STOKES PHENOMENON
and
DYNAMICS ON WILD CHARACTER VARIETIES
of
PAINLEVÉ EQUATIONS

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In the middle of the XX-th century, starting from the very simple notion of scheme, A. Grothendieck created a geometric theory unifying algebraic geometry and arithmetic. Now it is time to push forward the machinery and to create a larger geometric theory including also in the unification some “old complex analysis”, as classical special functions: Hypergeometric functions, Kummer functions, Bessel functions Mathieu and spheroïdal functions, and “the special functions of the XXI-st century”, the Painlevé functions (and their discrete analogs).

There are strong evidences of the fact that this picture is very near of some theories of theoretical physics. A simple example is the anomalous magnetic moment of the electron (QED), with Gevrey divergent series and the apparition of periods (in Kontsevich-Zagier sense) in the coefficients of the series. Cf. also the CFT litterature, in particular some Witten papers...

A big continent is just emerging from the mist...

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In the minicourse I will illustrate this picture with the exemple of Painlevé equations, using classical algebraic geometry (in a partially new way) but also a new geometry, the "wild geometry". I will not introduce a heavy and complicated formalism and I will remain at a quite elementary level. In a first step, I will return to simple and classical examples using a "new" look (in the line of Euler, Poincaré, Stokes, Ramanujan, Watson, Hardy...). It is important to get rid of bad habits inherited in particular from the use of classical asymptotics: the so-called Poincaré asymptotics, used without the original intuitions of Poincaré. A red thread to follow is to try to deal with "exact formulae" allowing precise numerical computations. The wild geometry is also related to a very powerful "combinatorics", the Ecalle resurgence theory.

The Painlevé equations

The Painlevé property

Paul Painlevé discovered (some of) its equations in an effort of classification of the second order algebraic differential equations (in the complex domain) :

$$y'' = R(x, y, y'),$$

with R rational, possessing the so called *Painlevé property*, which controls the "ramification points": *the only possible movable singularities of the solutions are poles*.

According to Painlevé and Gambier, such equations can be reduced to a list of 50 equations (canonical forms) and if one excludes "already known" equations (linear equations, Riccati equations, elliptic ODE...) it remains only 6 new families: P_I, P_{II}, P_{III}, P_{IV}, P_V, P_{VI}, the so called *Painlevé differential equations*.

The Painlevé differential equations

P_I: $\frac{d^2y}{dt^2} = 6y^2 + t.$

P_{II}: $\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha.$

P_{III}: $\frac{d^2y}{dt^2} = \frac{1}{y}(\frac{dy}{dt})^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}.$

P_{IV}: $\frac{d^2y}{dt^2} = \frac{1}{2y}(\frac{dy}{dt})^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}.$

P_V: $\frac{d^2y}{dt^2} = (\frac{1}{2y} + \frac{1}{y-1})(\frac{dy}{dt})^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t}(\alpha y + \frac{\beta}{y}) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}.$

P_{VI}: $\frac{d^2y}{dt^2} = \frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}))(\frac{dy}{dt})^2 - (\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t})\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}).$

With *parameters* $\alpha, \beta, \gamma, \delta \in \mathbf{C}$.

Non-linear second order ODEs whose all *moving singularities* are *poles* (Painlevé property).

Fixed singularities : ∞ for all equations, plus 0 for P_{II} (generic), $P_{III}, P_{IV}, P_V, P_{VI}$, plus 1 for P_{VI}.

Figure: Paul Painlevé 1863 – 1933

The Painlevé equations

Painlevé property or isomonodromic deformations ?

In fact, the equation P_{VI} was discovered by Richard Fuchs (son of Lazarus Fuchs), in relation with *isomonodromic deformations of linear O.D.E.* (1905-1907). A completely different property.

Later, in 1919, Garnier discovered that the others Painlevé equations translate similar phenomena: *iso-irregular deformations of linear O.D.E.*.

The equivalence between the two approaches: Painlevé property and iso-monodromy (or iso-irregularity) is only true for equations of order 2 (a miracle...).

All O.D.E. coming from isomonodromic (or more generally iso-irregular) deformations have the Painlevé property (Malgrange, Miwa...) but the converse can be false for equations of order > 2.

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Painlevé equations were forgotten for a long time. The subject was rediscovered and received new impulsions around 1980: Okamoto, Sato, Jimbo, Miwa, Ueno...

During the last 40 years the subject "exploded" and there are an enormous number of papers, from the mathematical viewpoint or in relation with some applications (in particular in theoretical physics).

Some works are very technical and the technics are very different. In this minicourse I will try to explain the main ideas and how to cross the technics. I will also present some approaches, simple, efficient and apparently new (it is a work in progress in collaboration with Martin Klimes and Emmanuel Paul).

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1. To use the methods of the holomorphic dynamical systems and holomorphic foliations. The main historical sources are Painlevé work (in particular the *Leçons de Stockholm* and Poincaré and Dulac work. The main tools are: blow-ups, ramified blow-ups (Briot and Bouquet, Chiba...) formal normal forms, analytic normal forms (Martinet-Ramis...), invariant subspaces (Hadamard-Perron), k-summability (Ramis), resurgence (Ecalle)...
   An essential breakthrough in these lines is the work of Kazuo Okamoto ?? (which began in Strasbourg under the impulsion of Raymond Gérard): the Okamoto space of initial conditions.
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The main tools are Schlesinger equations, Lax pairs, spaces of representations (character varieties), generalized monodromy data and wild monodromy representations (Jimbo-Miwa-Ueno, Martinet-Ramis, van der Put-Saito, Boalch...), Riemann-Hilbert methods and Deift-Zhu non-linear saddle method (Fokas, Its, Kapaev, Kitaev, Novokhchenov).

The main heuristic principle is :

All the asymptotics of the solutions of a Painlevé equation are parametrized by the (generalized) monodromy data of the linearized equation.

