A Yau--Tian--Donaldson correspondance on a class of toric fibrations

σ-Calabi Pbm:

(X, ω, J)

$\hookrightarrow$ smooth comp

- $J \in \mathcal{C}^\infty(\mathrm{End}(TX))$ / $J^2 = -\mathrm{Id}$ + int. cond.
- $\omega \in \Omega^2(X)$, $d\omega = 0$, ω non-degenerate
- $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$

$g := \omega(\cdot, J\cdot) > 0$

$[\omega_0] \in H^2(X, \mathbb{R})$

$\rightsquigarrow$ Kähler class

$\omega_\varphi := \omega_0 + dd^c\varphi > 0$

def° (Calabi 80')

ω is extremal if

$$\mathcal{L}_{\omega^{-1}(d\,\mathrm{Scal}(\omega))} J = 0$$

$$\mathrm{Scal}(\omega_\varphi) = c$$

Pbm (Calabi) Given $\alpha := [\omega_0]$,

$\exists ?\ \omega \in \alpha$ extremal.

ex:

1) $\mathbb{P}(\mathcal{O} \oplus \ldots) \longrightarrow \mathbb{P}^1$

1) $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}) \to \mathbb{P}^1$

does not admit cscK

but $\exists\, \omega \in \alpha$ extremal $\forall \alpha$.

2) $\dim_{\mathbb{C}} X = 1$ (Thm of unif° of Riemann)

TF

3) $\mathbb{P}(\mathbb{C} \oplus L) \xrightarrow{\pi} B_g,\ g \geq 2$.

$$\alpha_c := c_1(\mathcal{O}(1)) + c\, \pi^* \alpha_B$$

$\exists\, c_0 \mid \alpha_c$ does not admit extremal for $c \leq c_0$

and α_c admits extremal for $c > c_0$.

YTD Conjecture:

$\exists\, \omega \in \alpha$ extremal

$$\Updownarrow$$

α is "stable"

Thm (J. 2021)

(Y, J_Y, ω_Y) a semisimple principal toric fibration.
Then $\exists\, \omega \in [\omega_Y]$ extremal

$$\Updownarrow$$

The polytope Δ of the fiber is "stable"

II - Weighted toric geometry

(X, J, ω, T), $\dim_{\mathbb{C}} X = \dim_{\mathbb{R}} T$

$\forall\, \omega \in [\omega_0]$ T-inv.

$\mu_\omega : (X, \omega) \longrightarrow \Delta \subset \mathfrak{t}^*$

$\mathfrak{t} = \mathrm{Lie}(T)$

$$\mathcal{S}_{symp} = \{ u \in \mathcal{C}^\infty(\mathring{\Delta}) \cap \mathcal{C}^0(\Delta) \text{ convex + bdd cond}^{\underline{c}} \}$$

Abreu -- Guillemin

" $u \longleftrightarrow \omega_u \in [\omega_0]$ "
T- inv.

Thm (Abreu) ξ basis of $\mathfrak{t}$.

$$\mathrm{Scal}(\omega_u) = -\sum_{i,j=1}^{\ell} (H_{ij})_{,ij}$$

$$H_{ij} = \mathrm{Hess}^{-1}(u)$$

A Kähler metric ω is

(v,w)-cscK iff (if)

extremal

$$-\underbrace{\sum_{i,j=1}^{\ell} (V H_{ij})_{,ij}}_{\mathrm{Scal}_V(u_u)} = \overbrace{c_\Delta + \langle \xi_\Delta , x \rangle}^{\ell_{ext}} = w$$

$V > 0$ smooth

W smooth

$x_0 \in \mathring{\Delta}$

def $\underline{\circ}$

Δ is (v,w)-uniformly K-stable if $\exists\, c > 0 \,|$

$\forall\, f \in \mathcal{C}^\infty(\mathring{\Delta}) \cap \mathcal{C}^0(\Delta)$ convex

$$F_{v,w}(f) \geq \int_\Delta f^* \, dx \,,$$

- $F_{v,w}(f) := 2\int_{\partial\Delta} f v \, d\sigma - \int_\Delta w f \, dx$
- $f^*(x) \geq f(x_0) = 0$

$\exists\, w_u \in [w_0]$ (v,w)-csck

$F(f) = \quad \forall f$

$\Longrightarrow \quad H_{\text{vertical}}(f) = 0 \ldots$

affine-linear

III - Semisimple principal toric fibrations:

$A < GTF$.

