X given by the Bohr-Sommerfeld cycles of the limit polarization.

We will now describe a (more involved) non-abelian analog.

Cotangent bundles of compact Lie groups, $T^*K \cong K_{\mathbb{C}}$

Let K be a compact, connected, simply-connected Lie group and $h : Lie(K) \to \mathbb{R}$ an $Ad-$invariant strongly convex Hamiltonian function $h(\xi)$.

Theorem: (Kirwin-Mourão-N '13) Let $Im(\tau) > 0$. The diffeomorphism

$$\begin{array}{rcl} T^*K & \cong & K_{\mathbb{C}} \\ (x,\xi) & \mapsto & xe^{\tau u}, \end{array}$$

where $u = \frac{\partial h}{\partial \xi}$, defines a $K \times K$-invariant Kähler metric on T^*K, $\gamma_{(h,\tau)}$. Invariance of h guarantees that ξ and $u(\xi)$ Lie in the same Cartan subalgebra. (For simplicity, we will identify $LieK \cong (LieK)^*$).

Note that T^*K is equipped with the standard symplectic structure while $K_{\mathbb{C}}$ is equipped with the standard complex structure inherited from $LieK \otimes \mathbb{C}$.

The Kähler potential is the $K \times K-$invariant function determined by the Legendre transform of h. This generalizes the construction for toric manifolds.

For $\tau = is, s \in \mathbb{R}$, this family of $K \times K-$invariant metrics is a Mabuchi geodesic, generated by the Hamiltonian flow of h analytically continued to imaginary time.

Let $\mathcal{P}_{s,h}$ be the corresponding Kähler polarization.

The quantization in the vertical polarization gives, à la Peter-Weyl and including the half-form correction, with Ω_0 the Haar volume-form,

$$L^2(K, dx) \cong \mathcal{H}_{\mathcal{P}_{vert}} = \widehat{\bigoplus}_{\lambda \in \widehat{K}} W_\lambda^0,$$

$$W_\lambda^0 = \{\sigma_{\lambda,A}^0 = \mathrm{tr}(\pi_\lambda(x)A) \otimes \sqrt{\Omega_0},\ \lambda \in \widehat{K}, A \in End(V_\lambda)\}.$$

(Hall '02)

The quantization in the Kähler polarization polarization $\mathcal{P}_{s,h}, s > 0$, gives

$$\mathcal{H}_{\mathcal{P}_{s,h}} = \widehat{\bigoplus}_{\lambda \in \widehat{K}} W_\lambda^s,$$

where

$$W_\lambda^s = \left\{\sigma_{\lambda,A}^s = \mathrm{tr}(\pi_\lambda(xe^{isu})A)e^{-\frac{1}{2}\kappa_s} \otimes \sqrt{\Omega_s}, \lambda \in \widehat{K}, A \in End(V_\lambda)\right\},$$

where κ_s is the Kähler potential (which is, explicitly, the Legendre transform of sh) and, as above, $u = \nabla h$. Ω_s is a trivializing holomorphic section of the canonical bundle of (T^*K, J_s). (Kirwin-Mourão-N '13)

The quantizations of T^*K in the corresponding Kähler polarizations can be connected to the quantization in the vertical polarization explicitly by analogs of the coherent state transform of Hall (which is the case $h = \xi^2/2, \tau = i$)

$$\begin{aligned} C_s : L^2(K, dx) &\rightarrow \mathcal{H}L^2(K_{\mathbb{C}}, d\nu_s), \quad s > 0, \\ f &\mapsto \mathcal{C} \circ e^{-\frac{s}{2}\Delta} f, \end{aligned}$$

where $\mathcal{C}$ denotes analytic continuation and Δ is the Laplacian for a bi-invariant metric on K. (Hall '02, Florentino-Matias-Mourão-N '05 '06, Kirwin-Mourão-N '13'14)

In the present case we have linear isomorphisms

$$C_s : \mathcal{H}_{\mathcal{P}_{vert}} \to \mathcal{H}_{\mathcal{P}_{s,h}}, \ s > 0,$$

where

$$C_s = \exp(s h_{pQ}) \circ \exp(-s\widehat{h}),$$

As in the toric case, h_{pQ} is the Kostant-Souriau prequantum operator for h and $\widehat{h}_{|_{W^0_\lambda}} := h(\lambda + \rho)$ where ρ is the Weyl vector. (Kirwin-Mourão-N '13) (Again, I am hiding a term with a Lie derivative that takes care of the half-forms.)

Note that the points $\lambda + \rho$ in the weight lattice play the role of the integral points in the moment polytope in the toric case. (Recall we are identifying $LieK \cong (LieK)^*$.)

Note that C_s preserves the isotypical decompositions of the Hilbert spaces, so we have linear $K \times K-$equivariant isomorphisms

$$C_s : W_\lambda^0 \to W_\lambda^s, \ \lambda \in \widehat{K}.$$

Note also that

$$e^{isX_h} \cdot \mathrm{tr}(\pi_\lambda(x)A) = \mathrm{tr}(\pi_\lambda(xe^{isu})A),$$

and, for the trivializing holomorphic section of the canonical bundle,

$$e^{is\mathcal{L}_{X_h}}\Omega_0 = \Omega_s.$$

The prequantum operator $h_{pQ} = i\nabla_{X_h} + h$ is responsible for analytic continuation along the complex time flow, while $\widehat{h}$ asymptotically corrects the growth of norms. (Kirwin-Mourão-N '14).

