Completeness and linearization

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1. Complete connections

1.1 Ehresmann connections

An **Ehresmann connection** on a surjective submersion $p : E \to M$ is a smooth distribution $H \subset TE$ such that

$$TE = H \oplus \ker dp$$

For every p there always exists an Ehresmann connection H

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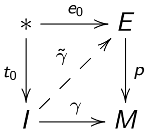
Fixed H, given $t_0 \in I \subset \mathbb{R}$, $\gamma : I \to M$ a curve and $e_0 \in E$ such that $p(e) = \gamma(t_0)$, there exists a locally defined **horizontal lift**

$$\tilde{\gamma} : J \to E,\ \tilde{\gamma}(t_0) = e_0,\ p\tilde{\gamma} = \gamma,\ \tilde{\gamma}'(t) \in H$$



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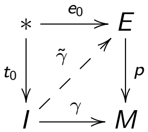

A connection H is **complete** if the horizontal lift is defined in the whole I

1.2 The proper case

Lemma

If $p : E \to M$ admits a complete connection then it is locally trivial.

Proof.

Parallel transport along radial curves on a ball $x \in B \subset M$. □

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- Preimage of compact sets are compact (*)
- Every base-change is closed
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Theorem [Ehresmann, 1950]

A proper submersion is locally trivial.

Proof.

If $p : E \to M$ is **proper** then every connection H is complete. □

1.3 A tricky exercise

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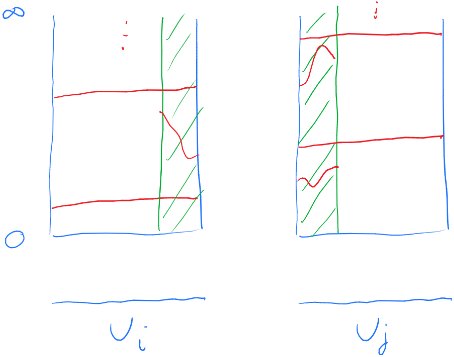
A shory history of the result:

- First appeared in [Wolf 1964] with first problematic proof
- Exercise in [Greub, Halperin, Vanstone 1972] without a proof
- Second problematic proof published in [Michor 1988, 1991, 2008] and [Kolar, Michor, Slovak 1993] is attributed to Halperin
  Complete connections are not closed under combex combinations
- Definite proof in [dH 2016]
- Generalization to Lie algebroid submersions [Frejlich 2019]

1.4 The proof



Passing from local to global by convex combination via partition of 1

1.5 The metric approach

A **Riemannian submersion** $p : (E, \eta_E) \to (M, \eta_M)$ is a surjective submersion such that $dp_e : (\ker d_e p)^\perp \to T_{p(e)}M$ is isometry $\forall e$.

$$p : (E, \eta_E) \to (M, \eta_M) \text{ Riemm. subm.} \iff \begin{cases} \eta_E \ p\text{-fibered} \\ \eta_M = p_*(\eta_E) \end{cases}$$

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- Every manifold admits a complete metric: $\tilde{\eta}_x=\frac{1}{d(x,\infty)}\eta_x$
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Theorem

A surjective submersion $p : E \to B$ admits a complete and fibered metric if and only if it is locally trivial.

2. Riemannian stacks

2.1 Groupoids and stacks

A **Lie groupoid** $G \rightrightarrows M$ consists of manifolds of objects M and arrows G, submersions $s, t : G \to M$ and a **multiplication** with unit and inverse

$$m : G \times_M G \to G \quad (z \xleftarrow{g_2} y, y \xleftarrow{g_1} x) \mapsto (z \xleftarrow{g_2 g_1} x)$$

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- **isotropy** groups: $G_x = \{x \leftarrow x\}$ are Lie groups
- **orbits**: $O_x = \{y | \exists y \leftarrow x\} \subset M$ define singular foliation
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Example

$E \to M$ surj subm	$E \times_M E \rightrightarrows E$	$[E/E \times_M E] = M$
$K \curvearrowright M$ group action	$K \times M \rightrightarrows M$	$[M/K \times M]$ orbit space
$F \subset TM$ foliation	$Hol(F) \rightrightarrows M$	$[M/Hol(F)]$ leaf space

2.2 Riemannian groupoids and stacks

A **Riemannian groupoid*** is $(G \rightrightarrows M, \eta)$, $\eta^{(i)}$ metric on $G^{(i)}$ such that the following are Riemannian submersions [dH, Fernandes 2018]

$$G^{(2)} \overset{\pi_1}{\underset{\pi_2}{\overset{m}{\rightrightarrows}}} G \overset{s}{\underset{t}{\rightrightarrows}} M$$

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Example

- Riemannian manifolds and orbifolds [Thurston 1980]
- orbit spaces of isometric actions [Alekseevsky, Michor et al 2003]
- leaf spaces of Riem. foliations* [Alexandrino, Briquet, Toben 2013]

2.3 Curves on Riemannian stacks

A **stacky curve** $\alpha : I \to [M/G]$ is given by a *cocycle* of curves

$$(I \rightrightarrows I) \leftarrow (\coprod I_{n+1} \cap I_n \rightrightarrows \coprod I_n) \xrightarrow{\coprod a_n} (G \rightrightarrows M)$$

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Theorem (dH, de Melo 2020)