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- Geometry of spaces of initial conditions of the Painlevé equations
- Character varieties
- Isomonodromic deformations and Riemann-Hilbert map
- The dynamics of P_{VI}

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EXAMPLES OF ALGEBRAIC ODEs

Basic bricks and basic tools

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Basic bricks and basic tools

LINEAR ALGEBRAIC ODEs

Linear differential equations on $P^1(\mathbf{C})$

Homogeneous equations :

$$Dy = a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_o(x)y = 0,$$

$a_n, a_{n-1}, \ldots, a_0 \in \mathbf{C}(x)$ ($\mathbf{C}[x]$ case is sufficient...), and :

$$D = a_n(d/dx)^n + a_{n-1}(d/dx)^{n-1} + \cdots + a_0 \in \mathbf{C}(x)[d/dx];$$

$\mathcal{D} := \mathbf{C}(x)[d/dx]$, *non commutative polynomials* (Oystein Ore 1933); a non-commutative version of the planar algebraic geometry (symplectic structure on the co-tangent space: $\mathbf{C}(x)[d/dx] \to \mathbf{C}(x)[\xi]$):

$$\left[\frac{d}{dx}, x\right] = \frac{d}{dx}x - x\frac{d}{dx} = 1, \quad x\xi - \xi x = 0.$$

Linear differential equations $\leftrightarrow$ *$\mathcal{D}$-modules*

(Malgrange 1949-1950, Bernstein 1971, Sato-Kawai-Kashiwara 1970, ...).

Linear ODEs on the projective line $P^1(\mathbf{C})$

Coordinates on $P^1(\mathbf{C})$: $xw = 1$, $wd/dw = -xd/dx$.

Singularities $\Sigma(D) \subset P^1(\mathbf{C})$ of $Dy = 0$: the zeroes of a_n and (perhaps...) ∞. They are *fixed singularities*: the only possible singularities of a solution are in $\Sigma(D)$, "the singularities of the equation".

if $x_0 \in P^1(\mathbf{C}) \setminus \Sigma(D)$, then there exists a fundamental system of solutions in a neighborhood of x_0 (Cauchy) and any germ of holomorphic solution can be uniquely continued along any continuous path γ in $P^1(\mathbf{C}) \setminus \Sigma(D)$ with origin x_0.

If γ is a loop at x_0 ($\gamma(0) = \gamma(1) = x_0$), then it induces a linear automorphism of the vector space $Sol_{x_0}(D)$ of germs of solutions at x_0. It is the *monodromy isomorphism*. it depends only of the homotopy class of γ, an element of the fundamental group $\pi_1\left(P^1(\mathbf{C}) \setminus \Sigma(D), x_0\right)$.

Linear equations: the basic bricks
(On **C** or on $P^1(\mathbf{C})$)

For a linear ODE there are two types of singular points :

- the regular-singular points, the "simple case", completely understood at the end of XIX-th century;
- the irregular singularities, completely understood only during the eighties.

The Fuchs equations are the equations with only regular-singular points. I will begin with very simple (and however very interesting !) examples.

Examples of Fuchs equations

1. Non homogeneous equation : $xy' = 1$. Solutions :

$$y = \log x + C,\ C \in \mathbf{C}.$$

We have :

$$Dy = \frac{d}{dx}\left(x\frac{d}{dx} - 1\right)y = \left(x\frac{d}{dx} - 1\right)\frac{d}{dx}y = xy'' + y' = 0$$

The space of solutions of the homogeneous equation $Dy = 0$ is $C + C' \log x$, $C, C' \in \mathbf{C}$.

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The monodromy along a simple loop around the origin is $\log x \mapsto \log x + 2i\pi$, $1 \mapsto 1$. The monodromy matrix is $\begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix}$. It is *unipotent*.

2. $Dy = (xd/dx - \alpha)y = xy' - \alpha y = 0, \quad \alpha \in \mathbf{C}$.
Then $y = Cx^\alpha = C\, e^{\alpha \log x}$ $(C \in \mathbf{C})$. Along a simple positive loop around 0, $\log x \mapsto \log x + 2i\pi$ and $y \mapsto e^{2i\pi\alpha} y$. The monodromy is the multiplication by $e^{2i\pi\alpha}$; α is the monodromy exponent. Particular cases: $\alpha \in \mathbf{Z}$, the monodromy is trivial, $\alpha \in \mathbf{Q}$, the monodromy is of finite order.

3.
$Dy = (xd/dx - \alpha)(xd/dx - \beta)y = y'' - (\alpha+\beta)y - \alpha\beta y = 0$, $\alpha,\ \beta \in \mathbf{C}$.

If $\alpha - \beta \notin \mathbf{Z}$, then the space of solutions is $\mathbf{C}_1 x^\alpha + C_2 x^\beta$.
If $\alpha - \beta \in \mathbf{Z}$ (the *resonant* case), $x^\alpha = x^\beta$.
We saw above that, for $\alpha = 0$, $\beta = 1$, the space of solutions is $C_1 + C_2 \log x$. The general case is similar : $x^\alpha (C_1 + C_2 \log x)$.

In the non-resonant case, the monodromy matrix is $\begin{pmatrix} e^{2i\pi\alpha} & 0 \\ 0 & e^{2i\pi\beta} \end{pmatrix}$. It is *semi-simple*.

In the resonant case, the monodromy matrix is $e^{2i\pi\alpha}\begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix}$.

Linear equations: singularities and monodromy
(Algebraic equations on $P^1(\mathbf{C})$)

$a_2y'' + a_1(x)y' + a_0(x) = 0$, $a_2, a_1, a_0 \in \mathbf{C}(x)$;
$y'' + p_1(x)y' + p_0(x) = 0$, $p_1, p_0 \in \mathbf{C}(x)$;
a is a regular point if p_1 and p_0 are analytic at $x = a$;
a is a regular singular point (singularity of Fuchs type) if p_1 has a pole of order ≤ 1 at a and p_0 has a pole of order ≤ 2 at a;
$x^2y'' + xq_1(x)y' + q_0(x) = 0$;
otherwise a is an irregular singular point.
At infinity: $w = 1/x$, $xd/dx = -wd/dw$.
Evident generalisations for equations of order n.