- (X, J_X, ω_X, T)
  Δ Delzant polytope
- $\pi : P \longrightarrow \prod_{a=1}^{k} (B_a, J_a, \omega_a)$

principal T-bundle

- ω_a csc K
- $\exists\, \theta \in \Omega^1(P) \otimes \mathfrak{t}$ |

$$\boxed{d\theta = \sum^{k} p_a \otimes \pi^* \omega_a}$$

$a=1$

$p_a \in \Lambda$, $T = E/\Lambda$.

<u>Rk</u>: $k=1$ $\leadsto$ basis of t

$$T = \prod_{i=1}^{\ell} S^1_i$$

$$P \longrightarrow (B_1, J_1, \omega_1)$$

$$\updownarrow$$

$$\bigoplus_{i=1}^{\ell} L_i \longrightarrow (B_1, J_1, \omega_1)$$

$$-c_1(L_i) = p_i [\omega_1]$$

<u>ex</u>: 1) B is Riemann surface

$k=1$

2) $L_i \longrightarrow B_i$

$$\bigoplus^{\ell} L_i \longrightarrow \prod^{\ell} B$$

$$\bigoplus_{i=1} L_i \longrightarrow \prod_{i=1} D_i$$

$c_1(L_i)$ admits a cock.

- $Y := X \times_T P \longrightarrow (B, J_B, \omega_B)$
  - $J_Y := J_X \oplus J_B$
  - $T \curvearrowright (Y, J_Y)$

<u>on $X \times P$</u>:

$$\omega_Y := \omega_X + \langle d\mu \wedge \theta \rangle_{\mathfrak{t}, \mathfrak{t}^*} + \sum_{a=1}^{k} \underbrace{\left(\langle p_a, \mu \rangle + c_a \right)}_{>0} \pi^* \omega_a$$

<u>def</u>: ω_Y is called a compatible Kähler metric

- $[\omega_Y]$ //

class

Thm (J. 21')

(Y, ω_Y, J_Y). TFSAE

1) $\exists\ \tilde{\omega} \in [\omega_Y]$ extremal.

2) $\exists\ \omega \in [\omega_X]$ (v,w)-csck.

$$v = \prod_{a=1}^{k} \left(\langle p_a, x \rangle + c_a \right)$$

$$w = v \left(\ell_{ext} - \sum \frac{\mathrm{Scal}(\omega_a)}{\langle p_a, x \rangle + c_a} \right)$$

3) Δ is (v,w)-uniformly k-stable.

Corollary (J. 21)

$\mathbb{P}(L_1 \oplus L_2 \oplus L_3) \longrightarrow \left| \mathbb{CP}^1 \right.$ elliptic curve]

admits an extremal metric in each Kähler class.

$2) \Rightarrow 1)$ ACGT.

$2) \Rightarrow 3)$ Li--Lian--Sheng

$\left.\begin{array}{l} 1) \Rightarrow 2) \\ 3) \Rightarrow 2) \end{array}\right\}$ Remains.

<u>Strategy for $1) \Rightarrow 2)$:</u>

$Aut(B, J) = \{1\}$

$T \underset{max}{\subset} Aut(Y)$.

$K(Y)^T := \{ \tilde{\omega} \in [\omega_Y]$ Kähler T-inv. $\}$

Thm: (Chen--Cheng, He / Berman--Darvas--Lu)

$\exists\ \tilde{u} \in [\omega_Y]$ extremal

$\Updownarrow$

$\exists\ C, D > 0 \mid \forall\ \tilde{u} \in \mathcal{K}(Y)^T$

$$M^T(\tilde{u}) \geq C \inf_{\gamma \in T^{\mathbb{C}}} J_{\omega_Y}(\gamma^* \tilde{u}) - D$$

$J_{\omega_Y}(\tilde{u}) \geq 0, \quad J_{\omega_Y}(\tilde{u}) = 0$

$\Leftrightarrow \omega_Y = \tilde{u}.$

<u>Continuity path of Chen:</u>

$t \in [0,1]$ $\qquad \tilde{u} \in \mathcal{K}(Y)^T$

$t\,(S\ \ell(\tilde{u}) \ldots$

$$t\left(\mathrm{Scal}(u) - e_{sat}\right)$$

$$\parallel$$

$$(1-t)\left(\Lambda_{\tilde{u}}(\tilde{x}) - n\right)$$

$$\tilde{x} \in [u_Y]$$

$$K(X)^T = \{\, u \in [u_X] \;\; T\text{-inv.} \,\}$$

$$K(X)^T \subset K(Y)^T$$

$$t\left(\mathrm{Scal}_Y(u) - e_{sat}\right)$$

$$\parallel$$

$$(1-t)\left(\Lambda^Y_X(u) - n\right)$$