Virasoro constraints for sheaf moduli spaces via wall-crossing

Miguel Moreira
ETHZ
Joint with Arkadij Bojko and Woonam Lim

Arxiv 2210.05266

Geometria em Lisboa
18 October 2022

History of Virasoro constraints

- In 1990, Witten proposed a conjecture saying that integrals of ψ-classes in the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ satisfy some relations which completely determine them:

$$L_k(Z) = 0 \quad \text{for } k \geqslant -1\,,$$

where Z is the generating function of these integrals and L_k are differential operators satisfying the Virasoro bracket

$$[L_k, L_\ell] = (\ell - k)L_{k+\ell}\,.$$

- Witten's conjecture was proven in 1992 by Kontsevich. Alternative proofs by Okounkov-Pandharipande and Mirzakhani were found later.
- Eguchi-Hori-Xiong propose in 1997 a generalization to the Gromov-Witten (GW) theory of a target variety X.

History of Virasoro constraints

- In 2006, Maulik-Nekrasov-Okounkov-Pandharipande (MNOP) propose a conjecture connecting Gromov-Witten invariants on 3-folds to Donaldson-Thomas (DT) invariants, defined using the moduli space of ideal sheaves.
- An analog of Virasoro constraints should exist in DT theory! Oblomkov-Okounkov-Pandharipande make a precise conjecture by calculations in $X = \mathbb{P}^3$.
- In 2020, with Oblomkov-Okounkov-Pandharipande we prove that the MNOP correspondence intertwines the GW Virasoro and the DT Virasoro constraints (in stationary regime).
- This proves Virasoro constraints for the DT theory of toric 3-folds with stationary descendents.

History of Virasoro constraints

- In 2020 I used the previous result to prove a version of Virasoro constraints for the Hilbert scheme of points on simply-connected surfaces.
- In 2021 D. van Bree conjectures a generalization of the Hilbert scheme result to moduli spaces of stable sheaves on surfaces.
- Much more general?...

Today

I will explain joint work with A. Bojko and W. Lim containing:

- Unified formulation of Virasoro constraints for moduli spaces of sheaves and pairs.
- How the Virasoro constraints are naturally formulated using the vertex algebra that D. Joyce introduced to study wall-crossing.
- Virasoro constraints are compatible with wall-crossing.
- A proof of the Virasoro constraints for moduli spaces of stable sheaves on curves and surfaces with $h^{0,1} = h^{0,2} = 0$ (either torsion-free or dimension 1 sheaves) by reducing everything to the rank 1 case.

Stable bundles on curves

Let C be a smooth projective curve of genus $g \geqslant 0$. Given a vector bundle G on C define its slope as

$$\mu(G) = \frac{\deg(G)}{\mathrm{rk}(G)} .$$

Definition

A vector bundle is called (semi)stable if for every subbundle $G' \subsetneq G$

$$\mu(G')(\leqslant)\mu(G)$$

where $(\leqslant)$ means $<$ in the stable case and $\leqslant$ in semistable.

We can form the moduli space $M = M_C(r, d)$ of semistable bundles of rank r and degree d.

Stable bundles on curves

If r, d are coprime then:

- Every semistable sheaf in $M_C(r, d)$ is stable.
- The moduli space $M_C(r, d)$ is a smooth projective variety of dimension $r^2(g-1)+1$.
- The tangent space at $[G] \in M_C(r, d)$ is given by

$$\mathrm{Ext}^1(G, G).$$

- There exists a universal bundle $\mathbb{G}$ on $M \times C$. Very important: $\mathbb{G}$ is not unique, it is defined only up to twisting by a line bundle pulled back from M.

Moduli spaces of Bradlow pairs

We want to define moduli spaces of pairs, that parametrize a vector bundle F together with a section (or many sections), i.e maps of vector bundles $\mathcal{O}_C^{\oplus m} \to F$.
Given $t \in \mathbb{R}_{>0}$ we define the μ_t-slope

$$\mu_t(\mathcal{O}_C^{\oplus m} \to F) = \frac{\deg(F) + t \cdot d}{\mathrm{rk}(F)}\ .$$

Definition

A pair $\mathcal{O}_C^{\oplus m} \to F$ is called μ_t-(semi)stable if for every subpair $\mathcal{O}^{\oplus m'} \to F'$ we have

$$\mu_t(\mathcal{O}_C^{\oplus m'} \to F')(\leqslant)\mu_t(\mathcal{O}_C^{\oplus m} \to F)$$

where $(\leqslant)$ means $<$ in the stable case and $\leqslant$ in semistable.