An example of Irregular equation

The Euler equation, Euler *De seriebus divergentibus* 1760

The basic brick of the linear (or non-linear !) *irregular* equations is the Euler equation :

$$x^2y' + y = x.$$

Using this example we can introduce some fundamental (inter-related) tools :

- Formal solutions (divergent);
- Poincaré asymptotic expansions;
- Gevrey asymptotic expansions and 1-summability,
- Borel-Laplace summation and resurgence;
- Summation in astronomers sense or at the smallest-term (Poincaré 1892).
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Formal solutions of the Euler equation

We can search a *formal* power series solution $\hat{f}(x) = a_0 + a_1 x + \ldots + a_n x^n + \cdots$. We get an unique solution: $a_0 = 0$, $a_1 = 1$, $a_{n+1} = (-1)^n n!$, that is the *Euler series* :

$$\hat{f}(x) = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}.$$

This power series is clearly *divergent*. However, following Euler, it is “meaningful” and in particular one can use it to compute a very good appoximation (for x “small”) of $f(x)$, the value at x of the unique bounded solution f of the ODE on $\mathbf{R}^+$.

The associated homogeneous equation is $x^2 y' + y = 0$, its solution: $y = Ce^{1/x}$. Therefore a *formal* solution of the Euler equation is $\hat{f}(x) + C\, e^{1/x}$.

Homogeneous version : $\frac{d}{dx}\left(x\frac{dy}{dx}+\frac{1}{x}y\right)=0.$

A formal fundamental system of solutions at 0 is $(\hat{f}, e^{1/x})$.

At ∞ $(xz=1)$: $zy''+(1-z)y'-y=0$. It is a Kummer confluent hypergeometric function $E(1,1)$.

Asymptotic solutions

One can obtain a particular solution of Euler equation by the method of variation of the constant. After some manipulations we get as Euler :

$$y = f(x) = \int_0^{+\infty} \frac{e^{-t/x}}{1+t} \mathrm{d}t.$$

In the variable $w = 1/x$ it is a *Laplace transform* :

$$f(1/w) = \int_0^{+\infty} \frac{e^{-tw}}{1+t} \mathrm{d}t = \mathcal{L}(\frac{1}{1+t})(w).$$

We set :

$$f_n(x) := x - 1!x^2 + \cdots + (-1)^{n-1} x^n,$$

$$R_n(x) := (-1)^n \int_0^{+\infty} \frac{t^n e^{-t/x}}{1+t} \mathrm{d}t.$$

Then $f(x) = f_n(x) + R_n(x)$ and, for $x \in R^+$

$$|f(x) - f_n(x)| \leq |R_n(x)| \leq \Gamma(n+1)x^{n+1} = n!x^{n+1}.$$

Let $\theta > 0$ such that $|\theta| < \pi/2$. If x belongs to the closed sector :

$$\overline{V}_{\mathbf{R}^+}(\theta) := \{-\theta \leq \arg x \leq \theta\},$$

then : $|R_n(x)| \leq (1/\cos\theta)^n n! |x|^n$.
We set $(M_n(\theta) := (1/\cos\theta)^n n!)_{n \in \mathbf{N}^*}$, then:

$$\forall x \in \overline{V}_{\mathbf{R}^+}(\theta),\ \forall n \in \mathbf{N}^*,\ \ |f(x) - f_n(x)| \leq |R_n(x)| \leq M_n(\theta)|x|^{n+1}.$$

According to Poincaré définition (1886), the Euler power series $\hat{f}$ is the *asymptotic expansion* at the origin of the actual solution f on the *open* sector $\Re x > 0$ (opening π). The asymptotic expansion is uniform on each closed subsector $\overline{V}_{\mathbf{R}^+}(\theta)$ (and on each closed disc $\{\Re 1/x \geq a > 0\}$).

Poincaré asymptotics

Poincaré asymptotics on an open sector V with its vertex at 0, on $\mathbf{C}$ (or on the Riemann surface of the logarithm).
Let : $\hat{g}(x) = \sum a_n x^n \in \mathbf{C}[[x]]$ be a *formal* power series,

$$\forall n \in \mathbf{N}, \quad \hat{g}_n(x) := \sum_{i=0}^{n} a_i x^i, \quad |g(x) - g_n(x)| \leq M_{W,n} |x^{n+1}|;$$

g holomorphic on V, $W \subset V$ arbitrary *strict* subsector.

Notations: $g \sim \hat{g} \in \mathbf{C}[[x]]$ on V, $g \in \mathcal{A}(V)$.

The Taylor map $J : \mathcal{A}(V) \to \mathbf{C}[[x]]$, $J : g \mapsto \hat{g}$, is a morphism of differential algebras; it is *surjective* (Borel-Ritt) but *never injective*; AN ESSENTIAL FLAW:

$$0 \to \mathcal{A}^{<0}(V) \to \mathcal{A}(V) \xrightarrow{J} \mathbf{C}[[x]] \to 0;$$

$\mathcal{A}^{<0}(V)$ holomorphic functions *infinitely flat* at the origin.

The fundamental theorem of asymptotic expansions

Le théorème fondamental des développements asymptotiques est extraordinaire (Pierre Deligne).

The fundamental theorem of asymptotic expansions says that, given a formal solution of an algebraic (or analytic) ODE, it is allways possible to represent it asymptotically by an actual solution on a sufficiently small sector (the bisecting line being arbitrary).

Theorem

Let $G(x, Y, Y_1, \ldots, Y_n)$ be a polynomial in $n+2$ variables and $\hat{f} \in \mathbf{C}[[x]]$ a formal power series solution of the ODE

$$G(x, y, y', \ldots, y^{(n)} = 0 \text{ i. e. } G(x, \hat{f}, \hat{f}', \ldots, \hat{f}^{(n)} = 0)). \quad (1)$$

There exists a real number $k > 0$ such that for every open sector V at the origin of opening $< \pi/k$ and with a sufficiently small radius, there exists an actual solution f of (1) asymptotic to $\hat{f}$ on V.

