Topological quantum field theories & homotopy cobordisms

arXiv:2208.14504

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16/11/22

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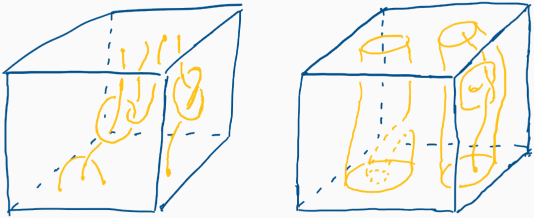

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- Such functors may factor through other categories that may be easier to work with - I will give a construction of a category of *cofibrant cospans* of topological spaces. Functors into this category are obtained roughly by taking the complement of particle trajectories.
- I will also show that Yetter's TQFTs associated to finite groups generalise to explicitly calculable functors from this category.

Talk Plan

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3. Family of functors $\mathsf{Z}_G \colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Cofibrant cospans and homotopy cobordisms

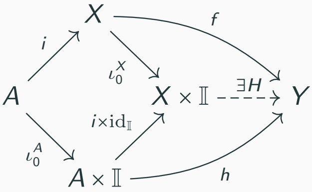
Definition
Let X, Y and M be spaces. A <u>cofibrant cospan</u> from X to Y is a diagram $i\colon X \to M \leftarrow Y : j$ such that $\langle i, j\rangle\colon X \sqcup Y \to M$ is a closed cofibration.
For spaces $X, Y \in \mathbf{Top}$, we define the set of all cofibrant cospans

$$\mathrm{CofCos}(X,Y) = \left\{ \begin{matrix} X & & & & Y \\ & {}_i\searrow & & \swarrow_j & \\ & & M & & \end{matrix} \;\middle|\; \langle i,j\rangle \text{ is a closed cofibration} \right\}.$$

Cofibrations

Definition

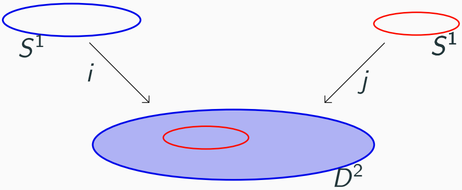
Let A and X be spaces. A map $i\colon A \to X$ has the homotopy extension property, with respect to the space Y, if for any pair of a homotopy $h\colon A \times \mathbb{I} \to Y$ and a map $f\colon X \to Y$ satisfying $(f \circ i)(a) = h(a, 0)$, there exists a homotopy $H : X \times \mathbb{I} \to Y$, extending h, with $H(x, 0) = f(x)$ and $H(i(a), t) = h(a, t)$. This is illustrated by the following diagram.



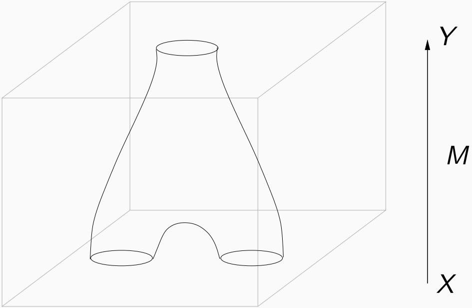
(Where for any space X, $\iota_0^X\colon X \to X \times \mathbb{I}$ is the map $x \mapsto (x, 0)$.)

We say that $i\colon A \to X$ is a cofibration if i satisfies the homotopy extension property for all spaces Y.

Cofibrant cospans



Cofibrant cospans



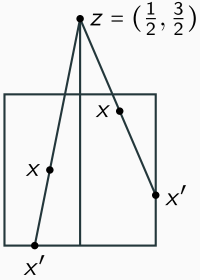
Cofibrant cospans

Example
Let X be a space. The cospan $\mathrm{id}_X\colon X \to X \leftarrow X :\!\mathrm{id}_X$ is not a cofibrant cospan, unless $X = \varnothing$.

Cofibrant cospans

Proposition
For X a topological space, the cospan $\iota_0^X\colon X \to X \times \mathbb{I} \leftarrow X : \iota_1^X$ is a cofibrant cospan (where $\iota_a^X\colon X \to X \times \mathbb{I}$ is the map $x \mapsto (x, a)$).