By the way, acting with complex time Hamiltonian flows of a function which is convex only along a Cartan subalgebra one can also generate $K \times T-$invariant, but not $K \times K-$invariant, Kähler structures on T^*K, one can get interesting mixed polarizations of T^*K and also study the corresponding quantizations. (Mourão-N-Pereira ’18)

The Kirwin-Wu polarization on T^*K

It turns out that the geodesic family of $K \times K-$invariant Kähler structures on T^*K, as defined above, defines a very interesting mixed polarization as $s \to +\infty$, as first noticed by Kirwin and Wu in unpublished work.

We will describe this polarization in a different approach, by using local algebras of polarized functions, and will describe how half-form corrected holomorphic sections converge to distributional sections supported on (partial) Bohr-Sommerfeld cycles.

Let $\mathfrak{t}_+$ be the (closed) positive Weyl chamber associated to a (fixed) choice of maximal torus $T \subset K$ and of simple roots $\{\alpha_1, \ldots, \alpha_r\}$. The sweeping map $s : LieK \to \mathfrak{t}_+$ conjugates Lie algebra elements to the positive Weyl chamber and gives an homeomorphism $(LieK)/Ad_K \cong \mathfrak{t}_+$. (Again, recall that we are identifying $LieK$ and its dual.)

To an Hamiltonian $K-$action on a symplectic manifold X one can associate a smooth Hamiltonian effective action of a torus $\widehat{T}$, with $Lie\widehat{T} \subset LieT$, on an open dense subset $\check{X} \subset X$ and whose moment map is $\widehat{\mu} = s \circ \mu$. This action is given, explicitly, by

$$t \star p = (g^{-1}tg) \cdot p,\ t \in \widehat{T}, g \in K, Ad_g\mu(p) \in \check{\mathfrak{t}}_+.$$

(In general, $\widehat{\mu}$ can take values on a positive codimension stratum of $\mathfrak{t}_+$, called the principal stratum, and whose linear span is $Lie\,\widehat{T}$). (Guillemin-Sternberg '83, Kirwan '84, Woodward '98, Lane '18, ...)

Recall that the (left) $K \times K-$action on T^*K is Hamiltonian with (equivariant) moment map

$$\mu = (\mu_L, \mu_R)(x, \xi) = (Ad_x \xi, -\xi), x \in K, \xi \in LieK.$$

For T^*K, $\hat{T} \cong T$. Consider KAK coordinates on $T^{\check{*}}K = K \times (LieK)_{reg}$,

$$T^*K \ni (x, \xi) \leftrightarrow xe^{i\xi} \leftrightarrow x_1 e^{is\xi_+} x_2^{-1} \in K_{\mathbb{C}},$$

where $x = x_1 x_2^{-1}$ and $\xi = Ad_{x_2}\xi_+$, $\xi_+ \in \check{\mathfrak{t}}_+$, $x_1, x_2 \in K$. Then,

$$(x_1, \xi_+, x_2) \star t = (x_1 t^{-1}, \xi_+, x_2) \sim (x_1, \xi_+, x_2 t).$$

(Since $s \circ \mu_L(x, \xi)$ and $s \circ \mu_R(x, \xi)$ take values in opposite Weyl chambers, we are now considering $\hat{T}$ to act on the left on x_1 or on the right on x_2).

The Kirwin-Wu polarization, $\mathcal{P}_{KW}$, on $T\check{^*}K = K \times (LieK)_{reg}$, is a $K \times K-$invariant polarization that can be described as follows. Its real directions are tangent to the $\hat{T}-$orbits. The subsets

$$\widehat{\mu}^{-1}(\xi_+) = K \times \mathcal{O}_{\xi_+}, \, \xi_+ \in \check{\mathfrak{t}}_+,$$

are $\hat{T}-$ principal bundles over a product of coadjoint orbits $\mathcal{O}_{\xi_+} \times \mathcal{O}_{\xi_+}$ with natural identifications $\mathcal{O}_{\xi_+} \cong K/T \cong K_{\mathbb{C}}/B$, where B is the Borel subgroup. The holomorphic directions of $\mathcal{P}_{KW}$ are then induced by the standard complex structure on the product of these orbits. (One of the orbits actually gets the conjugate complex structure due to the signs of the actions.)

