The Power of Analogue-digital Machines

(Extended abstract)

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The ARNN abstract computer,[3] extensively analysed in [28], introduces an analogue-digital model of computation in discrete time. When the parameters of the system (so-called weights) are real-valued,[4] the computations cannot be specified by finite means: we have computation without a program. Several other models of analogue-digital computation were introduced around the same time to explore the power of reals added to digital computation (see [17,27,29]). Under the polynomial time constraint, the ARNN efficiently performs not only all Turing machine efficient computations,[5] but also computes non-recursive functions such as (a unary encoding of) the halting problem (of Turing machines). The reals[6] are introduced into the computation by means of measurements made either by a few neurons that read a weight byte by byte, or by means of a real-valued probability of transition. In the first case, the ARNN decides P/poly in polynomial time and, in the second case, the ARNN decides $BPP//\log\star$ in polynomial time. However, in these systems, measurements sound physically unrealistic since the function involved in computing the so-called activation of the neurons (the physical processors) is the well-behaved piecewise linear function, exhibiting sharp vertices. In an attempt to recover the classical analytic sigmoid activation function, in [25], the power of the deterministic ARNN in polynomial time drops to $P/\log\star$ as shown in [7,19].

Criticism was addressed towards the possibility of engineering such machines. In [20], Martin Davis pointed clearly that the only way a machine can go beyond the Turing limit is being provided with non-computable information and in [21] he says that, even if a machine could compute beyond the Turing limit, we would not be able to certify that fact (a phenomenon that can be well understood in [24], since only the computable character of a function can be verified — but not decided — in the limit). In [30], Younger et al. discuss the realization of

[3] **A**nalogue **R**ecurrent **N**eural **N**et.
[4] Real weights are quite common in the neural net literature.
[5] A few rational weights being enough for the purpose.
[6] In fact, the truncated reals. The amount of precision depends on the size of the input.

$BPP//\log\star$ super-Turing machines with their electronic engineering project. In our paper, the general model is only intended to establish limits to abstract and ideal computing devices that, like the ARNN, have access to real numbers by means of an ideal measurement in Classical Physics. It should be noted that measurements of physical quantities are also the subject of well-developed theory that started with Hempel and Carnap (see [18,22,23]). Their theory explains how numerical representations of qualitative attributes are possible and is laid out in the work of Krantz et al. [26].

In order to understand the computations of new paradigms of computing involving real numbers, it was proposed in [5] and [6] to replace the classical oracle to a Turing machine by an analogue device like those in the hybrid models of the sixties (see [16] for those analogue-digital models).

The oracles that we considered are physical processes that enable the Turing machine to measure quantities. As far as we have investigated (see [15]), measurements can be classified in one of the three types:[7] one-sided or threshold measurement, two-sided measurement and vanishing measurement. A one-sided experiment is an experiment that approximates the unknown value y just from one side, i.e. it approximates an unknown value y either with values z from above or with values z from below, checking either if $y < z$ or if $z < y$, but not both. A two-sided experiment[8] is an experiment that approximates the unknown value y from both sides, i.e. it approximates the unknown value y with values z from above and with values z from below, checking if $y < z$ and $z < y$. A vanishing experiment is an experiment that approximates the unknown value y measuring the number of ticks of a (Turing machine) clock.[9] This type of experimental classification is neither in Hempel's original work in [23], nor in the fully developed theory in [26].

For the previous types of oracle, the communication between the Turing machine and the oracle is ruled by one of the following protocols, inter alia (see [8] for the other protocols):

- Infinite precision: the oracle answers to the queried word with infinite precision;
- Arbitrary precision: the oracle answers to the queried word with probability of error that can be made as small as desired but is never 0;
- Fixed precision $\varepsilon > 0$: the oracle answers to the queried word with probability of error ε.

It was then realised that the interaction between the analogue part – experiment to conduct or value to be measured – and the digital part – the scientist or the computer – takes (physical) time that is at least exponential in the desired number of bits of precision (see [10,11,12,13,14,15]). (This physical time is intrinsic to physical law and does not represent the time needed for the activity of measurement itself.) Having discovered such a timing restriction (that

[7] This is still conjecture.
[8] ARNN computes with a two-sided experiment.
[9] A time constructible function.

in the ARNN model corresponds to the replacement of the piecewise linear or saturated sigmoid by the analytical sigmoid), we engaged in an investigation on experimental apparatuses in order to answer the question *What can one compute with the help of a measurement of a magnitude?* (see [1,4,8,13,15]). In [1,2,4,8,9] the upper bounds of analogue-digital computation in polynomial time under ideal conditions were placed in $BPP//\log\star$ in the case of both deterministic, and of probabilistic, computation. In fact, the power of measurements has been $BPP//\log\star$ persistently, across all limited precision protocols, while it drops from P/poly to $P/\log\star$ in the case of the deterministic measurement. We wondered whether the barrier $BPP//\log\star$ would persist in more general conditions. In [3], we show that under the most general (ideal) conditions the upper bounds of computational power of measurements of (deterministic) infinite, arbitrary and limited precisions are $BPP//\log\star$.

Among a number of theorems, we have shown that if these measurements are used as an oracle to a Turing machine, then, in polynomial time, we can compute the complexity classes listed in Table 1.

NON-ANALYTIC ANALOGUE	**Infinite**	**Unbounded**	**Fixed**
Lower Bound	P/poly	P/poly	$BPP//\log\star$
Upper Bound	P/poly	P/poly	$BPP//\log\star$
ANALYTIC ANALOGUE	**Infinite**	**Unbounded**	**Fixed**
Lower Bound	$P/\log\star$	$BPP//\log\star$	$BPP//\log\star$
Upper Bound	$P/\log\star$	$BPP//\log^2\star$ or $BPP//\log\star$	$BPP//\log^2\star$ or $BPP//\log\star$

Table 1. The lower and upper bounds for the main complexity classes computed by the analogue-digital models characterised by either a non-analytic (C^0, but not C^2) or an analytic function (from C^2 to analytic). These results were presented in [5,6] for the first case and in [1,4,8] for the second case. Different classes such as $BPP//\log^2\star$ and $BPP//\log\star$ occur in further specialization of the protocols not considered in this extended abstract.

Recently, we have moved towards understanding the computational limits of analogue-digital machines operating in bounded space. Some new research will be summarised.

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