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A chromaticity-brightness model for color images denoising

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Global Portuguese Mathematicians

Técnico, Lisbon, July 13-14, 2017

We propose:

A **variational model** for **denoising color images** that combines

- ✓ Meyer's "$u + v$" decomposition
- ✓ chromaticity-brightness (CB) decomposition

It involves:

- ➢ **Minimization** of integral functionals • linear growth • maps taking values on a manifold • depend on a small parameter $\varepsilon > 0$
  - ➢ underlying manifold has boundary; the integrand and its recession function fail to satisfy hypotheses commonly assumed in literature

We prove:

- ✍ characterization of **asymptotic behavior** as $\varepsilon \to 0^+$
- ✍ **convergence** of infima, almost minimizers, and energies

Image Restoration

Deteriorated images

Images are **damaged** during creation, transmission, and recording:

- ✗ blur due to an incorrect lens adjustment or due to motion
- ✗ possible defects of the image system
- ✗ random phenomenon such as *noise* due to signal transmission

Variational PDE methods

- ✓ have proven to be **successful** in the restoration process, where the **desired clean** and **sharp image** is obtained as a **minimizer** of a certain energy **functional**

Variational PDE methods - cont.

The energy **functionals** proposed in the literature share the **common feature** of taking into **account a balance** between

- a certain distance to the given noisy image - fidelity term
- a filter acting as a regularization of the image

- Some notation:
  - $\Omega \subset \mathbb{R}^2$ image domain (typically, a rectangle)
  - $u : \Omega \to \mathbb{R}$ original (gray-scaled) image describing a real scene
  - u_0 observed (damaged) image of the same scene

- regularization term + fidelity term

Tikhonov & Arsenin ('77):

Find u that best fits the data, u_0, whose gradient is low (λ is a tuning parameter):

$$\min_{u \in W^{1,2}(\Omega)} \left\{ \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \lambda \int_\Omega |u_0 - u|^2 \,\mathrm{d}x \right\}$$

✓ L^2-norm of the gradient enhances **noise removal**

✗ but penalizes too much the gradient corresponding to edges - **oversmoothing**

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Rudin & Osher & Fatemi ('92): TV-model

Proposed to use the L^1-norm:

$$\inf_{\substack{u\in W^{1,1}(\Omega)\\ u_0-u\in L^2(\Omega)}} \left\{ \int_\Omega |\nabla u|\,\mathrm{d}x + \lambda \int_\Omega |u_0-u|^2\,\mathrm{d}x \right\}$$

or, equivalently,

$$\min_{\substack{u\in BV(\Omega)\\ u_0-u\in L^2(\Omega)}} \left\{ |Du|(\Omega) + \lambda \int_\Omega |u_0-u|^2\,\mathrm{d}x \right\}$$

Functions of Bounded Variation

- $\Omega \subset \mathbb{R}^N$ open set
- $\mathcal{M}(\Omega;\mathbb{R}^m)$ space of R^m-valued Radon measures $\lambda : \mathcal{B}(\Omega) \to \mathbb{R}^m$ endowed with the total variation norm $|\cdot|$

$$u \in BV(\Omega;\mathbb{R}^d) \;\Leftrightarrow\; u \in L^1(\Omega;\mathbb{R}^d) \;\&\; Du \equiv \lambda \in \mathcal{M}(\Omega;\mathbb{R}^{d\times N})$$

- $\|u\|_{BV(\Omega;\mathbb{R}^d)} := \|u\|_{L^1(\Omega;\mathbb{R}^d)} + |Du|(\Omega)$
- $u_j \overset{\star}{\rightharpoonup} u$ in $BV\big(\Omega;\mathbb{R}^d\big) \;\Leftrightarrow\; u_j \to u$ in $L^1\big(\Omega;\mathbb{R}^d\big) \;\&\; Du_j \overset{\star}{\rightharpoonup} Du$ weakly-$\star$ in $\mathcal{M}\big(\Omega;\mathbb{R}^{d\times N}\big)$
- if $u \in W^{1,1}(\Omega;\mathbb{R}^d)$, then $u \in BV(\Omega;\mathbb{R}^d)$ with $Du \equiv \nabla u \mathcal{L}^N$

TV-model/ROF's model

$$\min_{\substack{u \in BV(\Omega) \\ u_0 - u \in L^2(\Omega)}} \left\{ |Du|(\Omega) + \lambda \int_\Omega |u_0 - u|^2 \, \mathrm{d}x \right\}$$