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In this form, that is *without any restrictive hypothesis*, this result is due to Ramis-Sibuya 1989. The idea and the first particular cases are due to Poincaré and there are a lot of intermediate results in between due to various authors.

It is possible to compute a minoration of k using a Newton polygon algorithm (Malgrange, Ramis).

If $\hat{f}$ is convergent, then its sum f is an actual solution. We can ask if, in the divergent case, it is possible to get an actual solution f by a process of *resummation* of $\hat{f}$.
It is true as we will explain. In the "generic case" the k-summability works. In non generic situations it is necessary to use a more sophisticated process, the multisummability.
For applications to Painlevé equations, k summability suffices.

From Poincaré asymptotics to Gevrey asymptotics

In 1978, I rediscovered and extended some forgotten (and stupidely despised !) definitions of George Watson (1911). I introduced the notion of *Gevrey-s asymptotic expansion*.

Let $s \geq 0$. One modifies the Poincaré définition, replacing (M_n) by more precise estimates $(M_n = CA^n(n!)^s)$, for some $C, A > 0$ (which can depend on the subsector W). Notation $g \sim_s \hat{g}$.

Example: $\hat{f}$ is the *Gevrey-1 asymptotic expansion* of the actual solution f on the *open* sector $\Re x > 0$.

If $g \sim_s \hat{g}$, then the power series $\hat{g} = \sum_{n=0}^{\infty} a_n x^n$ is a *Gevrey-s power series* : $|a_n| \leq CA^n(n!)^s$.

Notations : $\hat{g} \in \mathbf{C}[[x]]_s$; $\mathbf{C}[[x]]_s$ is a *differential sub-algebra* of $\mathbf{C}[[x]]$.
(Gevrey filtration: $s \geq 0$; $\mathbf{C}[[x]]_0 = \mathbf{C}\{x\}$, $\mathbf{C}[[x]]_{+\infty} = \mathbf{C}[[x]]$), $g \in \mathcal{A}_s(V)$; a differential subalgebra of $\mathcal{A}(V)$.

There is an *essential difference* between the two notions of asymptotic expansions. It is related to the problem of *non unicity* of the asymptotic expansions (i. e. to the kernel of the Taylor map).

We suppose $s > 0$ and we set $k := 1/s$. Then we have *a dichotomy*, according to the opening $\mathrm{op}(V)$ of the sector V.

- if $\mathrm{op}(V) \leq \pi/k$ (a "small sector"), then the Taylor map :
$$J : \mathcal{A}_{1/k}(V) \to \mathbf{C}[[x]]_{1/k}$$
is *surjective* (Borel-Ritt-Gevrey theorem of Malgrange-Ramis) but it is *not injective*; it is similar to Poincaré asymptotics ("smoothness");
- if $\mathrm{op}(V) > \pi/k$ (a "big sector"), then the Taylor map
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k-summability
(Ramis 1980)

The idea of Poincaré was to define a notion of “sum” of a divergent power series but his method has a flaw. If the sum allways exists (good news), it is not unique (bad news): it is defined up to infinitely flat functions. Poincaré observed that in practical applications (as in celestial mechanics) its method was numerically efficient (in relation with the summation at the smallest term already used by Euler: “la sommation des astronomes”) but gave no explication.

If we use Gevrey-$(1/k)$ asymptotics, then the problem is the same on small sectors but on big sectors it is completely different: we have a notion of exact sum for *some* divergent series and this sum possesses very nice properties: it is compatible with the addition, the multiplication and the derivaton; moreover it coincides with the classical sum for convergent series.

The notion of sum *depends on a direction* $d = \mathbf{R}_+ e^{i\theta}$.

Definition. A power series $\hat{g} \in \mathbf{C}[[x]]$ is k-summable in a direction d if there exists an open sector V (on the Riemann surface of the logarithm) with vertex at 0 and bisected by d and an holomorphic function $g \in \mathcal{A}_{1/k}(V)$ such that $\hat{g} = J(g)$. Then g is the k-sum of $\hat{g}$ in the direction d.

A series k-summable in all the directions is convergent. A series k-summable in all the directions except a finite number is said k-summable. Notation: $\hat{g} \in \mathbf{C}\{x\}_{1/k}$.

Tauberian theorem (Ramis): if $k \neq k'$, then :

$$\mathbf{C}\{x\}_{1/k} \cap \mathbf{C}\{x\}_{1/k'} = \mathbf{C}\{x\}.$$

There are some (deep..) analogies with the Heisenberg uncertainty principle and also with arithmetic. There is a dictionary between exponential functions and prime numbers which motivates the name "wild" that I introduced in analogy with the *wild ramification* of the Galois groups in positive characteristic p :

exponential tori vs p-Sylows.

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1-summability of the Euler series

We recall :
$\hat{f}(x) = \sum_{n=0}^{\infty}(-1)^n n! x^{n+1}$ and $f(x) = \int_0^{+\infty} \frac{e^{-t/x}}{1+t} \mathrm{d}t$.
Let $\alpha > 0$ such that $|\alpha| < \pi/2$. We set $A(\alpha) := 1/\cos\alpha$,.
If x belongs to the closed sector :

$$\overline{V}_{\mathbf{R}^+}(\alpha) := \{-\alpha \leq \arg x \leq \alpha\},$$

then

$$\forall x \in \overline{V}_{\mathbf{R}^+}(\theta),\ \forall n \in \mathbf{N}^*,\ \ |f(x) - f_n(x)| \leq A(\alpha)^n n! x|^{n+1}.$$

Therefore f is Gevrey-1 asymptotic to $\hat{f}$ on the positive half-plane $\{\Re x > 0\}$.
Let d be a direction such that $d \neq \mathbf{R}_-$. We set

$$f_d(x) = \int_d \frac{e^{-t/x}}{1+t} \mathrm{d}t$$

When d moves in $\mathbf{C} \setminus \mathbf{R}_-$, the holomorphic functions f_d glue together by analytic continuation (residues formula and limit).

