The large-N limit of the Segal–Bargmann transform on unitary groups

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Credits for large-N limit

- Joint work with **Bruce Driver** and **Todd Kemp** of UCSD [*J. Funct. Anal.*, 2013].
- Expository article: arXiv:1308.0615.
- Results motivated by work of **Philippe Biane** [Segal-Bargmann transform, functional calculus, . . . , *J. Funct. Anal.*, 1997]
- Related results obtained independently by **Guillaume Cébron**: [*J. Funct. Anal.*, 2013]

Geometry of the unitary groups

- $U(N)$ = group of $N \times N$ unitary matrices ($U^*U = I$)
- Lie algebra = tangent space at $I = u(N)$
- $u(N)$ = space of $N \times N$ skew matrices ($X^* = -X$)
- Use on $u(N)$ some multiple of real Hilbert–Schmidt inner product:

$$\langle X, Y \rangle = C\,\mathrm{Re}[\mathrm{Trace}(X^*Y)], \quad C > 0.$$

- This inner product determines a bi-invariant Riemannian metric on $U(N)$
- Metric determines Laplacian Δ (with $\Delta \leq 0$)

Heat equation

- Study **heat equation**

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$$

- Introduce **heat kernel** ρ_t (based at I) on $U(N)$:

$$\begin{aligned} \frac{\partial \rho_t}{\partial t} &= \frac{1}{2}\Delta \rho_t \\ \lim_{t\to 0} \rho_t &= \delta_I \end{aligned}$$

- Introduce **heat operator** $e^{t\Delta/2}$:

$$(e^{t\Delta/2} f)(U) = \int_{U(N)} \rho_t(UV^{-1}) f(V)\ d\mathrm{vol}(V)$$

Segal–Bargmann transform

- $GL(N;\mathbb{C})$ = group of all $N \times N$ invertible matrices
- $GL(N;\mathbb{C})$ is "complexification" of $U(N)$ (complex manifold)

Theorem

For any fixed $t > 0$ and any reasonable function f on $U(N)$, the function

$$e^{t\Delta/2}f$$

admits a (unique) holomorphic extension from $U(N)$ to $GL(N;\mathbb{C})$.

Definition

The **Segal–Bargmann transform** for $U(N)$ is the map $B_t : L^2(U(N), \rho_t) \to \mathcal{H}(GL(N;\mathbb{C}))$ given by

$$B_t(f) = (e^{t\Delta/2}f)_{\mathbb{C}},$$

where $(\cdot)_{\mathbb{C}}$ denotes holomorphic extension.

Segal–Bargmann transform, cont'd

- Natural **heat kernel** μ_t on $GL(N;\mathbb{C})$ (based at I)
- $\mathcal{H}L^2(GL(N;\mathbb{C}),\mu_t)$ = space of **square-integrable holomorphic functions** with respect to the measure $\mu_t(Z)\ d\mathrm{vol}(Z)$

Theorem (H, '94)

For each $t>0$, the map B_t is a unitary map of $L^2(U(N),\rho_t)$ onto $\mathcal{H}L^2(GL(N;\mathbb{C}),\mu_t)$

- Same construction for $\mathbb{R}^n \subset \mathbb{C}^n$ yields the "classical" Segal–Bargmann transform

Large-N limit, first attempt

- Use **fixed** multiple of Hilbert–Schmidt inner product on Lie algebras
- Then inclusion of $u(N)$ into $u(N+1)$ is isometric
- Results of M. Gordina show that Segal–Bargmann transform **does not** have a reasonable limit with this approach
- Gordina shows (essentially) that in the large-N limit, all nonconstant functions in $\mathcal{H}L^2(GL(N;\mathbb{C}),\mu_t)$ have infinite norm
- E.g., the function $F(Z) = Z_{jk}$ has norm

$$\|Z_{jk}\|^2_{L^2(GL(N;\mathbb{C}),\mu_t)} = (1 - e^{-t})\frac{e^{Nt}}{N} \to \infty$$

Large-N limit, second attempt

- Use **scaled** Hilbert–Schmidt inner product on $u(N)$:

$$\langle X, Y \rangle_N := N \langle X, Y \rangle_{\mathrm{HS}} = N \,\mathrm{Re}[\mathrm{Trace}(X^* Y)]$$

- The associated Laplacian is then

$$\Delta_N = \frac{1}{N} \Delta_{\mathrm{HS}}$$

- This scaling was proposed by Biane.
- Consider, for example, the function $f(U) = U_{jk}$ on $U(N)$. Then

$$\Delta_N(U_{jk}) = -U_{jk}$$

(Eigenvalue is -1, **independent of N**.)