- ➢ It leads to a decomposition of the type $u_0 = u + v$, where
  - ➢ u **well-structured**, aimed at modeling homogeneous regions
  - ➢ v encodes **textures** and **noise**
- ✓ very successful as it **removes noise** while **preserving edges**
- ✗ in some cases, leads to undesirable phenomena like **blurring** and **stair-casing effect**
- ✗ may not provide a good decomposition - some **pure geometric images** are **treated** as **noise** or **textures**
- ☞ Reasons pointed out in the literature relate to **both** the fidelity term and the regularization term

Meyer ('01):

Proved:

- oscillating images are often treated as texture or noise
- replacing the L^2-norm in the fidelity term by a certain G-norm leads to better decompositions

Accordingly, he suggested the model

$$\inf_{\substack{u \in BV(\Omega) \\ u - u_0 \in G(\Omega)}} \left\{ |Du|(\Omega) + \lambda \, \|u - u_0\|_{G(\Omega)} \right\}$$

- $\Omega = \mathbb{R}^2$: Meyer '01
- $\Omega \subset \mathbb{R}^2$ open, bounded, connected, Lipschitz: Aubert & Aujol '05

$G(\Omega)$ is the subspace of $W^{-1,\infty}(\Omega) \simeq (W_0^{1,1}(\Omega))'$ given by

$$G(\Omega) := \left\{v \in L^2(\Omega)\colon\ v = \operatorname{div}\xi,\ \xi \in L^\infty(\Omega;\mathbb{R}^2),\ \xi\cdot n = 0 \text{ on } \partial\Omega\right\}$$

Banach space when endowed with the norm

$$\|v\|_{G(\Omega)} := \inf\left\{\|\xi\|_{L^\infty(\Omega;\mathbb{R}^2)}\colon\ \operatorname{div}\xi = v,\ \xi\cdot n = 0 \text{ on } \partial\Omega\right\}$$

Alternative characterization ($N = 2$):

$$G(\Omega) = \left\{v \in L^2(\Omega)\colon \int_\Omega v(x)\,\mathrm{d}x = 0\right\}$$

Main properties of $G(\Omega)$

Let $\{v_n\}_{n\in\mathbb{N}} \subset G(\Omega)$ be a sequence for which there exists $p > 2$ such that $v_n \rightharpoonup 0$ weakly in $L^p(\Omega)$ as $n \to \infty$. Then,

$$\lim_{n\to\infty} \|v_n\|_{G(\Omega)} = 0.$$

- a function in $G(\Omega)$ may have large oscillations but small $G(\Omega)$-norm

Let (u, v) be the unique solution of ROF's model. If $\|u_0\|_{G(\Omega)} \leqslant \frac{1}{2\lambda}$, then $u = 0$ and $v = u_0$.

- an oscillating image that has small $G(\Omega)$-norm will be treated by ROF's model as texture or noise, which is not what expected if u_0 were a pure geometric image (e.g., a characteristic function)

Back to Meyer's model

These two results show that the space G is well suited to capture oscillations of a function in an energy minimization method

$$\inf_{\substack{u \in BV(\Omega) \\ u-u_0 \in G(\Omega)}} \left\{ |Du|(\Omega) + \lambda \, \|u-u_0\|_{G(\Omega)} \right\}$$

✓ Existence: Yes

? Uniqueness: Open problem

➢ Meyer's model is **difficult** to **handle numerically** because of the form of the $G(\Omega)$-norm (it prevents to express directly the associated Euler–Lagrange equation with respect to u)

➢ Several models attempting to approximate Meyer's model have been proposed (Aubert, Blanc-Féraud, Chambolle, Osher, Solé, Vese, ...)

Chromaticity-Brightness (CB) models

- $u_0 : \Omega \to (\mathbb{R}_0^+)^3$ observed (damaged) **color** image (RGB system)
- $(u_0)_b := |u_0|$ **brightness** component of u_0
- $(u_0)_c := \dfrac{u_0}{|u_0|} = \dfrac{u_0}{(u_0)_b} \in S^2$ **chromaticity** component of u_0

General idea of the CB models:

- restore brightness and chromaticity components separately (get u_b and u_c)
- observe that brightness component behaves as a gray-scale image (use any of the previous models)
- assemble the two components to get the restored image: $u := u_b u_c$

☞ Considered as reducing shadowing and providing better simulation results

A CB model

Kang & March ('07): (inpainting) CB model

$$\min_{u_c \in W^{1,2}(\Omega;S^2)} \left\{ \int_\Omega g(|\nabla u_b^\sigma|)|\nabla u_c|^2 \, \mathrm{d}x + \lambda \int_D |u_c - (u_0)_c|^2 \, \mathrm{d}x \right\}$$

