Jones Polynomial and Gauge Theory

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Outline

1. Motivation
2. Jones Polynomial
   - Vertex model construction
   - Witten's construction
3. Analytic continuation
   - Review of Morse theory for finite-dimensional case
   - Application to our problem
4. Electric-magnetic duality

Jones polynomial $J(q, K)$ - Knot invariant discovered by J. F. R. Jones (1984). Assigns to each oriented knot/link a Laurent polynomial in q with integer coefficients.

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A. Tsuchiya and Y. Kanie (1987) – used 2D CFT to generalize Jones' construction to the choice of

- simple Lie group G;
- labeling of a knot by an irreducible representation R of G.

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Two new developments

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Two new developments

1. Khovanov homology
2. Volume conjecture

Summing up:

1. Interconnection between different areas of math and physics...
   - Differential Geometry
   - Knot Theory
   - Conformal Field Theory, String Theory, Quantum Gravity
   - Algebra
   - low-dimensional Topology
   - Functional Analysis
2. ... and has useful applications in these areas

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3. Sum over all possible labellings with certain weight functions of variable q
4. Sum is a Laurent polynomial in q – the Jones polynomial

Example - Trefoil knot

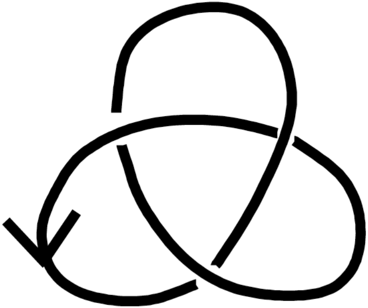
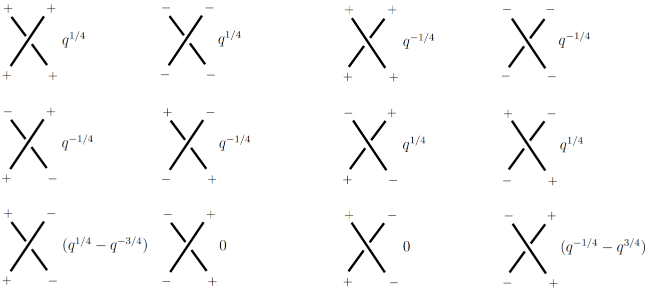
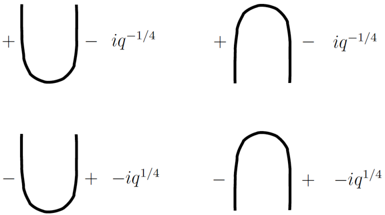


Figure: $\mathbb{R}^2$ projection of trefoil knot

Example - Weights of the vertex model





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- G compact, simple, 1-connected Lie group;
- trivial G-bundle $E \to W$, with W an oriented 3 dim'l manifold;
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Chern-Simons function:

$$\mathrm{CS}(A) = \frac{1}{4\pi} \int_W \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

with Tr an invariant, nondegenerate quadratic form on $\mathfrak{g} = \mathrm{Lie}(G)$, normalized for $\mathrm{CS}(A)$ to be gauge-invariant mod $2\pi\mathbb{Z}$.

For $G = SU(n)$ ($n \geq 2$), Tr taken to be the trace in the n-dim'l representation of $\mathfrak{g}$.

Feynman path integral – Partition function

Taking the Feynman path integral over the infinite dimensional space $\mathcal{U}$ of connections:

$$Z_k(W) = \frac{1}{\mathrm{vol}} \int_{\mathcal{U}} \mathcal{D}A \exp\Big(ik\mathrm{CS}(A)\Big)$$

with $k \in \mathbb{Z}$ for $G = SU(n)$ and $\mathcal{D}A$ represents an integral over all gauge orbits.

Problems of this approach:

- $\mathcal{D}A$ ill-defined as a measure;
- Oscillatory integrand.

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for $W = \mathbb{R}^3$, $G = SU(2)$ and R its 2-dim. representation, then $Z_k(W, K, R) = J(q, K)$ evaluated at $q = \exp(2\pi i/(k+2))$.