Bradlow pairs

We can form the moduli space $P = P^t_C(r,d)$ of μ_t-semistable pairs $\mathcal{O}_C \to F$ such that F has rank r and degree d. If $t \notin \frac{1}{r!}\mathbb{Z}$ then

- Every semistable pair in $P^t_C(r,d)$ is stable.
- If d is large enough, the moduli space $P^t_C(r,d)$ is a smooth projective variety of dimension $(r^2-r)(g-1)+d$ (for small d it is still virtually smooth).
- The tangent space at $[\mathcal{O}_X \to F]$ is given by

$$\mathrm{Ext}^0([\mathcal{O}_X \to F], F)\,.$$

- There exists a unique (!) universal pair $\mathcal{O}_{P\times C} \to \mathbb{F}$ on $P \times C$.

Example

If $r = 1$ then

1. $M_C(1,d)$ parametrizes degree d line bundles, i.e.

$$M_C(1,d) = \mathrm{Jac}^d(C)$$

is topologically a torus of (real) dimension $2g$.

2. $P_C^t(1,d)$ parametrizes surjective pairs of the form $\mathcal{O}_C \to \mathcal{O}_C(D)$ for some effective divisor D of degree d, i.e.

$$P_C^t(1,d) = C^{[d]} \cong C^{\times d}/\Sigma_d$$

is the symmetric power of C. In particular it does not depend on t.

General story...

More generally we can consider a smooth projective variety X of low dimension ($\leqslant 4$) and a moduli space M of semistable (for some notion of stability) sheaves on X. We don't need M smooth, but only virtually smooth i.e. have a 2-term perfect obstruction theory:

$$\mathrm{Ext}^1(G,G) = \mathrm{Tan}_{[G]}\,, \quad \mathrm{Ext}^2(G,G) = \mathrm{Ob}_{[G]}\,, \quad \mathrm{Ext}^{\geqslant 3}(G,G) = 0\,.$$

Then we get a virtual fundamental class $[M]^{\mathrm{vir}}$ and we can define enumerative invariants by

$$\int_{[M]^{\mathrm{vir}}} \dots$$

Includes many interesting invariants: Donaldson, Seiberg-Witten, Donaldson-Thomas, Pandharipande-Thomas. Another direction are moduli spaces of quiver representations.

Descendents

To get numerical invariants from M we integrate certain natural cohomology classes against the virtual fundamental class.

Definition (Descendent algebra)

Let $\mathbb{D}^X$ be the free (super)commutative $\mathbb{C}$-algebra generated by symbols

$$\mathrm{ch}_i^{\mathrm{H}}(\gamma) \quad \text{for } i \geqslant 0, \gamma \in H^\bullet(X)\,.$$

Definition (Geometric realization of descendents)

Let M be a moduli of sheaves with a universal sheaf $\mathbb{G}$ in $M \times X$. Define the geometric realization morphism $\xi_{\mathbb{G}} : \mathbb{D}^X \to H^\bullet(M)$ by

$$\xi_{\mathbb{G}}\left(\mathrm{ch}_i^{\mathrm{H}}(\gamma)\right) = p_*\left(\mathrm{ch}_{i+\dim(X)-s}(\mathbb{G})q^*\gamma\right) \in H^\bullet(M)$$

for $\gamma \in H^{s,t}(X)$. p, q are the projections of the product onto M and X, respectively.