- $D \subset \Omega$ set where the color is known
- $u_b^\sigma := G_\sigma * (u_0)_b$, $G_\sigma(x) := \frac{A}{\sigma} e^{-\frac{|x|^2}{4\sigma}}$, $A > 0$
- $g : \mathbb{R}_0^+ \to \mathbb{R}^+$, $g \searrow$, $g(0) = 1$, $g(+\infty) = 0$

$$\left(g(t) := \frac{1}{1 + \left(\frac{t}{a}\right)^2}, \quad g(t) := e^{-(\frac{t}{a})^2} \text{ with } a > 0 \right)$$

The value of the function $g(|\nabla u_b^\sigma|)$:

- ✓ is close to one in the regions where u_b^σ varies slowly
- ✓ is small at the edges of brightness (if both σ and a are small enough)

Kang & March's CB model

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$$\min_{u_c \in W^{1,2}(\Omega;S^2)} \left\{ \int_\Omega g(|\nabla u_b^\sigma|)|\nabla u_c|^2 \,\mathrm{d}x + \lambda \int_D |u_c - (u_0)_c|^2 \,\mathrm{d}x \right\}$$

- the first term acts as a regularization functional: the diffusion of chromaticity is inhibited across the edges of u_b^σ, yielding a sharp transition in the function u_c
- the second term requires the unitary vector field to be close to the chromaticity data $(u_0)_c$ in D

Therefore, the minimizer is a *piecewise smooth* color field, which is smooth in regions where the brightness u_b^σ varies slowly.

- If u_c is a solution of this problem, the colorized image u can is defined by $u := u_b u_c$.

A scheme towards our model

$u_0 : \Omega \longrightarrow (\mathbb{R}_0^+)^3$

- $u = u_b u_c$ $|Du|(\Omega) + \lambda_v \, \|u - u_0\|_{G(\Omega;\mathbb{R}^3)}$ v
  - u_b $|Du_b|(\Omega) + \lambda_b \, \|u_b - (u_0)_b\|_{G(\Omega)}$
  - u_c $\int_\Omega g(|\nabla u_b|)|\nabla u_c|^2 \,\mathrm{d}x + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \,\mathrm{d}x$

Because $1 \leqslant |u_0| \leqslant C$ a.e., we will consider u_b's such that $\alpha \leqslant |u_b| \leqslant \beta$ for some $0 < \alpha \leqslant \beta$. This condition plays an important role to obtain (uniform) estimates concerning $\int_\Omega |\nabla u_c| \,\mathrm{d}x$.

Towards our model

$$F^{reg}(u_b,u_c) := \int_\Omega |\nabla u_b|\,\mathrm{d}x + \int_\Omega g(|\nabla u_b|)|\nabla u_c|^2\,\mathrm{d}x + \int_\Omega |\nabla(u_b u_c)|\,\mathrm{d}x$$

$$F^{fid}(u_b,u_c) := \lambda_b\|u_b-(u_0)_b\|_{G(\Omega)} + \lambda_c\int_\Omega |u_c-(u_0)_c|^2\,\mathrm{d}x + \lambda_v\|u_bu_c-u_0\|_{G(\Omega;\mathbb{R}^3)}$$

$$\inf_{\substack{u_b\in W^{1,1}(\Omega;[\alpha,\beta]),\,u_c\in W^{1,2}(\Omega;S^2),\\ u_b-(u_0)_b\in G(\Omega),\,u_bu_c-u_0\in G(\Omega;\mathbb{R}^3)}} \left\{ F^{reg}(u_b,u_c) + F^{fid}(u_b,u_c) \right\}$$

A visit to the "Gap Problem"

$$F^{reg}(u_b, u_c) = \int_\Omega |\nabla u_b| \, \mathrm{d}x + \int_\Omega g(|\nabla u_b|)|\nabla u_c|^2 \, \mathrm{d}x + \int_\Omega |\nabla(u_b u_c)| \, \mathrm{d}x$$
$$= \int_\Omega h(u_b, u_c, \nabla u_b, \nabla u_c) \, \mathrm{d}x,$$

where

$$h(r, s, \xi, \eta) := |\xi| + g(|\xi|)|\eta|^2 + |s \otimes \xi + r\eta|$$

- $(\xi, \eta) \mapsto h(r, s, \xi, \eta)$ is **not** (in general) **quasiconvex**
- $\frac{1}{C}(|\xi| + |\eta|) \leqslant h(r, s, \xi, \eta) \leqslant C(1 + |\xi| + |\eta|^2)$ **non-standard growth**

$$\left(\tfrac{1}{2}|\xi| + \tfrac{\alpha}{2}|\eta| \leqslant \tfrac{1}{2}|\xi| + \tfrac{1}{2}(|r\eta + s \otimes \xi| + |\xi|) \leqslant h(r, s, \xi, \eta) \right)$$

GAP PROBLEM...!

