Semi-Supervised Learning and the ∞-Laplacian

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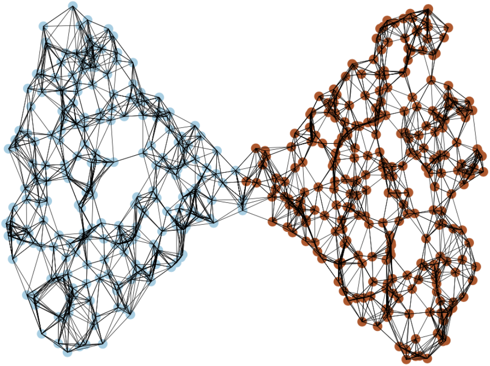
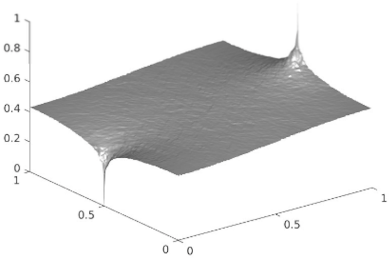
SSL and graph based algorithms



SSL - smoothness assumption

$$\min_{\substack{u:X\to\mathbb{R}\\ u=g \text{ on } \Gamma}} \sum_{x,y\in X} \omega_{xy} \left| u(x) - u(y) \right|^2$$

$$\min_u \int_\Omega |\nabla u|^2 \, dx \qquad \hookrightarrow \qquad -\mathrm{div}\left(Du\right) = 0$$

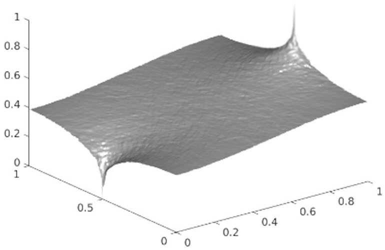


(a) $p = 2$

SSL - smoothness assumption

$$\min_{\substack{u:X\to\mathbb{R}\\ u=g \ \ on \ \ \Gamma}} \sum_{x,y\in X} \omega_{xy} \left| u(x) - u(y) \right|^{p}$$

$$\min_u \int_\Omega |\nabla u|^p \, dx \qquad \hookrightarrow \qquad -\mathrm{div}\left(|Du|^{p-2} Du\right) = 0$$

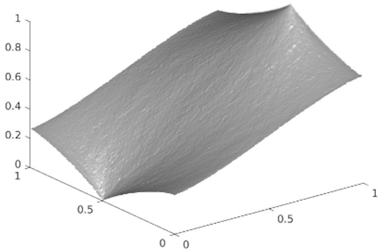


(b) $p = 2.5$

SSL - smoothness assumption

$$\min_{\substack{u:X\to\mathbb{R}\\ u=g \text{ on } \Gamma}} \max_{x,y\in X} \omega_{xy} \left| u(x) - u(y) \right|$$

$$\min_u \|\nabla u\|_\infty \qquad \hookrightarrow \qquad \Delta_\infty u = 0$$



(c) $p = \infty$

The infinity-Laplace equation

$$\begin{aligned}\Delta_\infty u &:= \langle D^2 u\, Du, Du \rangle \\ &= \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} \\ &= 0\end{aligned}$$

- nonlinear and degenerate
- not in divergence form

Lipschitz functions

Definition. Let $X \subset \mathbb{R}^n$. A function $f : X \to \mathbb{R}$ is Lipschitz continuous on X, equivalently $f \in Lip(X)$, if there exists a constant $L \in [0, \infty)$ such that

$$\left| f(x) - f(y) \right| \leq L |x - y|, \quad \forall x, y \in X. \tag{1}$$

Any $L \in [0, \infty)$ for which (1) holds is called a Lipschitz constant for f in X. The least constant $L \in [0, \infty)$ for which (1) holds is denoted by $\mathrm{Lip}_f(X)$.

If there is no L for which (1) holds, we write $\mathrm{Lip}_f(X) = \infty$.

The Lipschitz Extension Problem

Given $f \in Lip(\partial U)$, find $u \in Lip(\overline{U})$ such that

$$u = f \quad \text{on } \partial U$$

and

$$\mathrm{Lip}_u(\overline{U}) = \mathrm{Lip}_f(\partial U)$$

The McShane-Whitney extensions

Definition. The McShane-Whitney extensions of $f \in Lip(\partial U)$ are the functions defined in $\overline{U}$ by

$$\mathcal{MW}_*(f)(x) := \sup_{z\in\partial U} F_z(x) = \sup_{z\in\partial U} \left\{ f(z) - \mathrm{Lip}_f(\partial U)|x-z| \right\}$$

and

$$\mathcal{MW}^*(f)(x) := \inf_{y\in\partial U} G_y(x) = \inf_{y\in\partial U} \left\{ f(y) + \mathrm{Lip}_f(\partial U)|x-y| \right\}.$$

Problem solved?