Descendents for pairs

There is an analogous definition for pairs:

Definition (Pair descendent algebra)

Let $\mathbb{D}^{X,\mathrm{pa}} \cong \mathbb{D}^X \otimes \mathbb{D}^X$ be the free (super)commutative $\mathbb{C}$-algebra generated by symbols

$$\mathrm{ch}_i^{\mathrm{H},\mathcal{V}}(\gamma),\ \mathrm{ch}_i^{\mathrm{H},\mathcal{F}}(\gamma) \quad \text{for } i \geqslant 0, \gamma \in H^\bullet(X)\,.$$

Definition (Geometric realization of pair descendents)

Let P be a moduli of sheaves with a universal pair $q^*V \to \mathbb{F}$ in $X \times P$. Define the geometric realization morphism by

$$\xi_{(q^*V,\mathbb{F})}\left(\mathrm{ch}_i^{H,\mathcal{F}}(\gamma)\right) = p_*\left(\mathrm{ch}_{i+\dim(X)-s}(\mathbb{F})q^*\gamma\right)\,,$$

$$\xi_{(q^*V,\mathbb{F})}\left(\mathrm{ch}_i^{H,\mathcal{V}}(\gamma)\right) = p_*\left(\mathrm{ch}_{i+\dim(X)-s}(q^*V)q^*\gamma\right) = \delta_{i0}\int_X \mathrm{ch}(V)\gamma\,.$$

Virasoro operators

Definition

For $n \geqslant -1$ define the operators $\mathsf{L}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ by $\mathsf{L}_n = \mathsf{R}_n + \mathsf{T}_n$ where:

1. The operator $\mathsf{R}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ is a derivation defined on generators by
$$\mathsf{R}_n \mathrm{ch}_i^{\mathrm{H}}(\gamma) = \left(\prod_{j=0}^{n}(i+j)\right) \mathrm{ch}_{i+n}^{\mathrm{H}}(\gamma).$$

2. The operator $\mathsf{T}_n \colon \mathbb{D}^X \to \mathbb{D}^X$ is the multiplication by the element of $\mathbb{D}^X$ given by
$$\mathsf{T}_n = \sum_{i+j=n} i!j! \sum_s (-1)^{\dim X - p_s^L} \mathrm{ch}_i^{\mathrm{H}}(\gamma_s^L) \mathrm{ch}_j^{\mathrm{H}}(\gamma_s^R)\,,$$
where $\sum_s \gamma_s^L \otimes \gamma_s^R = \Delta_* \mathrm{td}(X)$.

Virasoro operators

They satisfy the Virasoro bracket:

$$[\mathsf{L}_n, \mathsf{L}_m] = (m-n)\mathsf{L}_{n+m}\,.$$

There is also a version $\mathsf{L}_n^{\mathrm{pa}} \colon \mathbb{D}^{X,\mathrm{pa}} \to \mathbb{D}^{X,\mathrm{pa}}$ for pairs. The main difference is in the T_n operator:

$$\mathsf{T}_n^{\mathrm{pa}} = \sum_{i+j=n} i!j! \sum_s (-1)^{\dim X - p_s^L} \mathrm{ch}_i^{H,\mathcal{F}-\mathcal{V}}(\gamma_s^L)\mathrm{ch}_j^{H,\mathcal{F}}(\gamma_s^R)\,.$$

Virasoro constraints for pairs

Conjecture (Virasoro for pairs)

Let P be a moduli space of pairs with universal pair $q^*V \to \mathbb{F}$. For any $D \in \mathbb{D}^{X,\mathrm{pa}}$ and $n \geqslant 0$ we have

$$\int_{[P]^{\mathrm{vir}}} \xi_{(q^*V,\mathbb{F})}\left(\mathsf{L}_n^{\mathrm{pa}}(D)\right) = 0\,.$$

Weight 0 descendents

The formulation of sheaf Virasoro constraints should be independent on the choice of universal sheaf. If $\mathbb{G}$ is a universal sheaf and L is a line bundle on M then $\mathbb{G}' = \mathbb{G} \otimes p^* L$ is another universal sheaf and

$$\xi_{\mathbb{G}'} = \sum_{j \geqslant 0} \frac{c_1(L)^j}{j!} \xi_{\mathbb{G}} \circ \mathrm{R}_{-1}^j \,.$$

Definition

We say that $D \in \mathbb{D}^X$ has weight 0 if $\mathrm{R}_{-1}(D) = 0$. We denote by $\mathbb{D}^X_{\mathrm{wt}_0} \subseteq \mathbb{D}^X$ the algebra of weight 0 descendents.