Towards our model - cont.

$$F^{reg}(u_b, u_c) := \int_\Omega |\nabla u_b| \, \mathrm{d}x + \int_\Omega g(|\nabla u_b|) |\nabla u_c| \, \mathrm{d}x + \int_\Omega |\nabla (u_b u_c)| \, \mathrm{d}x$$

$$F^{fid}(u_b, u_c) := \lambda_b \|u_b - (u_0)_b\|_{G(\Omega)} + \lambda_c \int_\Omega |u_c - (u_0)_c|^2 \, \mathrm{d}x + \lambda_v \|u_b u_c - u_0\|_{G(\Omega;\mathbb{R}^3)}$$

(Almost) Our model:

$$\inf_{\substack{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2), \\ u_b-(u_0)_b\in G(\Omega), u_b u_c - u_0 \in G(\Omega;\mathbb{R}^3)}} \left\{ F^{reg}(u_b, u_c) + F^{fid}(u_b, u_c) \right\}$$

- It is a **challenging** task to construct a recovery sequence that **simultaneously** satisfies the **manifold constraint** and the **average restrictions**

A penalized version of F^{fid}

$$
\begin{aligned}
&F_\varepsilon^{fid}(u_b, u_c) \\
&:= \lambda_b \left\| u_b - (u_0)_b - \fint_\Omega (u_b - (u_0)_b)\,\mathrm{d}x \right\|_{G(\Omega)} + \frac{1}{\varepsilon}\left| \int_\Omega (u_b - (u_0)_b)\,\mathrm{d}x \right| \\
&+ \lambda_c \int_\Omega |u_c - (u_0)_c|^2\,\mathrm{d}x \\
&+ \lambda_v \left\| u_b u_c - u_0 - \fint_\Omega (u_b u_c - u_0)\,\mathrm{d}x \right\|_{G(\Omega;\mathbb{R}^3)} + \frac{1}{\varepsilon}\left| \int_\Omega (u_b u_c - u_0)\,\mathrm{d}x \right|
\end{aligned}
$$

We will recover $F^{fid}(u_b, u_c)$ in the limit as $\varepsilon \to 0^+$.

Our model

Study the asymptotic behavior as $\varepsilon \to 0^+$ of:

$$\inf_{\substack{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2),\\ \sout{u_b-(u_0)_b\in G(\Omega),\, u_bu_c-u_0\in G(\Omega;\mathbb{R}^3)}}} \left\{ F^{reg}(u_b,u_c) + F^{fid}_\varepsilon(u_b,u_c) \right\}$$

$$F^{reg}(u_b,u_c) := \int_\Omega |\nabla u_b|\,\mathrm{d}x + \int_\Omega g(|\nabla u_b|)|\nabla u_c|\,\mathrm{d}x + \int_\Omega |\nabla(u_bu_c)|\,\mathrm{d}x$$

$$F^{fid}_\varepsilon(u_b,u_c) := \lambda_b \Big\| u_b-(u_0)_b - ⨍_\Omega (u_b-(u_0)_b)\,\mathrm{d}x \Big\|_{G(\Omega)}$$
$$+\frac{1}{\varepsilon}\Big|\int_\Omega (u_b-(u_0)_b)\,\mathrm{d}x\Big| + \lambda_c \int_\Omega |u_c-(u_0)_c|^2\,\mathrm{d}x$$
$$+\lambda_v \Big\| u_bu_c-u_0-⨍_\Omega (u_bu_c-u_0)\,\mathrm{d}x\Big\|_{G(\Omega;\mathbb{R}^3)} + \frac{1}{\varepsilon}\Big|\int_\Omega (u_bu_c-u_0)\,\mathrm{d}x\Big|$$

Approximate limit set A_u & Approximate discontinuity set S_u

$x \in A_u$ if u has an approximate limit at x, i.e., $\exists\, z =: \tilde{u}(x) \in \mathbb{R}^d$ such that

$$\lim_{\delta\to 0^+} \fint_{B(x,\delta)} |u(y) - z| \,\mathrm{d}y = 0.$$

$S_u := \Omega \setminus A_u$ is called the approximate discontinuity set.