Large-N behavior of Laplacian

- Consider normalized trace,

$$\mathrm{tr}(U) = \frac{1}{N}\mathrm{Trace}(U) = \frac{1}{N}\sum_{j=1}^{N} U_{jj}.$$

- Now consider **trace polynomials**, i.e., polynomials in traces of powers of U. E.g.

$$f(U) = 7\mathrm{tr}(U^2)\mathrm{tr}(U^3) - (\mathrm{tr}(U^2))^3.$$

- The action of Δ_N on trace polynomials decomposes as:

$$\Delta_N = \Delta_\infty + \frac{1}{N^2}L$$

for operators "Δ_∞" and "L" whose actions are **independent of** N.

Large-N behavior of Laplacian, cont'd

- Action of Δ_∞ (on trace polynomials) determined by two basic properties:
- First,

$$\Delta_\infty[\mathrm{tr}(U^k)] = -k\mathrm{tr}(U^k) - 2\sum_{j=1}^{k-1} j\mathrm{tr}(U^j)\mathrm{tr}(U^{k-j}).$$

- Second, Δ_∞ satisfies the **first-order product rule**:

$$\Delta_\infty(fg) = \Delta_\infty(f)g + f(\Delta_\infty g).$$

- Thus: cross terms in product rule for Δ_N are of order $1/N^2$.

Action as a vector field

- Let v_k denote the function

$$v_k = \mathrm{tr}(U^k), \quad k = 1, 2, 3, \ldots$$

- Any finite set of these functions is algebraically independent for sufficiently large N
- Action of Δ_∞ in these variables is given as a **first-order differential operator**:

$$-\sum_{k=1}^{\infty} k v_k \frac{\partial}{\partial v_k} - 2\sum_{k=2}^{\infty}\left(\sum_{j=1}^{k-1} j v_j v_{k-j}\right)\frac{\partial}{\partial v_k}$$

Concentration properties of heat kernels

- Look at **heat kernel measure**

$$d\rho_t^N := \rho_t(U)\ d\mathrm{vol}(U)$$

- Work of Biane, E. Rains, and T. Kemp shows (roughly) that the heat kernel measure on $U(N)$ **concentrates onto a single conjugacy class** in the large-N limit.
- Conjugacy classes in $U(N)$ are determined by list of eigenvalues.
- List $\{\lambda_1, \ldots, \lambda_N\}$ of eigenvalues for U can be encoded in the **empirical eigenvalue measure**

$$\mu_U = \frac{1}{N}(\delta_{\lambda_1} + \cdots + \delta_{\lambda_N})$$

- In the large-N limit, almost every U (with respect to ρ_t^N) has the **same empirical eigenvalue measure**, given by a certain fixed measure ν_t described by Biane (picture).

Concentration properties of heat kernels, cont'd

- As ρ_t^N concentrates, **all class functions become constant** (as elements of $L^2(U(N), \rho_t^N)$).
- E.g., normalized trace:

$$\left\| \mathrm{tr}(U) - e^{-t/2} \right\|_{L^2(U(N),\rho_t^N)} \longrightarrow 0$$

  as $N \longrightarrow \infty$.
- That is, ρ_t^N is concentrating onto the set where $\mathrm{tr}(U)$ has the constant value $e^{-t/2}$

Concentration properties of heat kernels, cont'd

- Concentration properties are closely related to the **first-order product rule** for Δ_∞.
- If Δ_∞ behaves like a first-order operator, then heat doesn't diffuse.
- Similar concentration results for trace polynomials on $GL(N;\mathbb{C})$ w.r.t.

$$d\mu_t^N(Z) := \mu_t(Z)\ d\mathrm{vol}(Z)$$

on $GL(N;\mathbb{C})$

Digression: "Master field" in plane

- In 2-d (Euclidean) Yang–Mills theory on $\mathbb{R}^2$, have **random connection**.
- Holonomies around loops are random variables with values in $U(N)$.
- Physicists: these holonomies are **nonrandom** in the large-N limit, come from "master field"
- For simple closed curve, holonomy distributed as ρ_t^N. Hence: concentration!
- "Master field" in plane has been studied mathematically by **Michael Anshelevitch** and **Ambar N. Sengupta** and by **Thierry Lévy**, based on proposals by **I. M. Singer**.

