

Insights into Standard Model and quantum gravity from higher gauge theory

Marko Vojinović

(in collaboration with Aleksandar Miković and Tijana Radenković)

Group for Gravitation, Particles and Fields, Institute of Physics Belgrade



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INTRODUCTION

A short recap of the spinfoam quantization method:

- Step 1: Rewrite the GR action — as a topological BF theory plus simplicity constraint,

$$S_{\text{Plebanski}}[B, \omega, \phi] = \int_{\mathcal{M}_4} \langle B \wedge F(\omega) \rangle_{\mathfrak{g}} + \langle \phi(B \wedge B) \rangle_{\mathfrak{g}},$$

where the Lie group G is Lorentz-like, and \mathfrak{g} is its Lie algebra.

- Step 2: Quantize the topological sector — a state sum over a triangulated manifold $T(\mathcal{M}_4)$,

$$Z_{BF} = \sum_{\Lambda} \prod_v \mathcal{A}_v(\Lambda) \prod_e \mathcal{A}_e(\Lambda) \prod_{\Delta} \mathcal{A}_{\Delta}(\Lambda) \prod_{\tau} \mathcal{A}_{\tau}(\Lambda) \prod_{\sigma} \mathcal{A}_{\sigma}(\Lambda).$$

“Colors” Λ are reps of G , amplitudes \mathcal{A} chosen so that Z_{BF} is invariant wrt. Pachner moves.

- Step 3: Impose the simplicity constraint — deform the invariant Z_{BF} by modifying the amplitudes and reps,

$$Z_{BF} \rightarrow Z_{GR} : \quad \mathcal{A}(\Lambda) \rightarrow W(j), \quad j = f(\Lambda),$$

obtaining a state sum Z_{GR} which defines a spinfoam model (Barret-Crane, EPRL/FK, etc).

Key question — how to add matter fields into the above?

HIGHER CATEGORY THEORY

A flash introduction to higher category theory:

- An n -category is a set of *objects* with:
 - *morphisms* (maps between objects),
 - *2-morphisms* (maps between morphisms),
 - *3-morphisms* (maps between 2-morphisms), ... up to *n -morphisms*,

along with certain axioms to provide suitable rules for composition, associativity, etc.

- An n -group is a special case of an n -category, which has only one object, and all morphisms are invertible.

An introduction to higher category theory (for physicists):

⇒ look up “**An Invitation to Higher Gauge Theory**” [Baez, Huerta (2011)]

The purpose of n -groups (for physicists):

⇒ *more fine-grained description of symmetry* using an n -group, than using a group,

⇒ generalization of differential geometry: *parallel transport, connection, holonomy, curvature*.

LIE 3-GROUPS

Focus on a strict Lie 3-group, isomorphic to a 2-crossed module:

[Faria Martins, Picken (2011); Wang (2014)]

$$(L \xrightarrow{\delta} H \xrightarrow{\partial} G , \triangleright , \{ -, - \})$$

- L, H, G — Lie groups,
- δ, ∂ — boundary morphisms,
- \triangleright — action of G , $\triangleright : G \times G \rightarrow G$, $\triangleright : G \times H \rightarrow H$, $\triangleright : G \times L \rightarrow L$,
- $\{ -, - \}$ — Peiffer lifting, $\{ -, - \} : H \times H \rightarrow L$.

Axioms that hold among these maps:

Chain complex:	$\partial\delta = 1_G$,
Conjugation:	$g \triangleright g_0 = g g_0 g^{-1}$,
G -equivariance of ∂ and δ :	$g \triangleright \partial h = \partial(g \triangleright h)$, $g \triangleright \delta l = \delta(g \triangleright l)$,
G -equivariance of lifting:	$g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, g \triangleright h_2\}$,
Peiffer commutator:	$\delta \{h_1, h_2\} = h_1 h_2 h_1^{-1} (\partial h_1) \triangleright h_2^{-1}$,
L -commutator:	$\{\delta l_1, \delta l_2\} = l_1 l_2 l_1^{-1} l_2^{-1}$,
δ -lifting relation:	$\{\delta l, h\} \{h, \delta l\} = l(\partial h \triangleright l^{-1})$,
Left product rule:	$\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\} \partial h_1 \triangleright \{h_2, h_3\}$.