Proof sketch
Suppose there exists a homotopy $h\colon (X \sqcup X) \times \mathbb{I} \to K$, and a map $f\colon X \times \mathbb{I} \to K$, such that $h((x,0),0) = f(x,0)$ and $h((x,1),0) = f(x,1)$. Composition with below retraction gives homotopy $H\colon (X \times \mathbb{I}) \times \mathbb{I} \to K$.



Cofibrant cospans

Proposition

A concrete cobordism canonically defines a cofibrant cospan.

Precisely, let X, Y and M be smooth oriented manifolds, and let M be a concrete cobordism from X to Y. Hence there exists a diffeomorphism $\phi\colon \bar{X} \sqcup Y \to \partial M$. Define maps $i(x) = \phi(x, 0)$ and $j(y) = \phi(y, 1)$. Then, using X, Y and M to denote the underlying topological spaces, $i\colon X \to M \leftarrow Y : j$ is a cofibrant cospan.

Example

Any CW complex together with a pair of disjoint subcomplexes and inclusions gives a cofibrant cospan.

Composition of cofibrant cospans

Lemma

(*I*) For any spaces X, Y and Z in $Ob(\mathbf{Top})$ there is a composition of cofibrant cospans

$$\bullet : \mathrm{CofCos}(X,Y) \times \mathrm{CofCos}(Y,Z) \to \mathrm{CofCos}(X,Z)$$

$$\left(\begin{array}{ccc} X & & Y \\ & {}_{i}\searrow \; M \; \swarrow_{j} & \end{array} , \begin{array}{ccc} Y & & Z \\ & {}_{k}\searrow \; N \; \swarrow_{l} & \end{array} \right) \mapsto \begin{array}{ccc} X & & Z \\ & {}_{\tilde{i}}\searrow \; M \sqcup_Y N \; \swarrow_{\tilde{l}} & \end{array}$$

where $\tilde{i} = p_M \circ i$ and $\tilde{l} = p_N \circ l$ are obtained via the following diagram

$$\begin{array}{ccccccc} X & & & Y & & & Z \\ & {}_{i}\searrow & \swarrow_{j} & & {}_{k}\searrow & \swarrow_{l} & \\ & & M & & N & & \\ & & {}_{p_M}\searrow & & \swarrow_{p_N} & & \\ & & & M \sqcup_Y N, & & & \end{array}$$

the middle square of which is the pushout of $j\colon M \leftarrow Y \to N\colon k$ in $\mathbf{Top}$.

Equivalence classes cofibrant cospans

Lemma

For each pair $X, Y \in Ob(\mathrm{CofCos})$, we define a relation on $\mathrm{CofCos}(X, Y)$ by

$$\begin{pmatrix} X & & Y \\ & {}_{i}\searrow \; M \; \swarrow_{j} & \end{pmatrix} \overset{ch}{\sim} \begin{pmatrix} X & & Y \\ & {}_{i'}\searrow \; N \; \swarrow_{j'} & \end{pmatrix}$$

if there exists a commuting diagram

$$\begin{array}{ccccc} & & M & & \\ & \overset{i}{\nearrow} & & \overset{j}{\nwarrow} & \\ X & & \downarrow \psi & & Y \\ & \underset{i'}{\searrow} & & \underset{j'}{\swarrow} & \\ & & M' & & \end{array}$$

where ψ is a homotopy equivalence. For each pair $X, Y \in \mathbf{Top}$ the relations $(\mathrm{CofCos}(X, Y), \overset{ch}{\sim})$ are a congruence on CofCos.

Category of cofibrant cospans

Theorem (T.)
The quadruple

$$\mathrm{CofCos} = \left(Ob(\mathbf{Top})\ ,\ \mathsf{CofCos}(X,Y)/\overset{ch}{\sim}\ ,\ \bullet\ ,\ \begin{bmatrix} X & & & & X \\ & {\scriptstyle \iota_0^X} \searrow & & \swarrow {\scriptstyle \iota_1^X} & \\ & & X \times \mathbb{I} & & \end{bmatrix}_{\mathsf{ch}} \right)$$

is a category.