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For $\mathrm{CS}(A)$ we obtain $F \equiv dA + A \wedge A = 0$ at critical point.

In the large k limit, the vol. conjecture arises when

$W = \mathbb{R}^3$, $G = SU(2)$ and R its n-dimensional representation, k doesn't need to be an integer.

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$\implies$ need to analytically continue $\mathrm{CS}(A)$

Analytic continuation of $\mathrm{CS}(A)$

Replace

- $G \longrightarrow G_{\mathbb{C}}$
- G-bundle $E \to W \longrightarrow G_{\mathbb{C}}$-bundle $E_{\mathbb{C}} \to W$
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$$\mathrm{CS}(\mathcal{A}) = \frac{1}{4\pi}\int_W \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$

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Critical points are complex-valued flat connections corresponding to a homomorphism

$$\rho : \pi_1(W) \to G_{\mathbb{C}}$$

Outline

Consider a general oscillatory integral in n dimensions:

$$I(k) = \int_{\mathbb{R}^n} \exp(ik\, f\,(x_1, \ldots, x_n))$$

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To do so, analytically continue from $\mathbb{R}^n$ to $\mathbb{C}^n$ and replace $\mathbb{R}^n \to \Gamma$, so that integral converges for any k

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Flow eq. solved on the half-line $(-\infty, 0]$ with condition $x^i(t)$ starts at p at $t = -\infty$.

Morse index of cycle Γ_p: number of directions one can flow downward.

For our integral, $h = \mathrm{Re}(ikf) = \mathrm{Re}(\mathcal{I})$ – real part of a holomorphic function $\implies$ all critical points have Morse index n.

Take $ds^2 = \sum_{i=1}^{n} d|x^i|^2$. Flow eqs. yield

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From which we obtain that $h = \mathrm{Re}(\mathcal{I})$ decreases and $\mathrm{Im}(\mathcal{I})$ is conserved along a flow.

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$$X_{-T} = \{p \in \mathbb{C}^n : h < -T\}$$

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For each $\alpha \in S$, $p_\alpha \mapsto \Gamma_\alpha$ which gives us a basis of $H_n(\mathbb{C}^n, X_{-T})$. Any reasonable integration cycle Γ will be given by

$$\Gamma = \sum_{\alpha \in S} n_\alpha \Gamma_\alpha \longrightarrow I(k) = \sum_{\alpha \in S} n_\alpha I_\alpha(k)$$

How to compute n_α?

Replace downward flow eqs. by upward flow equations

$$\frac{dx^i}{dt} = \frac{\partial \overline{\mathcal{I}}}{\partial \overline{x^i}}, \quad \frac{d\overline{x^i}}{dt} = \frac{\partial \mathcal{I}}{\partial x^i}$$

on the half-line $(-\infty, 0]$ with same boundary condition. Obtain $\mathcal{D}_\alpha$ upward flowing cycle associated to p_α.

There exists natural pairing $\langle \Gamma_\alpha, \mathcal{D}_\beta \rangle = \delta_{\alpha\beta} \implies n_\alpha = \langle \Gamma, \mathcal{D}_\alpha \rangle$.

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A Riemmanian metric on W induces a Kahler metric on $\mathcal{U}_\mathbb{C}$ that is invariant under G by

$$|\delta\mathcal{A}|^2 = -\int_W \mathrm{Tr}\delta\mathcal{A} \wedge \star_W \delta\overline{\mathcal{A}}$$

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Flow will be a differential equation on $M = W \times \mathbb{R}$.

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Moment map for the G-valued local gauge transformations

$$\mu = d_A \star_W \phi$$

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We will be interested only on Lefschetz thimbles on which $\mu = 0$.

KW equations

Finally, we apply the flow equation and simplify to obtain

$$\boxed{\begin{aligned} F - \phi \wedge \phi &= \star_M d_A \phi \\ d_A \star_M \phi &= 0 \end{aligned}}$$

Equations for a pair A, ϕ with A a real connection on G-bundle $E \to M$ and $\phi \in \Omega^1(M, \mathrm{ad}(E))$.