Set J_u of approximate jump points of u

$x \in J_u$ if $\exists\, a, b \in \mathbb{R}^d$, $a \neq b$, and $\nu \in S^{N-1}$ such that

$$\lim_{\delta\to 0^+} \fint_{B_\nu^+(x,\delta)} |u(y) - a| \,\mathrm{d}y = 0, \quad \lim_{\delta\to 0^+} \fint_{B_\nu^-(x,\delta)} |u(y) - b| \,\mathrm{d}y = 0.$$

The triplet (a, b, ν), uniquely determined up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

More on the space $BV(\Omega;\mathbb{R}^d)$ - cont.

Set D_u of approximate approximate differentiability points of u

$x \in D_u$ if $x \in A_u$ and $\exists\, L \in \mathbb{M}^{d\times N} =: \nabla u(x)$ such that

$$\lim_{\delta\to 0^+} \fint_{B(x,\delta)} \frac{|u(y) - \tilde{u}(x) - L(y-x)|}{\delta}\,\mathrm{d}y = 0.$$

Decomposition of Du with $u \in BV(\Omega;\mathbb{R}^d)$

$$\begin{aligned} Du &= D^a u + D^s u && D^a u \ll \mathcal{L}^N\lfloor_\Omega,\ D^s u \perp \mathcal{L}^N\lfloor_\Omega \\ &= D^a u + D^s u\lfloor_{J_u} + D^s u\lfloor_{A_u} && \mathcal{H}^{N-1}(S_u \setminus J_u) = 0 \\ &= \nabla u \mathcal{L}^N + (u^+ - u^-)\otimes \nu_u \mathcal{H}^{N-1}\lfloor_{J_u} + D^c u && D^c u := D^s u\lfloor_{A_u} \end{aligned}$$

First main result

A relaxation one:

$\Omega \subset \mathbb{R}^2$ open & bounded domain, $\partial\Omega$ Lipschitz

$F : L^1(\Omega) \times L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$

$$F(u_b,u_c) := \begin{cases} F^{reg}(u_b,u_c), & (u_b,u_c) \in W^{1,1}(\Omega;[\alpha,\beta]) \times W^{1,1}(\Omega;S^2) \\ +\infty, & \text{otherwise.} \end{cases}$$

Then, the **lower semicontinuous envelope** of F, $\mathcal{F} : L^1(\Omega) \times L^1(\Omega;\mathbb{R}^3) \to [0,+\infty]$, has the **integral representation**

$$\mathcal{F}(u_b,u_c) = \begin{cases} F^{reg,sc^-}(u_b,u_c), & (u_b,u_c) \in BV(\Omega;[\alpha,\beta]) \times BV(\Omega;S^2) \\ +\infty & \text{otherwise,} \end{cases}$$

where $F^{reg,sc^-} = F^{reg,sc^-}_{ac} + F^{reg,sc^-}_{j} + F^{reg,sc^-}_{c}$, with

$$F_{ac}^{reg,sc^-}(u_b,u_c) = \int_\Omega \mathcal{Q}_T f(u_b(x), u_c(x), \nabla u_b(x), \nabla u_c(x))\,\mathrm{d}x,$$

where $\mathcal{Q}_T f$ is the **tangential quasiconvex envelope** of f:

$$\mathcal{Q}_T f(r,s,\xi,\eta) = \inf\Big\{ \int_Q f(r,s,\xi+\nabla\varphi(y), \eta+\nabla\psi(y))\,\mathrm{d}y : \varphi \in W_0^{1,\infty}(Q),\ \psi \in W_0^{1,\infty}(Q; T_s(S^2)) \Big\}$$

- $T_s(S^2)$ tangential space to S^2 at s
- $r \in [\alpha,\beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, $\eta \in [T_s(S^2)]^2$
- $f(r,s,\xi,\eta) = |\xi| + g(|\xi|)|\eta| + |s\otimes\xi + r\eta|$