Category of cofibrant cospans

Proof uses classical theorem (E.g. Brown06, Thm7.2.8):

If $X \xrightarrow{i} M \xleftarrow{j} Y$, $X \xrightarrow{i'} N \xleftarrow{j'} Y$ are cospans such that $\langle i, j \rangle \colon X \sqcup Y \to M$ and $\langle i', j' \rangle \colon X \sqcup Y \to N$ are cofibrations, then the set of homotopy equivalences ψ such that

$$\begin{array}{ccccc} & \overset{i}{\nearrow} & M & \overset{j}{\nwarrow} & \\ X & & \downarrow{\scriptstyle \psi} & & Y \\ & \underset{i'}{\searrow} & M' & \underset{j'}{\swarrow} & \end{array}$$

commutes, is in bijective correspondence with the set of ψ' such that there exists $\phi \colon N \to M$ with $\psi' \circ \phi$ and $\phi \circ \psi'$ homotopic to identity through maps commuting with cospans.

Monoidal category of cofibrant cospans

There is a functor $\Phi\colon \mathbf{Top}^h \to \mathrm{CofCos}$ which sends a homeomorphism $f\colon X \to Y$ to the cospan $X \xrightarrow{\iota_0^Y \circ f} Y \times \mathbb{I} \xleftarrow{\iota_1^Y} Y$.

Theorem (T.)
There is a monoidal category $(\mathrm{CofCos}, \otimes, \varnothing, \alpha_{X,Y,Z}, \lambda_X, \rho_X, \beta_{X,Y})$ where

$$\left[W \xrightarrow{i} M \xleftarrow{j} X \right]_{\text{ch}} \otimes \left[Y \xrightarrow{k} N \xleftarrow{l} Z \right]_{\text{ch}} = \left[W \sqcup Y \xrightarrow{i \sqcup k} M \sqcup N \xleftarrow{j \sqcup l} X \sqcup Z \right]_{\text{ch}}.$$

All other maps are the images of the corresponding maps in $(\mathbf{Top}, \sqcup)$.

Definition
A space X is called *homotopically* 1*-finitely generated* if $\pi(X, A)$ is finitely generated for all finite sets of basepoints A.
Let χ denote the class of all homotopically 1-finitely generated spaces.

Theorem (T.)
There is a (symmetric monoidal) subcategory of CofCos

$$\mathrm{HomCob} = \left(\chi, \mathrm{HomCob}(X, Y), \bullet, \begin{bmatrix} X & & X \\ & {\scriptstyle \iota_0^X} \searrow \quad \swarrow {\scriptstyle \iota_1^X} & \\ & X \times \mathbb{I} & \end{bmatrix}_{\mathrm{ch}} \right).$$

Motion groupoids

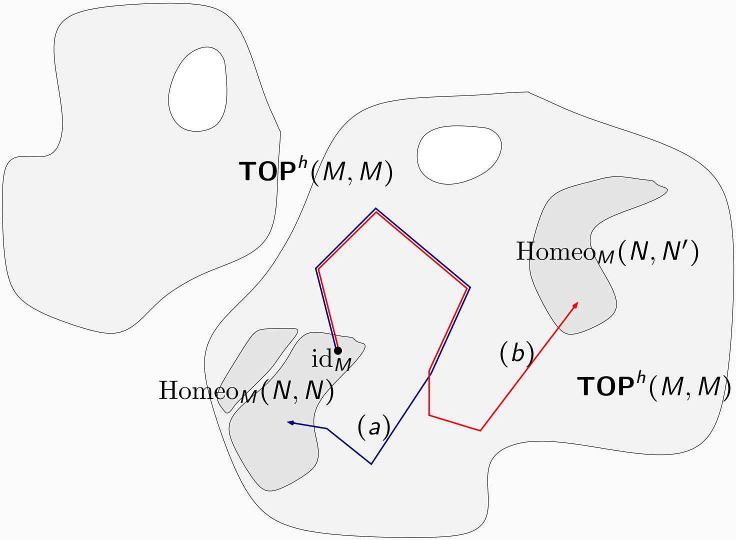
Motion groupoids

Definition
Fix a manifold, submanifold pair $\underline{M} = (M, A)$. A flow in $\underline{M}$ is a map $f \in \mathbf{Top}(\mathbb{I}, \mathbf{TOP}^h_A(M, M))$ with $f_0 = \mathrm{id}_M$. Define,