If $D \in \mathbb{D}^X_{\mathrm{wt}_0}$ then its geometric realization $\xi_{\mathbb{G}}(D)$ does not depend on the choice of $\mathbb{G}$, so we write

$$\int_{[M]^{\mathrm{vir}}} D = \int_{[M]^{\mathrm{vir}}} \xi_{\mathbb{G}}(D) \,.$$

Virasoro constraints for sheaves

Let

$$\mathsf{L}_{\mathrm{wt}_0} = \sum_{n \geqslant -1} \frac{(-1)^n}{(n+1)!} \mathsf{L}_n \mathsf{R}_{-1}^{n+1} .$$

Fact

$$\mathsf{L}_{\mathrm{wt}_0}(D) \in \mathbb{D}^X_{\mathrm{wt}_0} .$$

Conjecture (Virasoro for sheaves)

Let M be a moduli space of sheaves. For any $D \in \mathbb{D}^X$ we have

$$\int_{[M]^{\mathrm{vir}}} \mathsf{L}_{\mathrm{wt}_0}(D) = 0 .$$

Example – rank 2 sheaves on a curve

Let $M = M_C(2, \Delta)$ be the moduli space of stable bundles on a curve C of genus g with rank 2 and fixed determinant Δ of odd degree; this is a smooth moduli space of dimension $3g - 3$.
All integrals of descendents on M can be deduced from integrals of products of certain classes

$$\eta \in H^2(M), \quad \theta \in H^4(M), \quad \zeta \in H^6(M)\,.$$

Thaddeus proved:

$$\int_M \eta^m \theta^k \zeta^p = (-1)^{g-1-p} \frac{m!g!}{(g-p)!} 2^{2g-2-p} \frac{(2^q - 2)B_q}{q!}\,,$$

where $m + 2k + 3p = 3g - 3$ and $q = m + p - g + 1$.
The Virasoro constraints for M are equivalent to

$$(g-p) \int_M \eta^m \theta^k \zeta^p = -2m \int_M \eta^{m-1} \theta^{k-1} \zeta^{p+1}\,.$$

Wall-crossing

Wall-crossing=studying how a moduli space/enumerative invariants change when we change the stability condition. Let's study how $P^t(2,d)$ changes with $t \in \mathbb{R}_+$ for d odd:

1. When $t \gg 1$ there are no μ_t-semistable pairs, i.e. $P^t(2,d)$ becomes empty.
2. When $0 < t \ll 1$, a pair $[\mathcal{O}_C \xrightarrow{s} F]$ is μ_t-semistable if and only if F is stable and $s \neq 0$. Assuming d is large,

   $$P^t(2,d) \to M(2,d)$$

   is a projective bundle with fibers $\mathbb{P}(H^0(F))$.
3. The moduli space $P^t(2,d)$ changes when we cross a t for which $P^t(2,d)$ has strictly semistable objects. Such t is called a wall.

Wall-crossing

4. If $P^t(2,d)$ has strictly semistable objects then t is an odd integer $\leqslant d$. The strictly semistable pairs are (S-equivalent to)
$$(\mathcal{O}_X \to F_1) \oplus (0 \to F_2)$$
with
$$\mu_t(\mathcal{O}_X \to F_1) = \mu_t(0 \to F_2).$$
I.e.
$$(\mathcal{O}_X \to F_1) \in P^t\left(1, \frac{d-t}{2}\right), \quad (0 \to F_2) \in M\left(1, \frac{d+t}{2}\right)$$

Wall-crossing

5. (Thaddeus) Suppose t is a wall and $t_- < t < t_+$. Then there is a common blow-up

$$\begin{tikzcd} & \widetilde{P}^t(2,d) \arrow[dl] \arrow[dr] & \\ P^{t_-}(2,d) & & P^{t_+}(2,d) \end{tikzcd}$$