Nonetheless, these may continue to be viewed as flow equations for $\mathcal{A}$ on W.

We can now define the Lefschetz thimble for any choice of flat connection $\mathcal{A}_\rho$ (our "critical point") on $M = W \times \mathbb{R}^+$ associated to homomorphism $\rho : \pi_1(M) \to G_\mathbb{C}$.

Γ_ρ consists of all $\mathcal{A}$ that are boundary values (on $W \times \{0\} \subset M$) of solutions of the KW eqs. on M which approach $\mathcal{A}_\rho$ at infinity.

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Since $\pi_1(\mathbb{R}^3) = 0$ any flat $\mathcal{A}$ on $\mathbb{R}^3$ is gauge-equivalent to the trivial one $\implies \exists! \, \Gamma_0$.

The Jones polynomial then is

$$\boxed{Z_k(\mathbb{R}^3, K, R) = \frac{1}{\mathrm{vol}} \int_{\Gamma_0} \mathcal{D}\mathcal{A} \exp\left(ik\mathrm{CS}(\mathcal{A})\right) \mathcal{W}_R(K)}$$

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Here $\mathcal{W}_R(K)$ is evaluated on Γ_0.

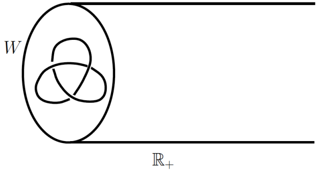

Figure: Knot embedded in boundary of M

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KW equations also arise in a twisted version of $\mathcal{N} = 4$ super Yang-Mills theory in 4D which localizes on the space of solutions of the KW eqs.

This space, if we require $\mathcal{A} \to 0$ at $\mathbb{R}^3 \times \{\infty\}$, is simply Γ_0.

$\implies$ Jones polynomial for K in $\mathbb{R}^3$ can be computed by a path integral of $\mathcal{N} = 4$ super Yang-Mills on M with a certain boundary condition on $\mathbb{R}^3 \times \{0\}$.

What can be said of this formula for the Jones polynomial?

$$\boxed{Z_k(\mathbb{R}^3, K, R) = \frac{1}{\text{vol}} \int_{\Gamma_0} \mathcal{D}\mathcal{A} \exp\left(ik\text{CS}(\mathcal{A})\right) \mathcal{W}_R(K)}$$

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$\Longrightarrow$ Jones polynomial for K in $\mathbb{R}^3$ can be computed by a path integral of $\mathcal{N} = 4$ super Yang-Mills on M with a certain boundary condition on $\mathbb{R}^3 \times \{0\}$.

However, this is still an infinite-dimensional integration.

Electric-magnetic duality

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory with gauge group G and coupling parameter τ

$$\Longleftrightarrow$$

Same theory with gauge group G^V (Langlands or GNO dual of G) and coupling parameter $\tau^V = -1/n_{\mathfrak{g}}\tau$.

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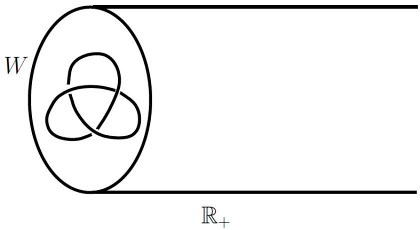
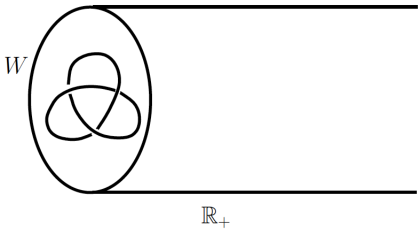
Dual boundary condition described by Gaiotto and Witten – has the effect of reducing to finite-dimensional spaces of solutions of the KW eqs.



In this case, after the duality transformation, the moduli space of solutions has dimension 0.

To evaluate $Z_k(K, R)$, count the number b_n, with signs, of solutions for a given value n of the instanton number (= second Chern class).



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Jones polynomial is

$$\boxed{Z_q(K, R) = \sum_n b_n q^n}$$

with $q = \exp(2\pi i/(k+2))$.

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