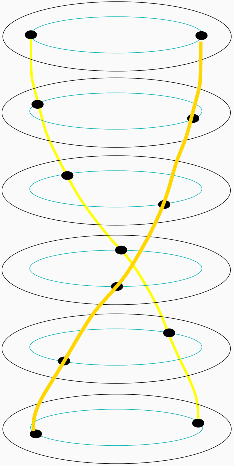
$$\mathrm{Flow}_{\underline{M}} = \{f \in \mathbf{Top}(\mathbb{I}, \mathbf{TOP}^h_A(M, M)) \mid f_0 = \mathrm{id}_M\}.$$

Definition
Fix a $\underline{M} = (M, A)$. A motion in $\underline{M}$ is a triple $f : N \rightharpoondown N'$ consisting of a pre-motion $f \in \mathrm{Flow}_{\underline{M}}$, a subset $N \subseteq M$ and the image of N at the endpoint of f, $f_1(N) = N'$.

Motion groupoids



Motion groupoids



Motion groupoids

Theorem (.T, Faria Martins, Martin)
Let $\underline{M} = (M, A)$ where M is a manifold and $A \subset M$ a subset. There is a groupoid

$$\mathrm{Mot}_{\underline{M}} = (\mathcal{P}M,\ \mathrm{Mt}_{\underline{M}}(N,N')/\overset{m}{\sim}, *, [\mathrm{Id}_M]_{\mathrm{m}},\ [f]_{\mathrm{m}} \mapsto [\bar{f}]_{\mathrm{m}}).$$

- The motion subgroupoid of a configuration of n points in the disk is isomorphic to the n strand Artin braid group.
- The motion subgroupoid of a configuration of n unknotted unlinked loops in the 3-ball is isomorphic to the loop braid group with n loops.

Definition
The worldline of a motion $f\colon N \rightsquigarrow N'$ in a manifold M is

$$\mathbf{W}\left(f\colon N \rightsquigarrow N'\right) = \bigcup_{t\in[0,1]} f_t(N) \times \{t\} \subseteq M \times \mathbb{I}.$$

Let $\mathbf{W}'(f\colon N \rightsquigarrow N') = (M \times \mathbb{I}) \smallsetminus \left(\mathbf{W}\left(f\colon N \rightsquigarrow N'\right)\right)$.

Theorem (T.)
Let M be a manifold. There is a well-defined functor

$$\mathcal{MOT}_M^A\colon \mathrm{hfMot}_{\underline{M}} \to \mathrm{HomCob}$$

which sends an object $N \in Ob(\mathrm{hfMot}_{\underline{M}})$ to $M \smallsetminus N$, and which sends a morphism $[f\colon N \rightsquigarrow N']_{\mathrm{m}}$ to the cospan homotopy equivalence class of

$$M \smallsetminus N \xrightarrow{\iota_{f_0}} \mathbf{W}'(f\colon N \rightsquigarrow N') \xleftarrow{\iota_{f_1}} M \smallsetminus N'$$

where $\iota_{f_t}\colon M \smallsetminus f_t(N) \to \mathbf{W}'(f\colon N \rightsquigarrow N')$, $m \mapsto (m, t)$.

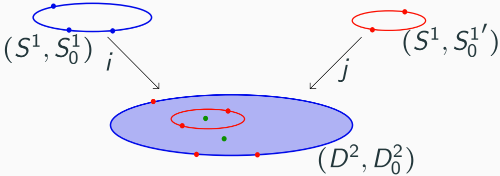
$Z_G \colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Definition

Let $\boldsymbol{\chi}$ be the set of pairs (X, X_0) such that X is in χ and X_0 is a finite representative subset.