Given $(G \rightrightarrows M, \eta)$ Riemannian gpd and d_N induced distance in M/G,

$$d_N([x],[y]) = \inf \left\{ \int_I ||\alpha'(t)|| dt : \ \alpha : I \to [M/G],\ \alpha(0) = [x],\ \alpha(1) = [y] \right\}$$

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- Local existence and local uniqueness hold easily
- Global uniqueness only if groupoid is **proper**
  Counter-example: $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$
- Minimizing geodesics stay in a same stratum: $G_{\alpha(t_0)} \cong G_{\alpha(t_1)}\ \forall t_0, t_1$

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Theorem (dH, de Melo 2020)

A stacky curve $\alpha : I \to [M/G]$ is a geodesic if and only if it is locally minimizing at every $t \in I$

2.5 Completeness of stacky metrics

A Riemannian stack $[M/G]$ is **geodesically complete** if geodesics can be extended to the whole real line.

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Every Lie groupoid $G \rightrightarrows M$ admits a groupoid metric
but not every Lie groupoid admits a complete groupoid metric!
Mistake in [Pflaum, Posthuma, Tang 2014]

$$(G \rightrightarrows M) \quad = \quad \begin{array}{c} M \\ \downarrow \\ [M/G] \end{array}$$

3. Invariant linearization

3.1 Linearization of groupoids

Given $G \rightrightarrows M$ a Lie groupoid and $S \subset M$ invariant submanifold, there is a **linear model** for G around S:

$$0 \to (TG_S \rightrightarrows TS) \to (TG|_{G_S} \rightrightarrows TM|_S) \to (NG_S \rightrightarrows NS) \to 0$$

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Theorem (dH, Fernandes 2018)

A proper groupoid is linearizable by exponential flows of groupoid metric.

- Simpler proof and stronger version of Linearization Theorem [Weinstein 2001], [Zung 2004], [Crainic, Struchiner 2011]
- Generalizes Ehresmann Thm for submersions, Tube Thm for actions, Reeb stability for foliations.

3.2 The open problem

A linearization is **invariant** if we can take U, V invariant opens (stability)

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Example

Proper actions of Lie groups $K \curvearrowright M$ (or their corresponding groupoids) can are invariantly linearizable. It doesn't follows from groupoid result!

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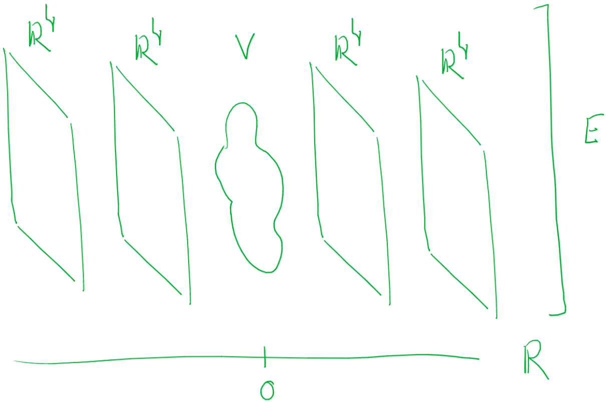
Question [Crainic, Struchiner 2011]

When a proper groupoid $G \rightrightarrows M$ is invariantly linearizable?

Conjecture [Crainic, Struchiner 2011]

When the source map $s : G \to M$ is trivial on an invariant nbh.

3.3 A counter-example [dH, de Melo 2021]



based on [Meigniez 1995, 2002] and [Weinstein 2002]

$$(G \rightrightarrows M) = (E \times_{\mathbb{R}} E \rightrightarrows E) \times (SU(2) \rightrightarrows *)$$

3.4 Sufficient condition

$p : E \to M$ Riemannian submersion, $S \subset M$, $S' = p^{-1}(S)$. Then:

- $\exp|_U$ étale $\Rightarrow$ $\exp|_{U'}$ étale ($\Leftarrow$ if $dp(U') = U$)
- $\exp|_U$ injective $\Rightarrow$ $\exp|_{U'}$ injective ($\Leftarrow$ if η^E complete and $U' = dp^{-1}(U)$)

$$\begin{array}{ccc} NS' \supset U' & \xrightarrow{\exp} & E \\ \big\downarrow{\scriptstyle dp} & & \big\downarrow{\scriptstyle p} \\ NS \supset U & \xrightarrow[\exp]{} & M \end{array}$$

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Theorem (dH, de Melo 2021)

If $(G \rightrightarrows M, \eta)$ is a proper Riemannian groupoid and $\eta^{(0)}$ is complete then it is invariantly linearizable.

- Proof uses geodesics on Riemannian stacks [dH, de Melo 2020]
- It includes the linearization of group actions as a particular case!
- Similar results for SRF appeared in [Mendes, Radeschi 2019] and [Alexandrino, Inagaki, de Melo, Struchiner 2022]

3.5 About necessity

Theorem (dH, de Melo 2021)

Let $G \rightrightarrows M$ be a proper groupoid that is invariantly linearizable around its orbits. Then it admits a complete 0-metric $\eta^{(0)}$ on $G^{(0)} = M$.

- Proof uses geodesics on Riemannian stacks [dH, de Melo 2020]
- The 0-metric $\eta^{(0)}$ may a priori not extend to a groupoid metric η!
- They do extend (and give a complete characterization of invariant linearization) for regular groupoids.

Obrigado!