Alternative characterization of $Q_T f$

For all $r \in [\alpha, \beta]$, $s \in S^2$, $\xi \in \mathbb{R}^2$, and $\eta \in [T_s(S^2)]^2$, we have that

$$\mathcal{Q}_T f(r, s, \xi, \eta) = \mathcal{Q}\tilde{f}(r, s, \xi, \eta),$$

where, for $(r, s, \xi, \eta) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 2}$,

$$\tilde{f}(r, s, \xi, \eta) := \begin{cases} f(\tilde{r}, \tilde{s}, \xi, P_{\tilde{s}}\eta)\,\phi(|s|) & \text{if } s \in \mathbb{R}^3 \backslash \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\tilde{r} := \begin{cases} \alpha & \text{if } r \leqslant \alpha, \\ r & \text{if } \alpha \leqslant r \leqslant \beta, \\ \beta & \text{if } r \geqslant \beta, \end{cases} \qquad \tilde{s} := \frac{s}{|s|},$$

and $\phi \in C^\infty(\mathbb{R}; [0,1])$ is a cut-off function such that $\phi(t) = 1$ if $t \geqslant 1$, and $\phi(t) = 0$ if $t \leqslant \frac{3}{4}$.

Relaxation - characterization of F_j^{reg,sc^-}

$$F_j^{reg,sc^-}(u_b,u_c) = \int_{S_{(u_b,u_c)}} K\big((u_b,u_c)^+(x),(u_b,u_c)^-(x),\nu_{(u_b,u_c)}(x)\big)\,\mathrm{d}\mathcal{H}^1(x),$$

where, for $a, b \in [\alpha,\beta]\times S^2$, $\nu \in S^1$, Q_ν unit cube centered at 0 and two faces orthogonal to ν,

$$K(a,b,\nu) = \inf_{(\varphi,\psi)\in\mathcal{P}(a,b,\nu)} \int_{Q_\nu} f^\infty(\varphi(y),\psi(y),\nabla\varphi(y),\nabla\psi(y))\,\mathrm{d}y$$

$$\begin{aligned} f^\infty(r,s,\xi,\eta) &:= \limsup_{t\to+\infty} \frac{f(r,s,t\xi,t\eta)}{t} \\ &= \limsup_{t\to+\infty} \big(|\xi| + g(t|\xi|)|\eta| + |r\eta + s\otimes\xi|\big) \\ &= |\xi| + \chi_{\{0\}}(|\xi|)|\eta| + |r\eta + s\otimes\xi| \end{aligned}$$

Relaxation - characterization of F_c^{reg,sc^-}

$$F_c^{reg,sc^-}(u_b, u_c) = \int_\Omega (\mathcal{Q}_T f)^\infty(\tilde{u}_b(x), \tilde{u}_c(x), W_b^c(x), W_c^c(x))\,\mathrm{d}|D^c(u_b, u_c)|$$

- $(Q_T f)^\infty(r, s, \xi, \eta) = \limsup_{t \to +\infty} \dfrac{Q_T f(r, s, t\xi, t\eta)}{t}$
- $\tilde{u}_b(x)$ and $\tilde{u}_c(x)$: approximate limits of u_b and u_c at x
- W^c: Radon-Nikodym derivative of $D^c(u_b, u_c)$ w.r.t. its total variation
- W_b^c : first row of W^c
- W_c^c: 3×2 matrix obtained from W^c by erasing its first row

Previous results - Sobolev setting

B. Dacorogna, I. Fonseca, J. Malý, K. Trivisa ('99)
Manifold constrained variational problems

- $f : \mathbb{R}^{d\times N} \to [0,\infty)$ continuous
- $0 \leqslant f(\zeta) \leqslant C(1+|\zeta|^p), \quad p \geqslant 1$
- $\mathcal{M} \subset \mathbb{R}^d$ ia a C^1 submanifold without boundary

$$\inf\left\{ \liminf_{n\to\infty} \int_\Omega f(\nabla u_n)\mathrm{d}x : u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathcal{M}) \right\} = \int_\Omega Q_T f(u,\nabla u)\mathrm{d}x$$

$$Q_T f(y,\zeta) = \inf\left\{ \int_\Omega f(\zeta + \nabla\varphi(x))\mathrm{d}x : \varphi \in W_0^{1,\infty}(Q;T_y(\mathcal{M})) \right\}$$

$$Q_T f(y,\zeta) = Q\bar{f}(y,\zeta), \quad \bar{f}(y,\zeta) := f(P_y\zeta)$$

($P_y\zeta$ is the orthogonal projection of $\mathbb{R}^d$ onto $T_y(\mathcal{M})$)

R. Alicandro, A. Corbo Esposito, C. Leone ('07)
Relaxation in BV of Integral Functionals Defined on Sobolev Functions with Values in the Unit Sphere

$$F(u) = \int_\Omega f(x, u, \nabla u)\, \mathrm{d}x, \quad u \in W^{1,1}(\Omega; S^{d-1})$$