Let (X, X_0), (Y, Y_0) and (M, M_0) be in $\boldsymbol{\chi}$. A *based homotopy cobordism* from (X, X_0) to (Y, Y_0) is a diagram $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) :j$ such that:

1. $i\colon X \to M \to Y\colon j$ is a homotopy cobordism.
2. i and j are maps of pairs.
3. $M_0 \cap i(X) = i(X_0)$ and $M_0 \cap j(Y) = j(Y_0)$.


(S^1, S^1_0)
i
j
$(S^1, {S^1_0}')$
(D^2, D^2_0)

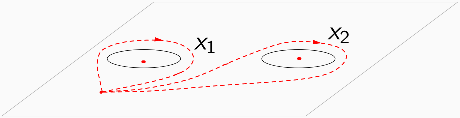
Let G be a group.

For a pair $(X, X_0) \in \chi$, define

$$Z^!_G(X, X_0) = \mathbb{C}\left(\mathbf{Grpd}\left(\pi(X, X_0), G\right)\right).$$

Example

$\pi(X, X_0) \cong (\mathbb{Z} * \mathbb{Z}) \sqcup \{*\} \sqcup \{*\}$. Maps from $\pi(X, X_0)$ to G are determined by pairs in $G \times G$, whose elements respectively denote the images of the equivalence classes of the loops marked x_1 and x_2 in the figure, so we have $\mathrm{Z}^!_G(X, X_0) \cong \mathbb{C}(G \times G)$.



$\mathrm{Z}_G \colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Let $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ be a based homotopy cobordism, we define a matrix

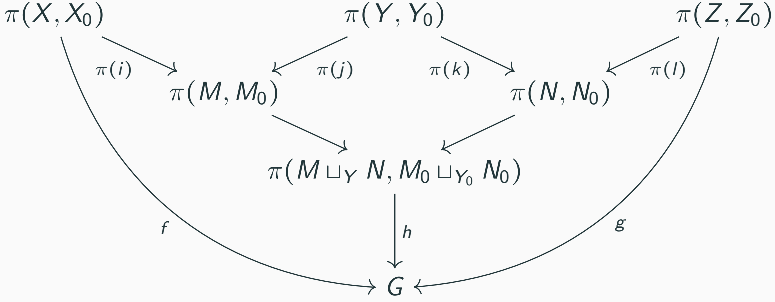
$$\mathrm{Z}^!_G\left(\begin{smallmatrix} (X,X_0) & & (Y,Y_0) \\ & {}_i\searrow \quad \swarrow_j & \\ & (M,M_0) & \end{smallmatrix}\right) : \mathrm{Z}^!_G(X, X_0) \to \mathrm{Z}^!_G(Y, Y_0)$$

as follows. Let $f \in \mathrm{Z}^!_G(X, X_0)$ and $g \in \mathrm{Z}^!_G(Y, Y_0)$ be basis elements, then

$$\left\langle g \,\middle|\, \mathrm{Z}^!_G\left(\begin{smallmatrix} (X,X_0) & & (Y,Y_0) \\ & {}_i\searrow \quad \swarrow_j & \\ & (M,M_0) & \end{smallmatrix}\right) \middle|\, f \right\rangle = \left| \left\{ h : \pi(M, M_0) \to G \;\middle|\; \begin{array}{ccccc} \pi(X,X_0) & & & & \pi(Y,Y_0) \\ & \overset{\pi(i)}{\searrow} & & \overset{\pi(j)}{\swarrow} & \\ & & \pi(M,M_0) & & \\ & f & \downarrow h & g & \\ & & G & & \end{array} \right\} \right|$$

Lemma

The map $Z^!_G$ preserves composition, extended in the obvious way to a composition of based cospans.



Proof

Thm.9.1.2, Topology and Groupoids, Brown gives that middle square is a push out.