Theorem. The McShane-Whitney extensions, $\mathscr{MW}_*(f)$ and $\mathscr{MW}^*(f)$, solve the Lipschitz extension problem for $f \in Lip(\partial U)$ and if u is any other solution to the problem then

$$\mathscr{MW}_*(f) \le u \le \mathscr{MW}^*(f) \quad \text{in} \quad \overline{U}.$$

Absolutely Minimising Lipschitz

Definition. A function $u \in C(U)$ is absolutely minimising Lipschitz on U, and we write $u \in \mathrm{AML}(U)$, if

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V), \quad \forall\, V \subset\subset U.$$

LEP: Given $f \in Lip(\partial U)$, find $u \in C(\overline{U})$ such that

$$u \in \mathrm{AML}(U) \qquad \text{and} \qquad u = f \ \text{ on } \partial U.$$

Comparison with Cones

Definition. A cone with vertex $x_0 \in \mathbb{R}^n$ is a function of the form

$$C(x) = a + b|x - x_0|, \quad a, b \in \mathbb{R}.$$

The height of C is a and its slope is b.

Definition. A function $w \in C(U)$ enjoys comparison with cones from above in U if, for every $V \subset\subset U$ and every cone C whose vertex is not in V,

$$w \leq C \text{ on } \partial V \quad \Longrightarrow \quad w \leq C \text{ in } V.$$

CWC and AML

Theorem. A function $u \in C(U)$ is absolutely minimising Lipschitz in U if, and only if, it enjoys comparison with cones in U.

Proof. Sufficiency only. Suppose u enjoys comparison with cones in U and let $V \subset\subset U$. We want to show that

$$\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\partial V).$$

Since $u \in C(\overline{V})$, we have $\mathrm{Lip}_u(V) = \mathrm{Lip}_u(\overline{V})$ (exercise!). Then, as $\partial V \subset \overline{V}$, we trivially have that $\mathrm{Lip}_u(V) \geq \mathrm{Lip}_u(\partial V)$ and it remains to prove the other inequality.

First, observe that, for any $x \in V$,

$$\mathrm{Lip}_u(\partial(V \setminus \{x\})) = \mathrm{Lip}_u(\partial V \cup \{x\}) = \mathrm{Lip}_u(\partial V). \tag{2}$$

To see this holds we need only check that, for any $y \in \partial V$,

$$|u(y) - u(x)| \leq \mathrm{Lip}_u(\partial V)\,|y - x|,$$

which is equivalent to

$$u(y) - \mathrm{Lip}_u(\partial V)\,|x - y| \leq u(x) \leq u(y) + \mathrm{Lip}_u(\partial V)\,|x - y|. \tag{3}$$

This clearly holds for any $x \in \partial V$ but what we want to prove is that it holds for $x \in V$. Let's focus on the second inequality in (3). The right-hand side can be regarded as the cone

$$C(x) = u(y) + \mathrm{Lip}_u(\partial V)\,|x - y|,$$

centred at $y \in \partial V$. Since $y \notin V$ and u enjoys comparison with cones from above in U, the inequality holds in V because it holds on ∂V. To obtain the first inequality in (3), we argue analogously, using comparison with cones from below.

Now let $x, y \in V$. Using (2) twice, we obtain

$$\mathrm{Lip}_u(\partial V) = \mathrm{Lip}_u(\partial(V \setminus \{x\})) = \mathrm{Lip}_u\left(\partial\left(V \setminus \{x, y\}\right)\right).$$

Since $x, y \in \partial\left(V \setminus \{x, y\}\right) = \partial V \cup \{x, y\}$, we have

$$|u(x) - u(y)| \le \mathrm{Lip}_u\left(\partial\left(V \setminus \{x, y\}\right)\right) |x - y| = \mathrm{Lip}_u(\partial V)\, |x - y|.$$

Thus

$$\mathrm{Lip}_u(V) \le \mathrm{Lip}_u(\partial V).$$

Viscosity solutions

Definition. A function $w \in C(U)$ is a viscosity subsolution of $\Delta_\infty u = 0$ (or a viscosity solution of $\Delta_\infty u \geq 0$ or ∞–subharmonic) in U if, for every $\hat{x} \in U$ and every $\varphi \in C^2(U)$ such that $w - \varphi$ has a local maximum at $\hat{x}$, we have

$$\Delta_\infty \varphi(\hat{x}) \geq 0.$$

A function $w \in C(U)$ is ∞–superharmonic in U if $-w$ is ∞–subharmonic in U. A function $w \in C(U)$ is ∞–harmonic in U if it is both ∞–subharmonic and ∞–superharmonic in U.