We get a function f holomorphic on the open sector $V := \{-3\pi/2 < \mathrm{Arg}(x) < 3\pi/2, \mathrm{op}(V) = 3\pi$.
The function f_d is Gevrey-1 asymptotic to $\hat{f}$ on the positive open half-plane bisected by d. Hence f is Gevrey-1 asymptotic to $\hat{f}$ on the sector V and it is k-summable in every direction $d \neq \mathbf{R}_-$: f_d is the sum in the direction d.

The function f has two branches f^+ and f^- on the half-plane $\Re x > 0$: we get f^- (resp.) using d^- "above" $\mathbf{R}^-$ (resp. d^+ "under" $\mathbf{R}^-$). When d crosses $\mathbf{R}^-$ there is a JUMP of the sum. We can compute it using the residues formula and a limit :

$$f^+(x) - f^-(x) = \int_{d^+ - d^-} \frac{e^{-t/x}}{1+t} \mathrm{d}t$$

$$= -2i\pi \,\mathrm{Res}_{t=-1} \left(\frac{e^{-t/x}}{1+t} \right) = -2i\pi e^{1/x}.$$

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Stokes lines or anti-Stokes lines

A source of confusion and of misunderstanding !

When, turning into the positive sense, we extends analytically f^- and f^+, it appears a radical change of asymptotic expansions (in generalized Poincaré sense) when one crosses the

ANTI-STOKES LINE

(or oscillationg line) $\text{Arg}(x) = -\pi/2$: $-2i\pi e^{1/x}$ "explodes".

For people who think in Poincaré asymptotics spirit (and they are a lot...) the Stokes-phenomena is the change of asymptotics when one crosses an oscillating line (and they name it a Stokes line !). We, as Stokes, think in terms of EXACT summation, therefore for us the central phenomena appears when one crosses a line of maximal decay of the exponential $e^{1/x}$ (a singular line): a exponentially small jump which is INVISIBLE with Poincaré asymptotics. The VISIBLE change of asymptotics is a by-product of a "HIDDEN JUMP".

Unfortunately a lot of (distinguished...) people stuck to a fundamental misunderstanding of Stokes work. It is not a good road towards summability and resurgence. In particular the celebrated “smoothing of a Victorian singularity” of John Berry (which can be reformulated with the exponential torus action) is not necessarily a good idea.

Borel-Laplace summation

Formal Borel transformation

Let $\hat{g} = \sum_{n=1}^{+\infty} \in \mathbf{C}[[x]]_1$. We define its formal Borel transform :

$$\hat{\mathcal{B}}\hat{g}(\zeta) = \mathbf{f}(\zeta) = \sum_{n=1}^{+\infty} a_n \frac{\zeta^{n-1}}{(n-1)!}$$

We have $\mathbf{g} \in \mathbf{C}\{\zeta\}$: a convergent series.
We define $\hat{\mathcal{B}}(1) = \delta$ (Dirac mass at 0).
The operator $\hat{\mathcal{B}}$ is linear and transforms multiplication into convolution.

Laplace transformation

If the sum of $\mathbf{g}$ extends analytically along the line d with an at most exponential growth at infinity, then we can define the Laplace transform in the direction d :

$$\mathcal{L}_d(\mathbf{g})(x) := \int_d \mathbf{g}(\zeta) e^{-\zeta/x} \mathrm{d}\zeta.$$

Borel-Laplace summation

We will say that $\hat{g}$ is Borel-Laplace summable in the direction d and that $\mathcal{L}_d(\mathbf{g})$ is its sum in the direction d.

This process extends the summation of the Euler series : $\hat{\mathcal{B}}(\hat{f}) = 1 - x + \cdots + (-1)^n x^n + \cdots = 1/(1+x)$.
The Borel-Laplace sum of a convergent series is its classical sum.

Theorem

Let $\hat{g} \in \mathbf{C}[[x]]_1$ it is Borel-Laplace summable in the direction d if and only if it is 1-summable in te direction d.

For the applications this result is very useful.
Borel-Laplace method gives an “explicit” formula for the sum but the summability is difficult to prove (this can explain why it remained quite impopular for nearly a century among mathematicians[1]...). On the contrary 1-summability is in many applications (in particular in dynamical systems) quite easy tp prove (cohomological methods...).

Boundary values

Let g be an holomorphic function on $\mathbf{C} \setminus \mathbf{R}$. One can define its boundary value in different senses (elementary, measures, distributions, Sato hyperfunctions) :

$$[g] = g(x+i0) - g(x-i0) = \lim_{\varepsilon \to 0} g(x+i\varepsilon) - g(x-i\varepsilon);$$

$[g]$ is a function, a measure, a distribution, an hyperfunction.

Let $T \in \mathcal{D}'(\mathbf{R})$ be a distribution with a compact support K. The Cauchy transform of T is :

$$g(z) := \frac{1}{2i\pi} < T_t, \frac{1}{t-z} >;$$

g is an holomorphic function on $\mathbf{C} \setminus K$.

We can recover T from g using a boundary value (in the distributions sense) : $T = [g]$.

Boundary values

Examples. Some basic bricks of resurgence

We change our notation, replacing z by ζ. We set $\Re\zeta := t$.

- $\left[\frac{1}{\zeta}\right] = -2i\pi\delta$, $\left[\frac{1}{\zeta+1}\right] = -2i\pi\delta_{-1}$
- $\left[\frac{1}{\zeta^{k+1}}\right] = 2i\pi\frac{(-1)^{k+1}}{k!}\delta^{(k)}$
- We consider the function $\log\zeta$ defined on $\mathbf{C}\setminus\mathbf{R}_+$ by the principal determination of the logarithm, then $[\log\zeta] = -2i\pi H$, H being the Heaviside function.
- Let $\alpha\in\mathbf{C}$, $\zeta^\alpha = e^{\alpha\log\zeta}$. Then $[\zeta^\alpha] = (1-e^{2i\pi\alpha})t^\alpha$.