Lemma
Let X be a topological space, G a group, $X_0 \subseteq X$ a finite representative subset and $y \in X$ a point with with $y \notin X_0$. There is a non-canonical bijection of sets

$$\Theta_\gamma\colon \mathbf{Grpd}(\pi(X, X_0), G) \times G \to \mathbf{Grpd}(\pi(X, X_0 \cup \{y\}), G)$$
$$(f, g) \mapsto F$$

where γ is a choice of a path from some $x \in X_0$ to y and F is the extension along γ and g.

$Z_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Consider a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$. It follows

$$Z^!_G(M, M_0 \cup \{m\}) = |G|\, Z^!_G(M, M_0).$$

It follows that for all M_0' and M_0, we can write

$$Z^!_G(M, M_0' \cup M_0) = |G|^{(|M_0' \cup M_0| - |M_0|)} Z^!_G(M, M_0)$$

and

$$Z^!_G(M, M_0' \cup M_0) = |G|^{(|M_0' \cup M_0| - |M_0'|)} Z^!_G(M, M_0')$$

which together imply

$$|G|^{-|M_0|} Z^!_G(M, M_0) = |G|^{-|M_0'|} Z^!_G(M, M_0')$$

and that

$$|G|^{-(|M_0| - |X_0|)} Z^!_G(M, M_0) = |G|^{-(|M_0'| - |X_0|)} Z^!_G(M, M_0').$$

Lemma

We redefine the linear map we assign to a concrete based homotopy cobordisms as

$$Z_G^{!!}\left(\begin{matrix} (X,X_0) & & (Y,Y_0) \\ & {}_i\searrow \quad \swarrow_j & \\ & (M,M_0) & \end{matrix}\right) = |G|^{-(|M_0|-|X_0|)} Z_G^{!}\left(\begin{matrix} (X,X_0) & & (Y,Y_0) \\ & {}_i\searrow \quad \swarrow_j & \\ & (M,M_0) & \end{matrix}\right).$$

The map $Z_G^{!!}$ does not depend on the choice of subset $M_0 \subseteq M$, and this preserves composition. When the relevant cospan is clear, we will refer to this as $Z_G^{!!}(M, X_0, Y_0)$ to highlight the dependence on X_0 and Y_0.

$Z_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$

Lemma
There is a contravariant functor

$$\mathcal{V}_X : \mathbf{FinSet}^*(X) \to \mathbf{Set}$$

constructed as follows. Let $X_\alpha, X_\beta \in Ob(\mathbf{FinSet}^*(X))$ with $X_\beta \subseteq X_\alpha$. Let $\mathcal{V}_X(X_\alpha) = \mathbf{Grpd}(\pi(X, X_\alpha), G)$. For any $v_\alpha \in \mathcal{V}_X(X_\alpha)$ we have a commuting triangle

$$\begin{array}{ccc} \pi(X, X_\beta) & \xrightarrow{\iota_{\beta\alpha}} & \pi(X, X_\alpha) \\ & \overset{v_\alpha \circ \iota_{\beta\alpha}}{\searrow} & \downarrow v_\alpha \\ & & G. \end{array}$$

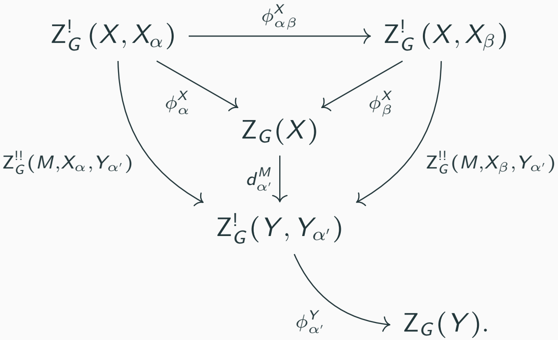
Now let $\mathcal{V}_X(\iota_{\beta\alpha}\colon X_\beta \to X_\alpha) = \phi_{\alpha\beta}$ where $\phi_{\alpha\beta} : \mathcal{V}_X(X_\alpha) \to \mathcal{V}_X(X_\beta)$, $v_\alpha \mapsto v_\alpha \circ \iota_{\alpha\beta}$.