For $\lambda \in \widehat{K}, A \in End(V_\lambda), u_+ = \nabla_{\xi_+} h$, let

$$f^s_{\lambda,A}(x,\xi) = \mathrm{tr}(\pi_\lambda(x_1 e^{isu_+} x_2^{-1})A),$$

(Recall the quantization of T^*K in the Kähler polarizations $\mathcal{P}_{s,h}$.) Consider, locally, for $v_\lambda \in V_\lambda$ a highest weight vector,

$$F_{\lambda,A}(x_1,x_2) = \mathrm{tr}(\pi_\lambda(x_1)v_\lambda \otimes v^*_\lambda \pi_\lambda(x_2^{-1})A).$$

The local rings generated by $\{F_{\lambda,A}\}_{\lambda\in\widehat{K},A\in End(V_\lambda)}$ are generated by products of the form, with $u \in V_\lambda, v^* \in V^*_\lambda$,

$$F_{\lambda,u\otimes v^*} = \mathrm{tr}(\pi_\lambda(x_1)v_\lambda \otimes v^*_\lambda \pi_\lambda(x_2^{-1})u \otimes v^*) = \mathrm{tr}(\pi_\lambda(x_1)v_\lambda \otimes v^*)\,\mathrm{tr}(\pi_\lambda(x_2^{-1})u \otimes v^*_\lambda).$$

Each of these factors transforms under the expected character of T as a section for a Borel-Weil-Bott line bundle over a coadjoint orbit. The quotients $F_{\lambda,A_1}/F_{\lambda,A_2}$, where defined, are $\widehat{T}-$invariant.

Note that $\mathcal{P}_{s,h}$ is generated pointwise by the Hamiltonian vector fields of J_s anti-holomorphic functions,

$$\mathcal{P}_{s,h} = \langle X_{\bar{f}^s_{\lambda,A}}, \lambda \in \widehat{K}, A \in End(V_\lambda)\rangle_{\mathbb{C}}.$$

Theorem (Baier-Hilgert-Kaya-Mourão-N)

i) Pointwise, on $T\check{^*}K$,

$$\lim_{s\to+\infty} X_{\frac{f^s_{\lambda,A_1}}{f^s_{\lambda,A_2}}} = X_{\frac{F_{\lambda,A_1}}{F_{\lambda,A_2}}}.$$

ii) Locally, on $T\check{^*}K$,

$$\mathcal{P}_{KW} = \langle X_{\frac{\bar{F}_{\lambda,A_1}}{\bar{F}_{\lambda,A_2}}}, X_{\widehat{\mu}_j}\rangle_{\mathbb{C}},$$

for $\lambda \in \widehat{K}, A \in End(V_\lambda)$ and $\widehat{\mu}_j$ are the components of $\widehat{\mu}$.

iii) Pointwise, in the Lagrangian Grassmannian on $T\check{^*}K$,

$$\lim_{s\to+\infty} \mathcal{P}_{s,h} = \mathcal{P}_{KW}.$$

One can check that the Bohr-Sommerfeld cycles for $\mathcal{P}_{KW}$ are precisely given by $\hat{\mu}^{-1}(\lambda+\rho)$, for highest weights λ, where ρ is the Weyl vector.

The canonical bundle $K_{\mathcal{P}_{KW}}$ is trivializable with a $K \times K-$invariant trivialization given by (with Lie algebra conventions such that $\rho(u_+) > 0$)

$$\Omega_\infty = \lim_{s \to +\infty} s^{-r} e^{-2s\rho(u_+)} \Omega_s.$$

One also has a polarized section of $K_{\mathcal{P}_{KW}}$, with $A_\rho = v_\rho \otimes v_\rho^* \in End(V_\rho)$,

$$\hat{\Omega}_\infty = \beta(\xi_+) F_{\rho, A_\rho}^{-2} \Omega_\infty,$$

where it is useful to include an appropriate ($\mathcal{P}_{KW}-$polarized) function $\beta(\xi_+)$.

Theorem (Baier-Hilgert-Kaya-Mourão-N)

$$\lim_{s\to+\infty} C_s\,\sigma^0_{\lambda,A} = \sigma^\infty_{\lambda,A}, \quad \lambda \in \hat{K}, A \in End(V_\lambda)$$

where

$$\sigma^\infty_{\lambda,A} = a_0^\lambda F_{\lambda,A} F_{\rho,A_\rho} \delta_{\xi_+}(\lambda+\rho) \otimes \sqrt{\hat{\Omega}_\infty},$$

is a $\mathcal{P}_{KW}-$polarized distributional section supported on the BS cycle $\hat{\mu}^{-1}(\lambda+\rho)$.

When h is quadratic, the CST transforms C_s are unitary for all $s > 0$. This gives a natural Hilbert space structure

$$\mathcal{H}_{\mathcal{P}_{KW}} = \widehat{\bigoplus}_{\lambda\in K, A\in End(V_\lambda)} \langle \sigma^\infty_{\lambda,A} \rangle_{\mathbb{C}},$$

such that the quantizations on $\mathcal{P}_{vert}$ and on $\mathcal{P}_{KW}$ become related, through the family $\mathcal{P}_{s,h}$ of Kähler polarizations, by a *unitary $K\times K-$equivariant non-abelian Fourier transform* $\mathcal{F}_K : \mathcal{H}_{\mathcal{P}_{vert}} \to \mathcal{H}_{\mathcal{P}_{KW}}$, defined by

$$\mathcal{F}_K(\sigma^0_{\lambda,A}) = \sigma^\infty_{\lambda,A}.$$

Future interesting generalizations are expected.

THANK YOU.