Back to Segal–Bargmann transform

- On class functions, Segal–Bargmann transform **makes sense** in the limit, but it is **trivial** (constants map to constants)
- To get something nontrivial, Biane proposes to consider **matrix-valued functions**: $f : U(N) \to M_N(\mathbb{C})$
- Laplacian and SBT extend to matrix-valued functions, by applying them **entrywise**
- Consider **matrix-valued trace polynomials**, e.g.,

$$f(U) = 2U^2 \mathrm{tr}(U^3) - 9U\mathrm{tr}(U^4).$$

- Product rule extends **only** if one of the polynomials is scalar:

$$\Delta_\infty(U^2 U^3) \neq \Delta_\infty(U^2)U^3 + U^2\Delta_\infty(U^3).$$

Single-variable polynomials

- Any function of the form $\mathrm{tr}(U^k)$ becomes constant (almost everywhere w.r.t. ρ_t^N or μ_t^N) in the large-N limit.
- Only **untraced powers of U** survive.
- In the large-N limit, **single-variable polynomials** (linear combinations of U^k) map to single-variable polynomials (linear combinations of Z^k) on $GL(N;\mathbb{C})$

Example

- Consider
$$f(U) = U^2, \quad U \in U(N).$$
- Apply (entrywise) Segal–Bargmann transform B_t^N for $U(N)$.
- Result is
$$\begin{aligned} B_t^N(f)(Z) &= e^{-t}\left[\cosh(t/N)Z^2 + t\frac{\sinh(t/N)}{t/N} Z\mathrm{tr}(Z)\right] \\ &= e^{-t}\left[Z^2 + tZ\mathrm{tr}(Z) + O(1/N^2)\right] \end{aligned}$$
- But on $GL(N;\mathbb{C})$ we have $\mathrm{tr}(Z) \approx 1$ (w.r.t. μ_t^N) in the large-N limit
- Thus,
$$\lim_{N\to\infty} B_t^N(f)(Z) = e^{-t}(Z^2 + tZ)$$

Main result

Theorem (Driver-H-Kemp, 2013)

Let p be a polynomial in a single variable and let

$$f(U) = p(U), \quad U \in U(N).$$

Then for each $t > 0$, there exists a unique polynomial q_t in a single variable such that

$$\left\| B_t^N(f)(Z) - q_t(Z) \right\|_{L^2(GL(N;\mathbb{C}),\mu_t^N)} \to 0$$

as $N \to \infty$.

- E.g., if $p(u) = u^2$, then

$$q_t(z) = e^{-t}(z^2 + tz).$$

Computing large-N limit on powers of U

- **Step 1**: Start with U^k and **apply heat operator** $e^{t\Delta_\infty/2}$. Result is a trace polynomial (on $GL(N;\mathbb{C})$).
- **Step 2**: **Evaluate the traces**. Actually, $\mathrm{tr}(Z^k) \approx 1$ for every k.
- After evaluating the traces, result is a **polynomial** in Z.
- Example: $f(U) = U^3$. Applying $e^{t\Delta_\infty/2}$ gives

$$e^{-3t/2}\left\{Z^3 + t[2Z^2\mathrm{tr}(Z) + Z\mathrm{tr}(Z^2)] + \frac{3t^2}{2}Z\mathrm{tr}(Z)^2\right\}$$

- Evaluating all traces to 1 gives

$$B_t^\infty(U^3) = e^{-3t/2}\left\{Z^3 + t(2Z^2 + Z) + \frac{3t^2}{2}Z\right\}$$

Comparing to Biane

- Biane defines a transform $\mathcal{G}^t$ mapping polynomials to polynomials, using "free probability theory"
- Biane conjectures that $\mathcal{G}^t$ is just the large-N limit of B_t.

Theorem (Driver-H-Kemp, 2013)

For every single-variable polynomial p, the polynomial q_t coincides with $\mathcal{G}^t(p)$.

Comparing to Biane, cont'd

- Let $p_{t,k}(u)$ = result of applying $(B_t^\infty)^{-1}$ to the function $F(Z) = Z^k$.
- Define **generating function**

$$\Pi(t,u,z) = \sum_{k=0}^{\infty} p_{t,k}(u) z^k$$

- We derive a PDE for Π and solve it by the **method of characteristics**
- Result:

$$\Pi(t,u,z) = \frac{1}{1 - uze^{\frac{t}{2}\frac{1+z}{1-z}}}.$$

- This is generating function for $(\mathcal{G}_t)^{-1}$!

Computing with the generating function

- Computer can compute Taylor series of Π in powers of z
- Coefficient of z^k is just $p_{t,k}(u)$
- This gives alternative method of computing explicitly with $B_t^\infty = \mathcal{G}_t$ (in addition to our recursion method).

Conclusion

- Thank you for your attention!