ON HIGHLY DEGENERATE ALT-PHILLIPS PROBLEMS

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Here we start mentioning the following classical works

- Phillips, D.: *A minimization problem and the regularity of solutions in the presence of a free boundary.* Indiana Univ. Math. J. 32, 1–17 (1983)
- Alt, H.M., Phillips D.: *A free boundary problem for semilinear elliptic equations.* J. Reine Angew. Math. 368, 63–107 (1986)

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with $0 < \gamma < 1$ (non-differentiable)

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Sharp regularity estimates for solutions in the variational case:

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- ✓ For the p-Laplacian case: $\alpha = \min\left\{\alpha_p^-, \dfrac{1-\gamma}{p-(1-\gamma)}\right\}$ (LTQ 2015)
- ✓ Obstacle p-Laplacian ($\gamma = 0$): $\alpha = \dfrac{1}{p-1}$ at $\partial\{u > 0\}$ (ALS 2015)

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✓ The non-singular case $\gamma < 0$ (Teixeira 2016, WY 2021)

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Degenerate ellipticity:

$$F(X) \leq F(Y) \quad \text{whenever } \; X \leq Y$$

What about to consider 2nd order degenerate operators?

Infinity Laplacian equations with singular absorptions.
with G. Sá
Calc. Var. 61, 132 (2022).

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where Δ_∞ is the Infinity Laplacian.

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the best Lispchitz extension	$\Leftrightarrow$	$\Delta_\infty u = 0$	$\Leftrightarrow$	comparison with cones

! One of the main issues here is the lack of methods for providing regularity results for PDE involving Δ_∞. Indeed, even for the homogeneous case

$$\Delta_\infty u = 0$$

one is known that in general solutions are only locally Lipschitz.

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- ✓ for any dimensions: differentiable (ES 2012)
- ✓ in two dimensions:
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  - solutions are locally $C^{1,\alpha}$ for small $\alpha > 0$ (ES 2008)

? **open question**: $C^{1,\alpha}$ for a optimal $0 < \alpha \leq 1/3$

in R^2 : $X^{4/3} + Y^{4/3}$ (Arronson's example 1960)

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OUR MAIN GOAL

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- Solutions are $C^{1,\alpha}$ at $\partial\{u > 0\}$ for $\alpha = \frac{1}{3}$

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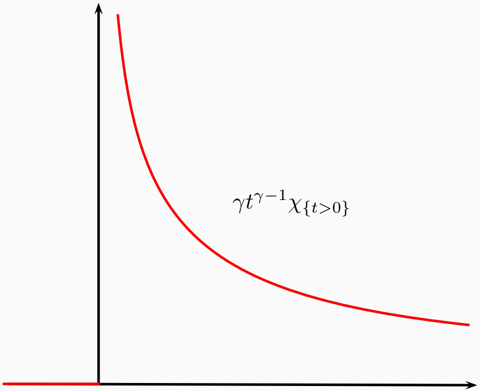
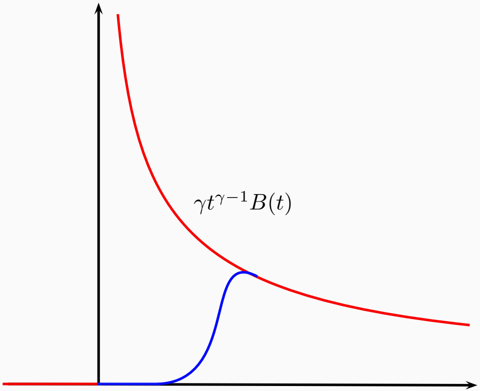

$\gamma t^{\gamma-1}\chi_{\{t>0\}}$

SINGULAR APPROACH STRATEGY



Here, for a given boundary datum $\varphi > 0$, we consider

$$\begin{aligned} -\Delta_\infty u &= B_\varepsilon(u) \cdot u^{-\gamma} \quad \text{in } \Omega \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned} \tag{E_ε}$$

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- ✓ Existence of positive (Perron's) solutions, denoted by u_ε
- ✓ Under $C^{0,1}$-compactness, $u_\varepsilon \xrightarrow[\varepsilon \to 0]{} u_0$ and by stability

$$-\Delta_\infty u_0 = u_0^{-\gamma} \quad \text{in} \quad \{u_0 > 0\}$$

in the viscosity sense.

THEOREM (______- G. Sá)
Let u be a positive solution of (E_ε). There exists constant $C > 0$, depending only on γ, $\|u\|_{L^\infty(B_1)}$ and dimension, such that, for points

$$x \in \partial\{u > 0\} \cap B_{1/2},$$

there holds

$$\sup_{B_r(x)} u \leq C r^\theta$$

for $\theta = \dfrac{4}{3+\gamma}$.