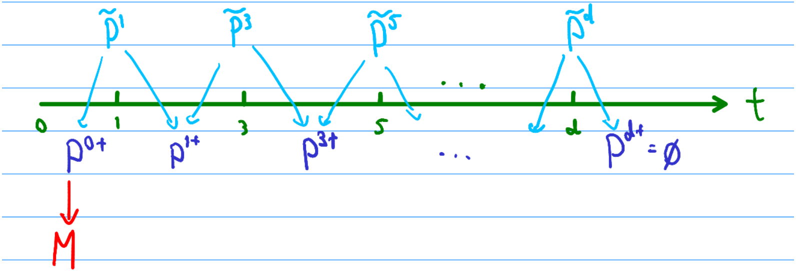
The exceptional divisor of the two blowups is the same and is a $\mathbb{P}^a \times \mathbb{P}^b$-bundle over

$$P^t\left(1, \frac{d-t}{2}\right) \times M\left(1, \frac{d+t}{2}\right).$$

6. Joyce's wall-crossing formula:

$$P^{t_-}(2,d) = P^{t_+}(2,d) - \left[M\left(1, \frac{d+t}{2}\right), P^t\left(1, \frac{d-t}{2}\right)\right].$$

Wall-crossing



Joyce's vertex algebra

1. Joyce defines a vertex algebra $V_\bullet$ and with an associated Lie algebra $\check{V}_\bullet = V^\bullet / T(V_\bullet)$. They are defined as homologies of the higher stack parametrizing complexes on X.
2. We can roughly think of $V_\bullet, \check{V}_\bullet$ as the duals to $\mathbb{D}^X$, $\mathbb{D}^X_{\mathrm{wt}_0}$, respectively. I.e an element in $V_\bullet$ carries the information of how to integrate descendents. Similarly, an element of $\check{V}_\bullet$ carries information of how to integrate weight 0 descendents.
3. A moduli space M defines a class $[M]^{\mathrm{vir}} \in \check{V}_\bullet$ and a moduli space with universal sheaf $\mathbb{G}$ an element $[M]^{\mathrm{vir}}_{\mathbb{G}} \in V_\bullet$.

Joyce's vertex algebra

1. Joyce extends the definition of the classes $[M]^{\mathrm{vir}} \in \check{V}_\bullet$ to the case when M has strictly semistable sheaves.
2. Wall-crossing formulas are written in terms of the Lie bracket on $\check{V}_\bullet$.
3. (J. Gross+BLM) For curves and surfaces, the vertex algebra $V_\bullet$ is isomorphic to a (generalized) lattice vertex algebra.
4. We define a pair version $V_\bullet^{\mathrm{pa}}$ of Joyce's vertex algebra. A moduli space of pairs naturally defines an element $[P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})}$ induced by a universal pair $q^*V \to \mathbb{F}$.

Conformal element

Vertex algebras often come with a conformal element $\omega \in V_\bullet$. The most important property of the conformal element is that it induces operators

$$\mathrm{L}_n \colon V_\bullet \to V_\bullet, \quad n \in \mathbb{Z}$$

via the state-field correspondence that form a representation of the Virasoro Lie algebra:

$$[\mathrm{L}_n, \mathrm{L}_m] = (n-m)\mathrm{L}_{m+n} + \delta_{m+n,0}\, c\, \frac{m^3 - m}{12}\, \mathrm{id}\ .$$

The constant $c \in \mathbb{C}$ is called the central charge of ω. A vertex algebra together with a conformal element is called a vertex operator algebra.

Conformal element in Joyce's VA

Theorem (Bojko-Lim-M)

Let X be a point, a curve or a surface with $h^{0,2} = 0$. Then there is a conformal element ω is the pair vertex algebra $V_\bullet^{\mathrm{pa}}$.
Under the duality between $V_\bullet^{\mathrm{pa}}$ and $\mathbb{D}^{X,\mathrm{pa}}$, the Virasoro fields L_n induced by ω are dual to the pair Virasoro operators $\mathsf{L}_n^{\mathrm{pa}}$ defined in the algebra of descendents $\mathbb{D}^{X,\mathrm{pa}}$.