Consistency

Lemma. If $u \in C^2(U)$ then u is ∞−harmonic in U if, and only if, $\Delta_\infty u = 0$ in the pointwise sense.

Proof. Suppose u is ∞−harmonic. Then it is ∞−subharmonic and we take $\varphi = u$ in the definition. Since every point $x \in U$ will then be a local maximum of $u - \varphi \equiv 0$, $\Delta_\infty u(x) \geq 0$, for every $x \in U$. Since also $-u$ is ∞−subharmonic, we get in addition

$$\Delta_\infty(-u)(x) \geq 0 \;\Leftrightarrow\; -\Delta_\infty u(x) \geq 0 \;\Leftrightarrow\; \Delta_\infty u(x) \leq 0, \quad \forall x \in U$$

and so $\Delta_\infty u = 0$ in the pointwise sense.

Reciprocally, suppose $\Delta_\infty u = 0$ in the pointwise sense and take $\hat{x} \in U$ and $\varphi \in C^2(U)$ such that $u - \varphi$ has a local maximum at $\hat{x}$. We want to prove that $\Delta_\infty \varphi(\hat{x}) \geq 0$, thus showing that u is ∞–subharmonic (the ∞–superharmonicity is obtained in an analogous way).

We have, since $u - \varphi \in C^2(U)$ and $\hat{x} \in U$ is a local maximum,

$$D(u-\varphi)(\hat{x}) = 0 \;\Leftrightarrow\; Du(\hat{x}) = D\varphi(\hat{x})$$

and

$$D^2(u-\varphi)(\hat{x}) \leq 0 \;\Leftrightarrow\; \langle D^2 u(\hat{x})\xi, \xi\rangle \leq \langle D^2\varphi(\hat{x})\xi, \xi\rangle, \quad \forall x \in \mathbb{R}^n.$$

Then

$$\begin{aligned}
\Delta_\infty \varphi(\hat{x}) &= \langle D^2\varphi(\hat{x}) D\varphi(\hat{x}), D\varphi(\hat{x})\rangle \\
&\geq \langle D^2 u(\hat{x}) D\varphi(\hat{x}), D\varphi(\hat{x})\rangle \\
&= \langle D^2 u(\hat{x}) Du(\hat{x}), Du(\hat{x})\rangle \\
&= \Delta_\infty u(\hat{x}) \\
&= 0.
\end{aligned}$$

Aronsson's example

$$u(x,y) = x^{\frac{4}{3}} - y^{\frac{4}{3}}$$

is ∞–subharmonic in $\mathbb{R}^2$. The proof that it is also ∞–superharmonic is analogous.

Take any point $(x_0, y_0) \in \mathbb{R}^2$ and $\varphi \in C^2(\mathbb{R}^2)$ such that $u - \varphi$ has a local maximum at (x_0, y_0). We start by observing that, since $u \in C^1(\mathbb{R}^2)$,

$$D(u-\varphi)(x_0, y_0) = 0$$

and, consequently,

$$\varphi_x(x_0, y_0) = u_x(x_0, y_0) = \frac{4}{3} x_0^{\frac{1}{3}} \tag{4}$$

and

$$\varphi_y(x_0, y_0) = u_y(x_0, y_0) = -\frac{4}{3} y_0^{\frac{1}{3}}. \tag{5}$$

We first exclude the case $x_0 = 0$. If $\varphi \in C^2(\mathbb{R}^2)$ is such that $u - \varphi$ has a local maximum at $(0, y_0)$, then

$$\begin{aligned} & (u-\varphi)(x, y_0) \leq (u-\varphi)(0, y_0) \\ \Leftrightarrow \quad & x^{\frac{4}{3}} \leq \varphi(x, y_0) - \varphi(0, y_0), \end{aligned} \tag{6}$$

for every x in a neighbourhood of 0 and this simply can not happen. In fact, letting $F(x) = \varphi(x, y_0) - \varphi(0, y_0)$, we have $F(0) = 0$ and also

$$F'(0) = \varphi_x(0, y_0) = u_x(0, y_0) = 0.$$

Then, by Taylor's theorem,

$$\lim_{x \to 0} \frac{F(x)}{x^2} = \frac{F''(0)}{2} = \frac{\varphi_{xx}(0, y_0)}{2} < +\infty.$$

On the other hand, if (6) would hold,

$$\lim_{x \to 0} \frac{F(x)}{x^2} \geq \lim_{x \to 0} \frac{x^{\frac{4}{3}}}{x^2} = \lim_{x \to 0} x^{-\frac{2}{3}} = +\infty,$$

a contradiction.

