Examples of explicit solutions to the cubic wave equation

by Giuseppe Negro

joint work with Thomas Duyckaerts (Sorbonne Paris Nord)

Instituto Superior Técnico, Lisbon

November 17th, 2022

Asymptotic behaviour. Basic facts

Cauchy initial value problem for the cubic wave equation

$$\partial_t^2 u - \Delta u = u^3, \qquad t \in I, x \in \mathbb{R}^3,$$
$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 = (u_0, \dot{u}_0) \quad \textit{(initial data)}.$$

Notation: $\boldsymbol{u}(t) = (u(t,\cdot), \partial_t u(t,\cdot)), \quad \mathcal{H}^s = \dot{H}^s \times \dot{H}^{s-1}.$

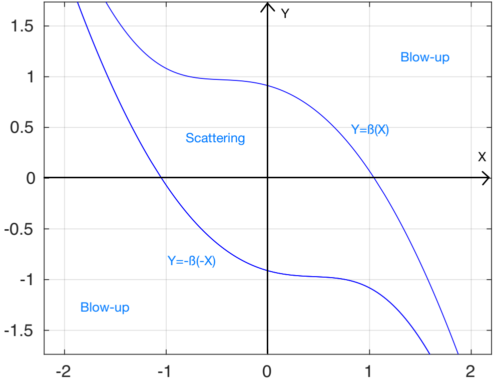
- $I \subseteq \mathbb{R}$ maximal time of existence. ***Blow-up*** occurs $\iff I \subsetneq \mathbb{R}$.
  Example: $u = \sqrt{2}(T \pm t)^{-1}$ for $T \geqslant 0$.
- $E = \frac{1}{2}\|\boldsymbol{u}(t)\|^2_{\mathcal{H}^1(\mathbb{R}^3)} - \frac{1}{4}\|u(t,\cdot)\|^4_{L^4(\mathbb{R}^3)}$ (*conserved energy*).
- L.W.P. for $\boldsymbol{u}_0 \in \mathcal{H}^s$ with $s \geqslant \frac{1}{2}$ ($s = \frac{1}{2}$ is *scaling-critical*).
- If $\|\boldsymbol{u}_0\|_{\mathcal{H}^{1/2}}$ is small, then $I = \mathbb{R}$ and u is asymptotic to a linear solution at $t \to \pm\infty$ (***scattering***).
  Those $\boldsymbol{u}_0$ that scatter are an open set in $\mathcal{H}^{1/2}$ (*scattering is stable*).
- (Dodson-Lawrie 2015) If $\|\boldsymbol{u}(t)\|_{\mathcal{H}^{1/2}} < C$ for all $t \in I$, then $I = \mathbb{R}$ and u scatters.

The new result in pictures. Future(+) times

There is a *threshold function* $\beta\colon \mathbb{R} \to \mathbb{R}$ such that, letting

$$\boldsymbol{u}_0^{X,Y}(x) \cong \left(\frac{X}{1+|x|^2}, \frac{Y}{(1+|x|^2)^2} \right), \qquad \text{for } (X,Y) \in \mathbb{R}^2,$$

we have the pictured behaviours for future times $t \geqslant 0$:



At the threshold $Y = \pm\beta(\pm X)$: **non-scattering, (+)-global** ($[0,\infty) \subseteq I$).

The complete picture. Future(+) & past(-)

Remark 1. $\sqrt{X^2 + Y^2} = \|\boldsymbol{u}_0^{X,Y}\|_{\mathcal{H}^{1/2}}$.

Remark 2. All our solutions are finite energy.

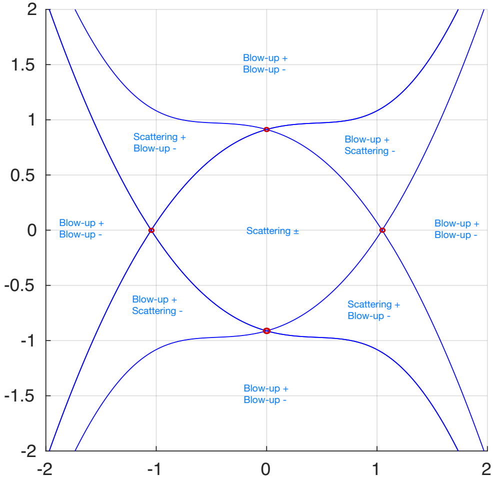


Figure: 9 behaviours. Red dots: $(\pm)$-**global**, **non-scattering** solutions.

The global, non-scattering solutions at the threshold

In this talk, we will focus on the threshold solutions only.

Let $u = u^{X,Y}$ be a threshold solution for $t \to +\infty$.

Inside light cone - ODE:

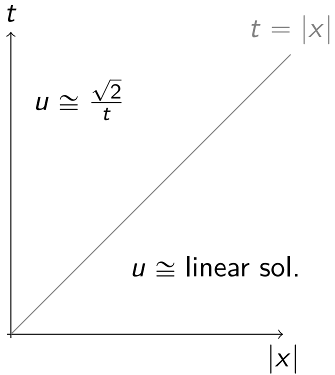
- $u = \frac{\sqrt{2}}{t} + O(t^{-3})$.
- $\left\| u(t,x) - \frac{\sqrt{2}}{t}\mathbf{1}_{|x|\leqslant t} \right\|_{L^p_x(\mathbb{R}^3)} \lesssim t^{\frac{2}{p}-1}$, $p > \frac{3}{2}$.