LIE 3-GROUPS

Purpose of all this — to generalize the notion of parallel transport, from curves to surfaces to volumes:

- Connection generalized to a 3-connection (α, β, γ) , a triple of algebra-valued differential forms:

$$\begin{aligned}\alpha &= \alpha^\alpha{}_\mu(x) \tau_\alpha \otimes \mathbf{d}x^\mu && \in \mathfrak{g} \otimes \Lambda^1(\mathcal{M}), \\ \beta &= \frac{1}{2} \beta^a{}_{\mu\nu}(x) t_a \otimes \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu && \in \mathfrak{h} \otimes \Lambda^2(\mathcal{M}), \\ \gamma &= \frac{1}{3!} \gamma^A{}_{\mu\nu\rho}(x) T_A \otimes \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^\rho && \in \mathfrak{l} \otimes \Lambda^3(\mathcal{M}).\end{aligned}$$

- Line holonomy generalized to surface and volume holonomies:

$$g = \mathcal{P}\exp \int_{\mathcal{P}_1} \alpha, \quad h = \mathcal{S}\exp \int_{\mathcal{S}_2} \beta, \quad l = \mathcal{V}\exp \int_{\mathcal{V}_3} \gamma.$$

- Ordinary curvature generalized to 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, where:

$$\begin{aligned}\mathcal{F} &= \mathbf{d}\alpha + \alpha \wedge \alpha - \partial\beta, \\ \mathcal{G} &= \mathbf{d}\beta + \alpha \wedge^\triangleright \beta - \delta\gamma, \\ \mathcal{H} &= \mathbf{d}\gamma + \alpha \wedge^\triangleright \gamma - \{\beta \wedge \beta\}.\end{aligned}$$

HIGHER GAUGE THEORY

At this point one can construct the action for a higher gauge theory:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}.$$

\Rightarrow Topological $3BF$ theory, based on the 3-group $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{-, -\})$.

The physical interpretation of the Lagrange multipliers C and D :

- for $H = \mathbb{R}^4$, multiplier C can be **interpreted as the tetrad 1-form**:

$$C \rightarrow e = e^a{}_{\mu}(x) t_a \otimes dx^{\mu}, \quad [\text{Miković, MV (2012)}]$$

- for given L , multiplier D can be **interpreted as the set of matter fields**:

$$D \rightarrow \phi = \phi^A(x) T_A. \quad [\text{Radenković, MV (2019)}]$$

\Rightarrow The action thus becomes:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle e \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle \phi \wedge \mathcal{H} \rangle_{\mathfrak{l}}.$$

THE STANDARD MODEL

How many real-valued field components do we have in the Standard Model?

The fermion sector gives us:

$$\left. \begin{array}{cccc} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L & \begin{pmatrix} u_r \\ d_r \end{pmatrix}_L & \begin{pmatrix} u_g \\ d_g \end{pmatrix}_L & \begin{pmatrix} u_b \\ d_b \end{pmatrix}_L \\ \nu_e)_R & (u_r)_R & (u_g)_R & (u_b)_R \\ (e^-)_R & (d_r)_R & (d_g)_R & (d_b)_R \end{array} \right\} = 16 \frac{\text{spinors}}{\text{family}} \times$$

$$\times 3 \text{ families} \times 4 \frac{\text{real-valued components}}{\text{spinor}} = 192 \text{ real-valued components } \phi^A.$$

The Higgs sector gives us:

$$\left. \begin{array}{c} \phi^+ \\ \phi_0 \end{array} \right\} = 2 \text{ complex scalar fields} = 4 \text{ real-valued components } \phi^A.$$

This suggests the structure for L in the form:

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}),$$

where \mathbb{G} is the Grassmann algebra.

THE STANDARD MODEL

The actions $\triangleright : G \times L \rightarrow L$ and $\triangleright : G \times H \rightarrow H$ specify the transformation properties of matter ϕ^A and tetrad e^a_μ with respect to Lorentz and internal symmetries:

- Choose the group $G = SO(3,1) \times SU(3) \times SU(2) \times U(1)$. Then, for example, given any $g \in G$ and a doublet

$$\begin{pmatrix} u_b \\ d_b \end{pmatrix}_L,$$

the action $g \triangleright u_b$ encodes that u_b consists of 4 real-valued fields which transform as:

- a left-handed spinor wrt. $SO(3,1)$,
 - as a “blue” component of the fundamental representation of $SU(3)$,
 - and as “isospin $+\frac{1}{2}$ ” of the left doublet wrt. $SU(2) \times U(1)$.
- Next choose the group $H = \mathbb{R}^4$. The action \triangleright of G on H is via vector representation for the $SO(3,1)$ part and via trivial representation for the $SU(3) \times SU(2) \times U(1)$ part.
 - Finally, the other maps in the 3-group are chosen to be trivial. For all $l \in L$ and $\vec{u}, \vec{v} \in H$,

$$\delta l = 1_H = 0, \quad \partial \vec{v} = 1_G, \quad \{\vec{u}, \vec{v}\} = 1_L.$$