The proof relies on Gross' isomorphism between $V_\bullet^{\mathrm{pa}}$ and a lattice vertex algebra and on a construction by Kac. Kac construction needs a choice of a maximal isotropic decomposition of the fermionic part, which in our case is

$$H^{\mathrm{odd}}(X) = H^{\bullet,\bullet+1}(X) \oplus H^{\bullet+1,\bullet}(X)\,.$$

Physical states

There is a vertex algebra notion of physical states that roughly corresponds to elements of $V_\bullet$ or $\check{V}_\bullet$ that satisfy Virasoro constraints:

$$P_i = \{v \in V_\bullet \colon \mathrm{L}_n(v) = \delta_{n0} i v \,, n \geqslant 0\} \subseteq V_\bullet \,,$$
$$\check{P}_0 = P_1 / T(P_0) \subseteq \check{V}_\bullet \,.$$

Proposition

Under some conditions, $\overline{u} \in \check{P}_0$ if and only if

$$0 = [\overline{u}, \omega] = \sum_{n \geqslant -1} \frac{(-1)^n}{(n+1)!} T^{n+1} \mathrm{L}_n(u) \,.$$

Physical states

Corollary (Bojko-Lim-M)

1. *A moduli of sheaves M satisfies the sheaf Virasoro constraints if and only if*
$$[M]^{\mathrm{vir}} \in \check{P}_0$$
is a physical state.

2. *A moduli of pairs P with universal pair $q^*V \to \mathbb{F}$ satisfies the pair Virasoro constraints if and only if*
$$[P]^{\mathrm{vir}}_{(q^*V,\mathbb{F})} \in P_0^{\mathrm{pa}}$$
is a physical state.

Wall-crossing compatibility

Proposition

1. *The subspace* $\check{P}_0 \subseteq \check{V}_\bullet$ *is a Lie subalgebra, i.e.*

$$\overline{u}, \overline{v} \in \check{P}_0 \Rightarrow [\overline{u}, \overline{v}] \in \check{P}_0 \,.$$

2. *The subspace* $P_0 \subseteq V_\bullet$ *is a Lie algebra subrepresentation of* $\check{P}_0 \subseteq \check{V}_\bullet$*, i.e.*

$$\overline{u} \in \check{P}_0,\ v \in P_0 \Rightarrow [\overline{u}, v] \in P_0 \,.$$

This proposition translates to a compatibility between the Virasoro constraints and wall-crossing in moduli spaces of sheaves!

Results

Theorem (Bojko-Lim-M)

The Virasoro constraints hold for the following moduli spaces:

1. *Moduli spaces of stable bundles on curves* $M_C(r,d)$*;*
2. *Moduli spaces of stable torsion-free sheaves* $M_S^H(r,\beta,n)$ *on surfaces* S *with* $h^{0,1}=h^{0,2}=0$ *and a polarization* H*;*
3. *Moduli spaces of stable 1 dimensional sheaves* $M_S^H(\beta,n)$ *on surfaces* S *with* $h^{0,1}=h^{0,2}=0$ *and a polarization* H*.*

Sketch of proof

I will focus on the case of curves. The proof goes through the strategy that was described before:

1. We prove by induction on r that $M(r,d)$ and $P^t(r,d)$ satisfy the sheaf and the pair Virasoro constraints, respectively.
2. In the base case $r = 1$,

$$M(1,d) = \mathrm{Jac}(C) \text{ and } P^t(1,d) = C^{[d]}.$$

   Both cases can be proven "by hand". For surfaces, everything can be reduced to the Hilbert scheme of points which was proven earlier (M-Oblomkov-Okounkov-Pandharipande, M).
3. For $r > 1$ the moduli space $P^t(r,d)$ becomes empty for large t, so it trivially satisfies Virasoro constraints.

Sketch of proof

4. Using the wall-crossing compatibility, $P^t(r,d)$ satisfies the pair Virasoro constraints for every t. Induction guarantees that all the wall-crossing terms already satisfy Virasoro, e.g.

$$[P^{t_-}(2,d)] = [P^{t_+}(2,d)] - \left[M\left(1, \frac{d+t}{2}\right), P^t\left(1, \frac{d-t}{2}\right)\right].$$

5. If $\gcd(r,d) = 1$ then $P^t(r,d) \to M(r,d)$ is a projective bundle for t close to 0. If $\gcd(r,d) > 1$ it "looks like" a projective bundle up to corrections by lower rank wall-crossing terms.
6. The projective bundle structure can be used to prove that if $P^t(r,d)$ satisfies pair Virasoro for t close to 0 then $M(r,d)$ satisfies sheaf Virasoro.

Thanks!