- $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, \infty)$ continuous, ...
- $C^{-1}|\zeta| \leqslant f(x, y, \zeta) \leqslant C(1 + |\zeta|)$
- $\mathcal{M} = S^{d-1}$
- $f(x, \cdot, \cdot)$ is a tangential quasiconvex function
- $|f^\infty(x, y, \zeta) - f(x, y, \zeta)| \leqslant C(1 + |\zeta|^{1-m}), \quad 0 < m < 1$

Previous results - BV setting

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D. Mucci ('09)
Relaxation of isotropic functionals with linear growth defined on manifold constrained Sobolev mappings

$$F(u) = \int_{B^N} f(x, u, \nabla u)\, \mathrm{d}x, \quad u \in W^{1,1}(\Omega; \mathcal{M})$$

- $f : B^N \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \to [0, \infty)$ continuous, ...
- $C^{-1}|\zeta| \leqslant f(x, y, \zeta) \leqslant C(1 + |\zeta|)$
- $\mathcal{M}$ smooth, compact, connected, without boundary
- $f(x, \cdot, \cdot)$ is a tangential quasiconvex function
- $|f^\infty(x, y, \zeta) - f(x, y, \zeta)| \leqslant C(1 + |\zeta|^{1-m}), \quad 0 < m < 1$

Previous results - BV setting

J.-F. Babadjian, V. Millot ('09)
Homogenization of variational problems in manifold valued BV-spaces

$$F(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u\right) \mathrm{d}x, \quad u \in W^{1,1}(\Omega; \mathcal{M})$$

- $f : \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,\infty)$ Carathéodory, ...
- $C^{-1}|\zeta| \leqslant f(x,\zeta) \leqslant C(1+|\zeta|)$
- $\mathcal{M}$ smooth, compact, connected, without boundary
- $|f^\infty(x,y,\zeta) - f(x,y,\zeta)| \leqslant C(1+|\zeta|^{1-m}), \quad 0 < m < 1$

Our case

$$F(w) = \int_\Omega f(w, \nabla w)\,\mathrm{d}x, \quad w \in W^{1,1}(\Omega; \mathcal{M}), \quad w = (u_b, u_c)$$

- $\mathcal{M} = [\alpha, \beta] \times S^2$ has boundary
- $f(r, s, \xi, \eta) = |\xi| + g(|\xi|)|\eta| + |s \otimes \xi + r\eta|$
  - is not tangentially quasiconvex!

    $$f^\infty(r, s, \xi, \eta) = |\xi| + \chi_{\{0\}}(|\xi|)|\eta| + |r\eta + s \otimes \xi|$$

  - does not satisfy a condition of the type

    $$|f^\infty(x, y, \xi, \eta) - f(x, y, \xi, \eta)| \leqslant C(1 + |(\xi, \eta)|^{1-m}), \;\; 0 < m < 1$$

    $$|f^\infty(x, y, \xi, \eta) - f(x, y, \xi, \eta)| = |\chi_{\{0\}}(|\xi|) - g(|\xi|)||\eta|$$

We anticipate that our arguments may be used to treat **more general manifolds with boundary** and **integrands**.

Back to our original problem

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Study the asymptotic behavior as $\varepsilon \to 0^+$ of:

$$\inf_{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2)} \left\{ F^{reg}(u_b,u_c) + F^{fid}_\varepsilon(u_b,u_c) \right\}$$

$$F^{reg}(u_b,u_c) := \int_\Omega |\nabla u_b|\,\mathrm{d}x + \int_\Omega g(|\nabla u_b|)|\nabla u_c|\,\mathrm{d}x + \int_\Omega |\nabla(u_b u_c)|\,\mathrm{d}x$$

$$\begin{aligned} F^{fid}_\varepsilon(u_b,u_c) &:= \lambda_b \Big\| u_b - (u_0)_b - \fint_\Omega (u_b-(u_0)_b)\,\mathrm{d}x \Big\|_{G(\Omega)} \\ &+ \frac{1}{\varepsilon}\Big|\int_\Omega (u_b-(u_0)_b)\,\mathrm{d}x\Big| + \lambda_c \int_\Omega |u_c-(u_0)_c|^2\,\mathrm{d}x \\ &+ \lambda_v \Big\| u_b u_c - u_0 - \fint_\Omega (u_b u_c - u_0)\,\mathrm{d}x \Big\|_{G(\Omega;\mathbb{R}^3)} + \frac{1}{\varepsilon}\Big|\int_\Omega (u_b u_c - u_0)\,\mathrm{d}x\Big| \end{aligned}$$

$$\begin{aligned} X = \Big\{ (u_b,u_c) \in BV(\Omega;[\alpha,\beta]) \times BV(\Omega;S^2) \colon\, & u_b-(u_0)_b \in G(\Omega), \\ & u_b u_c - u_0 \in G(\Omega;\mathbb{R}^3) \Big\} \end{aligned}$$