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For this, we analyze regularity aspects for the corrector function

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which solves in the viscosity sense

$$\Delta_\infty v = \left[(1-\theta)|Dv|^4 + \theta^{-3} f(x)\right] v^{-1}$$

for $f = B_\varepsilon(u_\gamma) \in [0,1]$.

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For $\rho = |x_m - y_m|$ and $\omega(\rho) = L\rho^\mu$, we have

$$\langle M_x \xi_x, \xi_x \rangle - \langle M_y \xi_y, \xi_y \rangle \lesssim -[\omega''(\omega')^2](\rho) \sim -\omega^{-1}(\rho).$$

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This would imply that

$$u_\gamma(y_m)^{-1} - u_\gamma(x_m)^{-1} < 0 \qquad \boxed{!}$$

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Proof: For each integer $k > 1$, we find $\varepsilon_k > 0$ and u_k, such that

$$s_k := \sup_{B_{r_k}(x_k)} u_k \geq k(r_k^\theta + u_k(x_k)), \tag{1}$$

for some radii $0 < r_k < 1/2$ and $x_k \in B_{1/2}$.

Define

$$\varphi_k(x) = \frac{u_k(x_k + r_k x)}{s_k} \quad \text{in} \;\; B_1.$$

SKETCH OF THE PROOF

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Note that

$$\sup_{B_1} \varphi_k = 1 \quad \text{and} \quad \varphi_k(0) + \frac{r_k^\theta}{s_k} \leq \frac{1}{k}.$$

In addition, for each $k > 0$, φ_k solves a singular equation.

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Proof: $\varphi_k \to \varphi_0$ which solves

$$\varphi_0^\gamma \Delta_\infty \varphi_0 = 0 \quad \text{in} \;\; B_1,$$

satisfies

$$\varphi_0 \geq 0 \;\text{ in } B_1, \quad \sup_{B_1} \varphi_0 = 1, \quad \varphi_0(0) = 0$$

and, for each $0 < \mu < 1$, there holds

$$\sup_{B_r(x)} \varphi_0 \leq \left(Cr^\mu + \varphi_0(x)^{\frac{1}{\theta}}\right)^\theta, \;\; \text{in particular} \;\; D\varphi_0 = 0 \text{ at } \partial\{\varphi_0 > 0\}.$$

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Proof: Finally, we find $z_0 \in \{\varphi_0 = 0\}$ and $z_+ \in \{\varphi_0 > 0\}$, satisfying

$$d := \operatorname{dist}(z_+, \{\varphi_0 = 0\}) = |z_+ - z_0|.$$

Note that φ_0 is infinity-harmonic in $B_d(z_+)$. By the Hopf lemma

$$0 < \liminf_{s \to 0^+} \frac{\varphi_0(z_0 + s(z_+ - z_0)) - \varphi_0(z_0)}{s}.$$

On the other hand, choosing $1/\theta < \mu < 1$ and letting $s \to 0^+$

$$\frac{\varphi_0(s(z_+ - z_0) + z_0)}{s} = \frac{\varphi_0(s(z_+ - z_0) + z_0)}{s^{\mu\theta}} \cdot s^{\mu\theta - 1} \leq Cs^{\mu\theta-1} \to 0.$$

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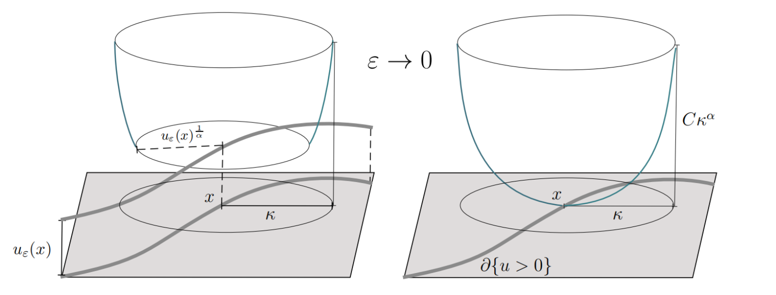

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NONDEGENERACY ESTIMATES

THEOREM (______- G. Sá)
Let u be a limiting Perron's solution. There exists $c > 0$, depending only on γ, such that for

$$x \in \overline{\{u > 0\}} \cap B_{1/2},$$

there holds

$$\sup_{B_r(x)} u \geq c\, r^{\theta}.$$

OTHER RELATED SCENARIOS

Sharp regularity for singular obstacle problems.
with R. Teymurazyan and V. Voskanyan
Math. Ann., to appear.

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Singular fully nonlinear parabolic equations
with G. Sá and J.M. Urbano
Submitted.

Muito obrigado!