We next consider the case $x_0 \neq 0$ and $y_0 = 0$. If $\varphi \in C^2(\mathbb{R}^2)$ is such that $u - \varphi$ has a local maximum at $(x_0, 0)$, then

$$\begin{aligned} & (u-\varphi)(x,0) \leq (u-\varphi)(x_0,0) \\ \Leftrightarrow \quad & x^{\frac{4}{3}} - \varphi(x,0) \leq x_0^{\frac{4}{3}} - \varphi(x_0,0), \end{aligned} \tag{7}$$

for every x in a neighbourhood of x_0. This means that the function

$$G(x) = x^{\frac{4}{3}} - \varphi(x,0)$$

has a local maximum at the point x_0. Since it is of class C^2 in a neighbourhood of x_0 (small enough that it does not contain 0), we have $G'(x_0) = 0$ and

$$G''(x_0) \leq 0 \quad \Leftrightarrow \quad \varphi_{xx}(x_0,0) \geq \frac{4}{9} x_0^{-\frac{2}{3}} \geq 0. \tag{8}$$

Then, using (4), (5) and (8),

$$\begin{aligned}\Delta_\infty \varphi(x_0,0) &= \left(\varphi_x^2\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + \varphi_y^2\varphi_{yy}\right)(x_0,0)\\ &= \varphi_x^2(x_0,0)\varphi_{xx}(x_0,0) \geq 0\end{aligned}$$

as required.

Finally, if both $x_0 \neq 0$ and $y_0 \neq 0$, u is C^2 in a neighbourhood of (x_0, y_0) and the equation is satisfied in the pointwise sense, the calculation being trivial.

CWC and ∞-harmonic

Theorem. A function $u \in C(U)$ is ∞–subharmonic if, and only if, it enjoys comparison with cones from above.

$$\boxed{AML \Longleftrightarrow CWC \Longleftrightarrow \infty - \text{harmonic}}$$

Regularity

Theorem [Harnack Inequality]. Let $0 \geq u \in C(U)$ satisfy

$$u(x) \leq u(y) + \max_{w \in \partial B_r(y)} \left(\frac{u(w) - u(y)}{r} \right) |x - y|, \qquad (9)$$

for $x \in B_r(y) \subset\subset U$.

If $z \in U$ and $R < d(z)/4$, then

$$\sup_{B_R(z)} u \leq \frac{1}{3} \inf_{B_R(z)} u.$$

Proof. Take arbitrary $x, y \in B_R(z)$. Then (9) holds for r sufficiently large. Let $r \uparrow d(y)$ to get, using the fact that $u \leq 0$,

$$u(x) \leq u(y)\left(1 - \frac{|x-y|}{d(y)}\right). \tag{10}$$

We have

$$d(y) \geq 3R \qquad \text{and} \qquad |x-y| \leq 2R$$

and thus, from (10), we obtain

$$u(x) \leq u(y)\left(1 - \frac{2R}{3R}\right) = \frac{1}{3}u(y)$$

and the result follows.

Local Lipschitz regularity

Theorem. If $u \in C(U)$ is ∞−harmonic then it is locally Lipschitz and hence (by Rademacher's theorem) differentiable almost everywhere.

Proof. We know u satisfies (9), since it enjoys comparison with cones from above.

Take $z \in U$, $R < d(z)/4$ and $x, y \in B_R(z)$.

Assume first that $u \leq 0$.

Then (10) and the Harnack inequality hold, and we get

$$\begin{aligned} u(x)-u(y) &\leq -u(y)\frac{|x-y|}{d(y)} \\ &\leq -\inf_{B_R(z)} u \ \frac{|x-y|}{3R} \\ &\leq -\sup_{B_R(z)} u \ \frac{|x-y|}{R}. \end{aligned}$$

If u is not non-positive, then this holds with u replaced by

$$v = u - \sup_{B_{4R}(z)} u \leq 0,$$

since $v = u + C$ still enjoys comparison with cones from above. We thus obtain

$$\begin{aligned} u(x) - u(y) = v(x) - v(y) &\leq -\sup_{B_R(z)} v \, \frac{|x-y|}{R} \\ &= \left(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u \right) \frac{|x-y|}{R} \end{aligned}$$

and, interchanging x and y,

$$|u(x) - u(y)| \leq \frac{1}{R} \left(\sup_{B_{4R}(z)} u - \sup_{B_R(z)} u \right) |x-y|.$$