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Resurgence of the Euler series

$$St_{\mathbf{R}_-}\hat{f}(x) = \int_{d_+-d_-} \mathbf{f}(\zeta)e^{-\zeta/x}d\zeta = 2i\pi e^{1/x};$$

Other computation using a boundary value :

$$\mathbf{f}(\zeta) = \frac{1}{1+\zeta} \text{ and :}$$

$$\int_{d_+-d_-} \mathbf{f}(\zeta)e^{-\zeta/x}d\zeta = -2i\pi \int_{\mathbf{R}_-} \left[\frac{1}{1+\zeta}\right] e^{-\zeta/x}d\zeta$$
$$= -2i\pi\delta_{-1}e^{-\zeta/x} = -2i\pi e^{1/x}.$$

This second computation is the very beginning of the resurgence. It is a basic brick of the alien derivations.

Roughly speaking the idea is to suppose that the sum **g** of the Borel transform can be analytically continued on the Riemann surface of $\mathbf{C} \setminus \Omega$ ($\Omega \subset \mathbf{C}$ a discrete subset). Then one can take at each point $\tilde{\omega}$ above $\omega \in \Omega$ an "holomorphic" variant of the boundary value "living locally" at $\tilde{\omega}$.

Hypergeometric equations

Euler, Gauss, Riemann

The Euler and Gauss hypergeometric equations are the simplest non-trivial linear ODEs on the Riemann sphere with only regular singular points. Following a seminal idea of B. Riemann (1857), one can associate to an hypergeometric equation its monodromy representation, that is a representation of the free group Γ_2 of rank 2 into the linear group $GL_2(\mathbf{C})$ (up to equivalence).

Then one gets a “dictionary” between the hypergeometric ODEs and *purely algebraic* objects, explicitely computable from the parameters (a, b, c) of the equation. It is the first version of the Riemann-Hilbert map.

There are confluent versions of the hypergeometric ODEs (Kummer, Whittaker) and a “confluent version” of the Riemann idea. An essential tool for us.

Hypergeometric equations
Euler, Gauss, Riemann

The Euler and Gauss hypergeometric equations are the simplest non-trivial linear ODEs on the Riemann sphere with only regular singular points. Following a seminal idea of B. Riemann (1857), one can associate to an hypergeometric equation its monodromy representation, that is a representation of the free group Γ_2 of rank 2 into the linear group $GL_2(\mathbf{C})$ (up to equivalence).

Then one gets a "dictionary" between the hypergeometric ODEs and *purely algebraic* objects, explicitely computable from the parameters (a, b, c) of the equation. It is the first version of the Riemann-Hilbert map.

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Hypergeometric series

Euler and Gauss defined (for $a, b, c \in \mathbf{R}$, $c \notin -\mathbf{Z}$) the *hypergeomeric series* :

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n, $$

where $(u)_n := u(u+1)\dots(u+n-1)$ (Pochhammer symbol).

One extends to the cases $a, b, c \in \mathbf{C}$, $c \notin -\mathbf{Z}$.

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Hypergeometric equations

Second order linear equations on $P^1(\mathbf{C})$ with 3 *regular singular points* : $0,\ 1,\ \infty$.

$$E_{a,b,c} \ : \quad z(1-z)y'' + (c-(a+b+1)z)\,y' - ab = 0,$$

$a,\ b,\ ,c \in \mathbf{C}$. Other expression :

$$\delta(\delta+c-1)y - z(\delta+a)(\delta+b)y = 0, \quad \delta := zd/dz$$

Fundamental system of solutions at the origin:

$$\Phi(a,b;c;z) := \left({}_2F_1(a,b;c;z), z_2^{1-c}F_1(a-c+1, b-c+1; 2-c; z)\right)$$

(generic case).
Monodromy around the origin: $z \to e^{2i\pi}z$ (analytic continuation along a simple loop around 0):

$$M_0 := \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\pi c} \end{pmatrix}.$$

$$\Phi(a,b;c;e^{2i\pi}z) = \Phi(a,b;c;z)M_0.$$

Monodromy : linear map $\mathcal{M}_\gamma : Sol_{z_0} \to Sol_{z_0}$ associated to the homotopy class of a loop γ at z_0 (by analytic continuation).

We choose 3 monodromy (simple) loops :

$$\gamma_0,\ \gamma_1,\ \gamma_\infty, \quad \gamma_0\gamma_1\gamma_\infty = 1.$$

After the choice of a fundamental solution Φ at z_0, we get 3 monodromy matrices $M_0,\ M_1,\ M_\infty$,

$$M_0 M_1 M_\infty = I.$$

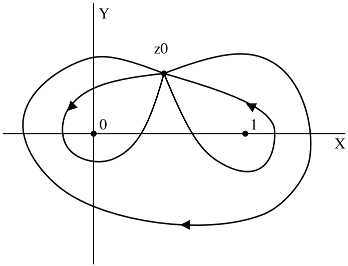


Figure: Monodromy loops of hypergeometic equations

Monodromy exponents

$$\left(\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & 0 & a \\ 1-c & c-a-b & b \end{array}\right)$$

Riemann P-symbol of the general Riemann equation (bijection) :

$$y(z) = P\begin{pmatrix} m_1 & m_2 & m_3 \\ \rho_1 & \sigma_1 & \tau_1 \\ \rho_2 & \sigma_2 & \tau_2 \end{pmatrix};$$

Riemann equation *regular-singular points* at $m_1,\ m_2,\ m_3$, with respective pairs of exponents $(\rho_1, \rho_2), (\sigma_1, (\sigma_2), (\tau_1, \tau_2)$ satisfying the *Fuchs relation* :

$$\sum_{i=1,2}(\rho_1 + \sigma_1 + \tau_i) = 1.$$

Reduction to the hypergeometric case by Möbius transforms on z: $(m_1, m_2, m_3) \to (0, 1, \infty)$ and afterwards :

$$y = z^{\rho_1}(z-1)^{\rho_2}u.$$

Irreducibility of representations

A representation :

$$\varpi : G := \pi_1 \left(P^1(\mathbf{C}) \setminus \{0, 1, \infty\} \right) \to Gl_2(\mathbf{C})$$

is determined by the two matrices :

$$M_0 := \varpi(\gamma_0) \text{ and } M_1 := \varpi(\gamma_1).$$

If there exists a one-dimensional subspace of $\mathbf{C}^2$ invariant by $\varpi(G)$, then ϖ is said to be *reducible*. If not ϖ is said to be *irreducible*.