Auxiliary lemma

Lemma

The set

$$X = \Big\{(u_b, u_c) \in BV(\Omega;[\alpha,\beta]) \times BV(\Omega;S^2) \colon u_b - (u_0)_b \in G(\Omega),\\ u_b u_c - u_0 \in G(\Omega;\mathbb{R}^3)\Big\}$$

is **non-empty**.

- Recall: $(u_0)_b = |u_0| \in [\alpha,\beta]$, $(u_0)_c = \frac{u_0}{|u_0|} = \frac{u_0}{(u_0)_b} \in S^2$, and $u_0 = (u_0)_b (u_0)_c$.
- Set $u_b(x) := c_0$, $x \in \Omega$, where $c_0 := \fint_\Omega (u_0)_b \, \mathrm{d}x$.
  - ✓ $u_b \in BV(\Omega;[\alpha,\beta])$
  - ✓ $u_b - (u_0)_b \in G(\Omega)$

Auxiliary lemma - cont.

- Observe: $\left| \fint_\Omega (u_0)_b (u_0)_c \,\mathrm{d}x \right| \leqslant \fint_\Omega (u_0)_b \,\mathrm{d}x = c_0$
- Thus: $\exists\, \theta \in [0,1], s_1, s_2 \in \partial B(0, c_0)$ such that

$$\fint_\Omega (u_0)_b (u_0)_c \,\mathrm{d}x = \theta s_1 + (1-\theta) s_2$$

- Let $\{\Omega_1, \Omega_2\}$ be a Lipschitz partition of Ω satisfying

$$\mathcal{L}^2(\Omega_1) = \theta \mathcal{L}^2(\Omega), \quad \mathcal{L}^2(\Omega_2) = (1-\theta)\mathcal{L}^2(\Omega)$$

- Set, for $x \in \Omega$, $u_c(x) := \frac{s_1}{c_0}$ if $x \in \Omega_1$ and $u_c(x) := \frac{s_2}{c_0}$ if $x \in \Omega_2$.
  - ✓ $u_c \in BV(\Omega; S^2)$
  - ✓ $u_b u_c - u_0 \in G(\Omega; \mathbb{R}^3)$

$\Omega \subset \mathbb{R}^2$ open & bounded domain, $\partial\Omega$ Lipschitz; $\varepsilon_n, \delta_n \to 0^+$

The imaging problem:

❶ $$\lim_{n\to\infty} \inf_{(u_b,u_c)\in W^{1,1}(\Omega;[\alpha,\beta])\times W^{1,1}(\Omega;S^2)} \left(F^{reg}(u_b,u_c) + F^{fid}_{\varepsilon_n}(u_b,u_c)\right)$$
$$= \min_{(u_b,u_c)\in X} \left(F^{reg,sc^-}(u_b,u_c) + F^{fid}(u_b,u_c) \right)$$

❷ If $(u_b^n, u_c^n) \in W^{1,1}(\Omega;[\alpha,\beta]) \times W^{1,1}(\Omega;S^2)$ is a δ_n-minimizer of $\left(F^{reg} + F^{fid}_{\varepsilon_n}\right)$ in $W^{1,1}(\Omega;[\alpha,\beta]) \times W^{1,1}(\Omega;S^2)$, then

- ✓ $(u_b^n, u_c^n)_{n\in\mathbb{N}}$ bounded in $BV(\Omega) \times BV(\Omega;\mathbb{R}^3)$
- ✓ A cluster point (u_b, u_c) of $(u_b^n, u_c^n)_{n\in\mathbb{N}}$ • belongs to X, • is a minimizer of $\left(F^{reg,sc^-} + F^{fid} \right)$ in X, and

  • $$\lim_{n\to\infty} \left(F^{reg}(u_b^n,u_c^n) + F^{fid}_{\varepsilon_n}(u_b^n,u_c^n)\right)$$
  $$= F^{reg,sc^-}(u_b,u_c) + F^{fid}(u_b,u_c)$$

Thank you!

Manuscript can be downloaded at:
http://arxiv.org/abs/1603.07647
http://www.ritaferreira.pt