Definition
For $X \in \chi$ define

$$\mathrm{Z}_G(X) = \mathrm{colim}(\mathcal{V}'_X) = \mathbb{C}(\mathrm{colim}(\mathcal{V}_X))$$

where $\mathcal{V}'_X = F_{V_{\mathbb{C}}} \circ \mathcal{V}_X$ and $\mathcal{V}_X\colon \mathbf{FinSet}^*(X) \to \mathbf{Set}$.

Let $i: X \to M \leftarrow Y : j$ be a concrete homotopy cobordism. Fix a choice of $Y_{\alpha'} \subseteq Y$ such that $(Y, Y_{\alpha'}) \in \chi$. For each pair $X_\alpha, X_\beta \subseteq X$ such that $(X, X_\alpha), (X, X_\beta) \in \chi$ we have the following diagram



(1)

Lemma
The assignment

$$Z_G\left(\begin{matrix} X & & & & Y \\ & {}_i\searrow & & \swarrow_j & \\ & & M & & \end{matrix}\right) = \phi^Y_{\alpha'} d^M_{\alpha'}$$

does not depend on the choice of $Y_{\alpha'}$.

Theorem (T.)
Z_G is a functor.

Lemma
Let $i\colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ a choice of concrete based homotopy cobordism, and $[f] \in Z_G(X)$ and $[g] \in Z_G(Y)$ be basis elements (so $[f]$, for example, is an equivalence class in $\mathrm{colim}(\mathcal{V}_X)$), then

$$\langle [g] | Z_G(M) | [f] \rangle = |G|^{-(|M_0| - |X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left| \left\{ h\colon \pi(M, M_0) \to G \mid h|_{\pi(X, X_0)} = f \wedge h|_{\pi(Y, Y_0)} = g \right\} \right|$$

$$= |G|^{-(|M_0| - |X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left\langle g \,\middle|\, Z_G^!(M, M_0) \,\middle|\, f \right\rangle$$

where $\phi_0^Y\colon Z_G^!(Y, Y_0) \to Z_G(Y)$ is the map into $\mathrm{colim}(\mathcal{V}'_Y)$.

Lemma

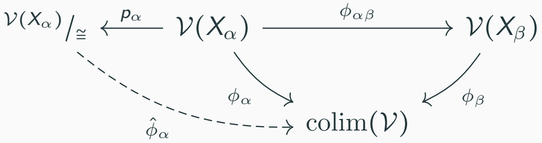
Let $i\colon X \to M \leftarrow Y : j$ be a concrete homotopy cobordism, $i\colon (X, X_0) \to (M, M_0) \leftarrow (Y, Y_0) : j$ a choice of concrete based homotopy cobordism, and $[f] \in \mathrm{Z}_G(X)$ and $[g] \in \mathrm{Z}_G(Y)$ be basis elements (so $[f]$, for example, is an equivalence class in $\mathrm{colim}(\mathcal{V}_X)$), then

$$\begin{aligned}\langle [g] | \mathrm{Z}_G(M) | [f] \rangle &= |G|^{-(|M_0| - |X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left| \left\{ h\colon \pi(M, M_0) \to G \mid h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} = g \right\} \right| \\ &= |G|^{-(|M_0| - |X_0|)} \sum_{g \in \phi_0^{Y-1}([g])} \left\langle g \,\middle|\, \mathrm{Z}^!_G(M, M_0) \,\middle|\, f \right\rangle\end{aligned}$$

where $\phi_0^Y\colon \mathrm{Z}^!_G(Y, Y_0) \to \mathrm{Z}_G(Y)$ is the map into $\mathrm{colim}(\mathcal{V}'_Y)$. Equivalently

$$\langle [g] | \mathrm{Z}_G(M) | [f] \rangle = |G|^{-(|M_0| - |X_0|)} \left| \left\{ h : \pi(M, M_0) \to G \mid h|_{\pi(X,X_0)} = f \wedge h|_{\pi(Y,Y_0)} \sim g \right\} \right|$$