We denote respectively (λ_1, λ_2), (μ_1, μ_2), (ν_1, ν_2) the pairs of eigenvalues of $M_0,\ M_1$ and $M_1 M_2 = \varpi(\gamma_0\gamma_1)$.

The representation ϖ is *irreducible* if and only if :

$$\forall i, j, k = 1, 2, \quad \lambda_i \mu_j \neq \nu_k.$$

Moreover the representation ϖ is up to conjugation determined by the two matrices :

$$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \mu_1 & 0 \\ \nu_1 + \nu_2 - (\lambda_1\mu_1 + \lambda_2\mu_2 & \mu_2 \end{pmatrix},$$

Irreducibility implies UNICITY for a given triple of pairs of eigenvalues $(\underline{\lambda}, \underline{\mu}, \underline{\nu})$.

Irreducibility of Riemann equations

Finding the monodromy from the exponents

B. Riemann ??

We denote $R(\underline{\rho}, \underline{\sigma}, \underline{\tau})$ the Riemann equation with singularities at 0, 1, ∞ with respective pairs of exponents $\underline{\rho}, \underline{\sigma}, \underline{\tau}$.

The equation $R(\underline{\rho}, \underline{\sigma}, \underline{\tau})$ is said to be (ir)reducibleif its monodromy representation is (ir)reducible.

$R(\underline{\rho}, \underline{\sigma}, \underline{\tau})$ is irreducible if and only if :

$$\rho_i + \sigma_j + \tau_k \notin \mathbf{Z} \quad (i, j, k = 1, 2).$$

Under this condition, the representation ϖ is expressed, *up to conjugacy* by the following matrices :

$$M_0 = \varpi(\gamma_0) = \begin{pmatrix} e^{2i\pi\rho_1} & 1 \\ 0 & e^{2i\pi\rho_2} \end{pmatrix} \quad M_1 = \varpi(\gamma_1) = \begin{pmatrix} e^{2i\pi\sigma_1} & 0 \\ b & e^{2i\pi\sigma_2} \end{pmatrix},$$

with : $b = e^{-2i\pi\tau_1} + e^{-2i\pi\tau_2} - e^{2i\pi(\rho_1+\sigma_1)} - e^{2i\pi(\rho_2+\sigma_2)}$.

How to compute the monodromy of $E(a, b; c)$?

Riemann computed the monodromy (in the irreducible case) *up to conjugacy*. The result is a *transcendental* function of the exponents: exponential (or trigonometric) functions.

If we want to compute the monodromy into the basis at 0

$$\left({}_2F_1(a, b; c; z), z_2^{1-c}F_1(a-c+1, b-c+1; 2-c; z)\right),$$

or into the similar basis at 1 or ∞ and more generally the connection formulae by analytic continuation between such basis, the exponential function is no longer sufficient, *we need the Γ function*. (Euler or Barnes integral formulae.)

Confluent Hypergeometric equations

Kummer confluent hypergeometric equations :

$$E(a,c;z) \;:\quad zy'' + (c-z)y' - ay = 0; \quad a, c \in \mathbf{C}.$$

Notation :

$${}_2F_0(a,b;z) = \sum_{n=0}^{\infty} \frac{(a)_n (c)_n}{n!} z^n,$$

This divergent power series is 1-summable.
A fundamental system of formal solutions at ∞ (an irregular point) is :

$$\left(z^{-a}\,{}_2F_0(a, 1+a-c; -1/z), z^{a-c} e^z\,{}_2F_0(c-a, 1-a; 1/z)\right).$$

Generically there are 2 Stokes lines $\mathbf{R}_{\pm}$.

$E(a, c; z)$ can be interpreted as a "limit", a *confluence*, of $E_{a,b,c}$. In $E_{a,b,c}$ we set $z = t/b$ and we take the limit $b \to \infty$.

$$E_{a,b,c} \quad : \quad z(1-z)y'' + (c - (a+b+1)z)\, y' - ab = 0,$$

Barnes integral formula
The classical case

The Euler formula $\Gamma(z) = \int_0^\infty z^{s-1} e^{-s}\,\mathrm{d}s$ translates a dictionary between the *differential* equation $y' - y = 0$ and the *difference equation* $u(z+1) - zu(z) = 0$.

Similarly the Barnes integral (1908) translates a dictionary between the hypergeometric equation $E(a, b; c)$ and a first order rational difference equation :

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(a+s)}{\Gamma(-s)} (-z)^s \mathrm{d}s $$

Using this formula we can compute *effectively* the representation of a fundamental *groupoïd* of $P^1(\mathbf{C}) \setminus \{0, 1, \infty\}$ associated to $E(a, b; c)$ (based at 3 points “near” the 3 singular points), that is the monodromy representation *and the connection formulae* associated to the *links*.

Method of computation: deformation of the “vertical” integration contour and residues formula.

Barnes integral formula

The *confluent* case

We set :

$$\psi(a;c;z)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(1-c-s)\Gamma(-s)}{\Gamma(a)\Gamma(a-c+1)}(-z)^s ds.$$

Then $(\psi(a;c;z), e^{-z}\psi(c-a,c;-z))$ is a fundamental system of solutions of the confluent hypergeometric equation $E(a,c)$.

Formally :

$$\psi(a;c;z)=z^{-a}\,{}_2F_0(a,1+a-c;1/z);$$

The power series ${}_2F_0$ is 1-summable.

Using the formula we can compute *effectively* the representation of a fundamental *groupoïd* of $P^1(\mathbf{C})\setminus\{0,\infty\}$ associated to $E(a,c)$ (based at 2 points near ∞), that is the monodromy representation *and the Stokes multipliers*.