Outside light cone - Scattering:

- $\| \boldsymbol{u}(t) - \boldsymbol{v}_L(t) \|_{\mathcal{H}^1(|x|>t)} \to 0$, where v_L solves $\partial_t^2 v_L = \Delta v_L$.

Grow-up at ∞ (recall Dodson–Lawrie):

- $\| \boldsymbol{u}(t) \|^2_{\mathcal{H}^{1/2}} \geqslant C \log t + O(\sqrt{\log t})$.



Comparison with known results (only for $\partial_t^2 u - \Delta u = u^3$ on $\mathbb{R}^{1+3}$)

Bizoń–Zenginoğlu's 2009 conjecture (numerical)

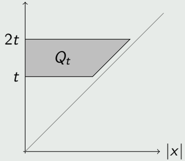
The general threshold between scattering and blowup should be given by codimension-1 global solutions u such that, (up to symmetries)

$$u = \frac{\sqrt{2}}{t} + O(t^{-4}), \qquad t \to \infty.$$

Donninger–Zenginoğlu 2014

There is a codimension-4 manifold of global non-scattering u such that

$$\left\| u - \frac{\sqrt{2}}{t} \right\|_{L^4(Q_t)} = O(t^{-\frac{1}{2}+\epsilon}).$$



Remark 1. DZ solutions: not finite energy, other Initial Value Problem.
Remark 2. Our solutions: first theoretical example of global, non-scattering solutions for Cauchy IVP (to the best of our knowledge).

Main ingredient of proof: conformal invariance

Recall. *Conformal* =*(Lorentzian) angle-preserving*.
Consider a conformal coordinate change $\mathcal{P}$ with factor Ω, i.e.:

$$(\tilde{t}, \tilde{x}) = \mathcal{P}(t, x), \quad \Omega = |\det D\mathcal{P}|^{\frac{1}{4}}, \quad \mathcal{P} \colon \mathbb{R}^{1+3} \to \mathbb{R}^{1+3}.$$

Fundamental property:

Conformal change of the D'Alembert operator

$$(\partial_t^2 - \Delta)u = \Omega^3(\partial_{\tilde{t}}^2 - \tilde{\Delta})(\Omega^{-1}u).$$

So letting $\tilde{u}(\tilde{t}, \tilde{x}) := (\Omega^{-1}u)(t, x)$, we have $u^3 = \Omega^3\tilde{u}^3$ and

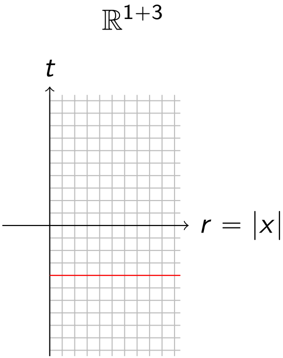
$$(\partial_t^2 - \Delta)u = u^3 \iff \Omega^3(\partial_{\tilde{t}}^2 - \tilde{\Delta})\tilde{u} = \Omega^3\tilde{u}^3.$$

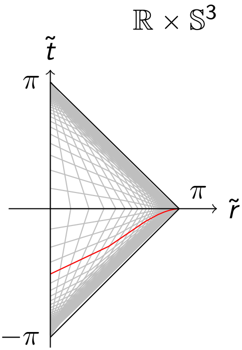
This can be done with manifold-valued $\mathcal{P} \colon \mathbb{R}^{1+3} \to \mathbb{R} \times M^3$, too.
We will have $M^3 = \mathbb{S}^3$. Thus the D'Alembertian is $\partial_{\tilde{t}}^2 - \Delta_{\mathbb{S}^3} + 1$.

Remark

Cubic wave equation = conformal wave equation.

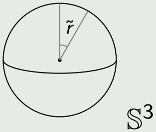
The Penrose map $\mathcal{P}\colon \mathbb{R}^{1+3} \to \mathbb{R} \times \mathbb{S}^3$



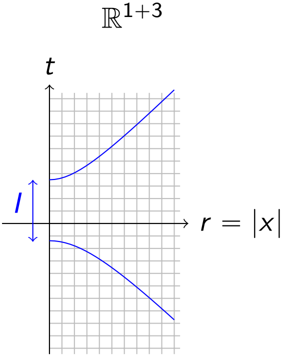


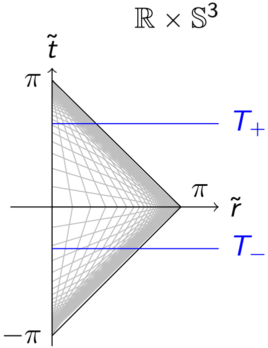
The map $(\tilde{t}, \tilde{r}) = \mathcal{P}(t, r)$ and the definition of $\tilde{r}$

$$\tilde{t} = \arctan(t + r) + \arctan(t - r),$$
$$\tilde{r} = \arctan(t + r) - \arctan(t - r).$$



Constructing our solutions





Recall: $\tilde{u} = \Omega^{-1} u$. Ansatz: $\tilde{u} = \tilde{u}(\tilde{t})$. We get an ODE:

$$\partial_t^2 u - \Delta u = u^3 \iff \tilde{u}'' + \tilde{u} = \tilde{u}^3.$$

Let (T_-, T_+) ODE time of existence of $\tilde{u}$. Then u exists for $t \in I$ (picture).

Behaviour of u as $t \to +\infty$

$T_+ < \pi$: blows-up. $T_+ > \pi$: scatters. $T_+ = \pi$: threshold.