$Z_G\colon \mathrm{HomCob} \to \mathbf{Vect}_{\mathbb{C}}$



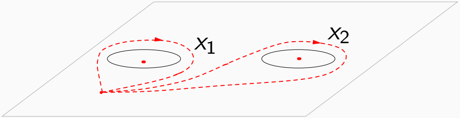
Theorem (T.)
For X a space, the map $\hat{\phi}_\alpha$ is an isomorphism. Hence, for a homotopically 1-finitely generated space $X \in \chi$

$$Z_G(X) = \mathbb{C}((\mathbf{Grpd}(\pi(X, X_0), G)/\cong),$$

for any choice $X_0 \subset X$ of finite representative subset, where $\cong$ denotes taking maps up to natural transformation.
Further,

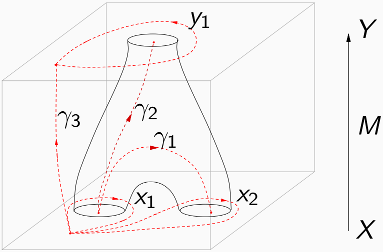
$$Z_G(X) = \mathbb{C}((\mathbf{Grpd}(\pi(X), G)/\cong).$$

Example



Let X be the complement of the embedding of two circles shown. Letting $X_0 \subset X$ be the subset shown, $\mathbf{Grpd}(\pi(X, X_0), G) = G \times G$ as discussed previously. Since all objects are mapped to the unique object in G, taking maps up to natural transformation is means taking maps up to conjugation by elements of G at each basepoint, hence in this case maps are labelled by pairs of elements of G, up to simultaneous conjugation, so we have $Z_G(X) = \mathbb{C}((G \times G)/G)$.

Example



Basis elements in $Z_G(X)$ are given by equivalence classes $[(f_1, f_2)]$ where $f_1, f_2 \in G$ and $[]$ denotes simultaneous conjugation by the same element of G.
Basis elements in $Z_G(Y)$ are given by elements of g taken up to conjugation, denoted $[g_1]$. We have

$$\begin{aligned} \langle [g_1] | Z_G(M) | [(f_1, f_2)] \rangle &= |G|^{-2} \left\{ a, b, c, d, e \in G \mid a = f_1, b = f_2, g_1 \sim ebae^{-1} \right\} \\ &= \left\{ e \in G \mid g_1 \sim ef_1 f_2 e^{-1} \right\} \\ &= \begin{cases} |G| & \text{if } g_1 \sim f_1 f_2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Example

Let $x \in X$ be the basepoint which is in the connected component of X homotopy equivalent to the punctured disk, and $x' \in X$ some choice of basepoint in the other connected component. There is a bijection sending a map $h \in \mathbf{Grpd}(\pi(M, M_0), G)$ to a quadruple $(h', h(\gamma_1), h(\gamma_2), h(\gamma_3)) \in \mathbf{Grpd}(\pi(M, \{x, x'\}) \times G \times G \times G$, where h' is the restriction of h to $\pi(M, \{x, x'\})$. Now $\pi(M, \{x, x'\})$ is the disjoint union of the groupoids $\pi(M_1, \{x\})$ and $\pi(M_2, \{x'\})$ where M_1 is the path connected component of M containing x, and M_2 is the path connected component containing x'. The group $\pi(M_2, \{x'\})$ is trivial, so there is one unique map into G. The group $\pi(M_1, \{x\})$ is isomorphic to the twice punctured disk, which has fundamental group isomorphic to the free product $\mathbb{Z} * \mathbb{Z}$. This isomorphism can be realised by sending the loop x_1 to the 1 in the first copy of $\mathbb{Z}$ and x_2 to the 1 in the second copy of $\mathbb{Z}$. Thus we can label elements in $\mathbf{Grpd}(\pi(M_1, \{x\}), G)$ by elements of $G \times G$ where $g_1 \in (g_1, g_2)$ corresponds to the image of x_1, and g_2 the image of x_2. Hence a map in $\mathbf{Grpd}(\pi(M, M_0), G)$ is determined by a five tuple $(a, b, c, d, e) \in G \times G \times G \times G \times G$ where a corresponds to the image of x_1, b to the image of x_2, and c, d and e correspond to the images of γ_1, γ_2 and γ_3 respectively.