THE STANDARD MODEL

The Standard Model 3-group, $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{-, -\})$, defined as:

$$G = SO(3, 1) \times SU(3) \times SU(2) \times U(1), \quad H = \mathbb{R}^4,$$

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}).$$

The constrained 3BF action for the Standard Model coupled to Einstein-Cartan gravity:

$$\begin{aligned} S_{GR+SM} = & \int_{\mathcal{M}_4} B_{\hat{\alpha}} \wedge \mathcal{F}^{\hat{\alpha}} + e_{\hat{a}} \wedge \mathcal{G}^{\hat{a}} + \phi_{\hat{A}} \wedge \mathcal{H}^{\hat{A}} \\ & + \left(B_{\hat{\alpha}} - C_{\hat{\alpha}}^{\hat{\beta}} M_{cd\hat{\beta}} e^c \wedge e^d \right) \wedge \lambda^{\hat{\alpha}} - \left(\gamma_{\hat{A}} - e^a \wedge e^b \wedge e^c C_{\hat{A}}^{\hat{B}} M_{abc\hat{B}} \right) \wedge \lambda^{\hat{A}} - 4\pi i l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \phi_{\hat{A}} T^{d\hat{A}}_{\hat{B}} \phi^{\hat{B}} \\ & + \zeta^{ab}_{\hat{\alpha}} \wedge \left(M_{ab}^{\hat{\alpha}} \varepsilon^{cdef} e_c \wedge e_d \wedge e_e \wedge e_f - F^{\hat{\alpha}} \wedge e_c \wedge e_d \right) + \zeta^{ab}_{\hat{A}} \wedge \left(M_{abc}^{\hat{A}} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat{A}} \wedge e_a \wedge e_b \right) \\ & - \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \left(\Lambda + M_{\hat{A}\hat{B}} \phi^{\hat{A}} \phi^{\hat{B}} + Y_{\hat{A}\hat{B}\hat{C}} \phi^{\hat{A}} \phi^{\hat{B}} \phi^{\hat{C}} + L_{\hat{A}\hat{B}\hat{C}\hat{D}} \phi^{\hat{A}} \phi^{\hat{B}} \phi^{\hat{C}} \phi^{\hat{D}} \right). \end{aligned}$$

\Rightarrow Finally, one can go even further and separate scalar and fermion fields into distinct groups, employing the structure of a 4-group and a 4BF action. [Miković, MV (2021)]

QUANTIZATION

Revisit the spinfoam quantization method:

- Step 1: Rewrite the GR+SM action... **done!**
- Step 2: Quantize the topological sector... **done!** (almost) [Radenković, MV (2022)]

$$\begin{aligned}
 Z = & |G|^{-|v|+|e|-|\Delta|} |H|^{|v|-|e|+|\Delta|-|\tau|} |L|^{-|v|+|e|-|\Delta|+|\tau|-|\sigma|} \\
 & \prod_{e \in T} \int_G dg_e \prod_{\Delta \in T} \int_H dh_\Delta \prod_{\tau \in T} \int_L dl_\tau \\
 & \prod_{\Delta \in T} \delta_G(\partial(h_\Delta) g_1 g_2 g_3^{-1}) \prod_{\tau \in T} \delta_H(\delta(l_\tau) h_1 (g \triangleright h_2) h_3^{-1} h_4^{-1}) \\
 & \prod_{\sigma \in T} \delta_L(l_1^{-1} h_1 \triangleright' \{h_2, (g_1 g_2) \triangleright h_3\} l_2^{-1} (h_4 \triangleright' l_3) l_4 h_5 \triangleright' (g_3 \triangleright l_5))
 \end{aligned}$$

Invariant wrt. 4D Pachner moves!

- Step 3: Impose the simplicity constraints... **work in progress!**

QUANTIZATION

GR without matter can be described using 2-groups ($H \xrightarrow{\partial} G, \triangleright$):

- Topological $2BF$ theory developed and studied: [Girelli, Pfeiffer, Popescu (2008)]
[Miković, Martins (2011)]

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge \mathcal{F}_{ab} + C^a \wedge \mathcal{G}_a .$$

- The choice $G = SO(3, 1)$, $H = \mathbb{R}^4$, is called the *Poincaré 2-group*. The action for GR is

$$S_{GR} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab}(\omega) + e^a \wedge G_a - \phi_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) .$$

One possible quantization prescription leads to the *spincube model*. [Miković, MV (2012)]

- Representation theory for 2-groups (including the Poincaré 2-group), has been developed in great detail. [Baez, Baratin, Freidel, Wise (2012)]
- The topological invariant and TQFT for the *Euclidean 2-group* ($G = SO(4)$, $H = \mathbb{R}^4$) has also been studied in detail. [Baratin, Freidel (2015)]
[Asante, Dittrich, Girelli, Riello, Tsimiklis (2020)]

CONCLUSIONS

- Higher gauge theory represents a formalism where gravity, gauge fields, fermions and Higgs are treated on an equal footing.
- Resulting generalized spinfoam models naturally include matter fields coupled to gravity.
- The underlying algebraic structure of a 3-group classifies all fundamental fields by specifying groups L, H, G and their maps $\delta, \partial, \triangleright, \{-, -\}$.
- This structure has natural geometrical interpretation of parallel transport along a curve, a surface, and a volume.
- The gauge group L specifies the complete matter sector of the Standard Model if one chooses

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) .$$

- The action \triangleright of G on L specifies the transformation properties of matter fields.
- Nontrivial choices of the 3-group structure may provide new avenues for research on unification of all fields.

THANK YOU!