The wild monodromy of the confluent hypergeometric equation (Martinet-Ramis 1989)

We give the wild monodomy *group* representation associated to $E(a,c)$ (expressed using a formal basis).
Stokes multipliers (unipotent matrices) :

$$\begin{pmatrix} 1 & 0 \\ \lambda(a,c) & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \mu(a,c) \\ 0 & 1 \end{pmatrix},$$

$$\lambda(a,c) = -2i\pi \frac{e^{i\pi(c-2a)}}{\Gamma(a)\Gamma(1-c+a)} \quad \mu(a,c) = -2i\pi \frac{1}{\Gamma(1-a)\Gamma(c-a)}$$

Formal monodromy :

$$\widehat{M}(a,c) = \begin{pmatrix} e^{-2i\pi a} & 0 \\ 0 & e^{2i\pi(a-c)} \end{pmatrix}$$

Actual monodromy :

$$M(a,c) = \begin{pmatrix} 1 & \mu(a,c) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda(a,c) & 1 \end{pmatrix} \widehat{M}(a,c)$$

The analytic halo
P. Deligne, J.P. Ramis

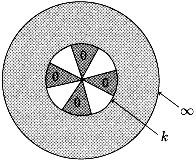
In a letter to, Deligne proposed a geometric picture of the Gevrey asymptotic theory.

One introduces a "blow-up" of the origin into **C**, the *analytic halo*. One performs a real blow up of the origin (polar coordinates), replacing $\{0\}$ by S^1. Afterwards one fills the hole with a disc. More precisely, one replaces $\{0\}$ by $\{0\} \cup]0, +\infty[\times S^1$ ($k \in]0, +\infty[$, $\theta \in S^1$).
Then one can extends the sheaf $\mathcal{O}$ of holomorphic functions on $\mathbf{C}^*$ by a sheaf on $\widetilde{\mathbf{C}} := \mathbf{C} \cup$ halo: an holomorphic function on a sector extends if its growth towards 0 is not too big.
One gets a sheaf $\widetilde{\mathcal{O}}$ of complex vector spaces (*no multiplication !*).
The small point of the algebraic geometry is at the heart of $\{0\}$ and the singular point of log at the heart of the algebraic point.

The analytic halo

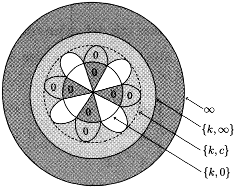
Pictures from M. Loday-Richaud and G. Pourcin

Analytic extension of the function e^{-1/x^4} on the analytic halo ($k = 4$.



Analytic extension of the function e^{-c/x^4} on the analytic halo; the circle $k = 4) \times S^1$ is blowed up into :

$$k = 4 \times (\mathbf{C}^* \cup (\{0\} \cup \{\infty\}) \times S^1.$$



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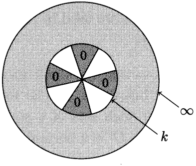

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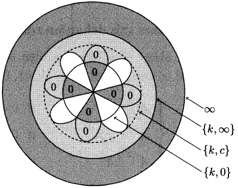

Analytic continuation across the halo

Martinet-Ramis, Malgrange-Ramis

The Deligne sheaf of analytic functions on the halo has two flaws :

- no multiplication;
- no analytic continuation.

This sheaf is “too smooth”. The situation is quite similar to the case of the analytic geometry over a nonarchimedean field. The remedy is similar: one must *rigidify* the sheaf. The rigidication is based on k-summability and multisummability and uses a notion of “big point”. After rigidification we recover a multiplication and a good notion of analytic continuation along some continuous paths across the halo. There are versions of Cauchy integral and residues theorem into the halo.

In this minicourse I will only use this as

AN HEURISTIC PICTURE.

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Heuristics

The linear case.

Let $D \in \mathbf{C}\{x\}[d/dx]$. It admits some singularities (a finite number) in the analytic halo. (There are algorithms (implemented in computer algebra) to compute them.) These singularities are "fixed", therefore we can follow analytically a solution along a continuous path entering into the analytic halo and when the path leave the halo we get a new solution in the ordinary world. It is a component of the *wild monodromy*.

This component is "discrete" as the ordinary monodromy, but there exists also another component which is "continuous". In particular the wild monodromy group possesses a :

NON TRIVIAL LIE ALGEBRA.

With J. Martinet we called it the *resurgence algebra* because it is generated by some operators $\dot{\Delta}$ which are the algebraic counterpart of (generalizations) of the Ecalle dotted alien derivations.

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This resurgence algebra comes from the exponential tori (rescaling of the exponentials) by Fourier analysis of the adjoint actions on the Stokes operators.

In the minicourse I will define and describe the wild dynamics for some “simple” families of algebraic ODEs: the linearized equations of the Painlevé equations.

The wild dynamics of the Painlevé equations

My initial intuition (seven years ago...) is that the Painlevé property can be extended in some sense into the analytic halo. Therefore there exists a wild dynamics as in the linear case: following a (wild) path, a solution can disappear into the halo at an irregular point and reappears (resurges as a resurgent stream) after a travel into the halo (underground).

In the minicourse I will define and describe this dynamics. It is not easy...

A microscope

The analytic halo is a “multilayered onion peel”. It is possible to zoom on a layer using a good microscope: the Borel transform. More precisely a “conjugate” of a Borel transform by a ramification operator $x \mapsto x^k$ ($k \in \mathbf{Q}+^*$): k is the magnification parameter. Moving k one finds critical layers $\{k_i\} \times S^1$. Each critical layer can be explored using the corresponding Borel plane. In most applications (and in all the resurgence litterature...) there is only one critical layer and the story ends. But in some cases it can be necessary to zoom again on the analytic halo of a singular point of the Borel plane... You can recognize a process of desingularization.

NON-LINEAR ALGEBRAIC ODEs

Movable singularities
Simple examples

If an ODE is *non-linear* we can in general predict neither *where the singularities of solutions appear* nor *of what kind the singularities are*.

Examples :

- Movable pole : $x' - x^2 = 0$, $y = -\frac{1}{t - C_1}$.
- Movable algebraic branch point : $my^{m-1}y' = 1$, $y = (x - C)^{1/m}$.
- Movable logarithmic branch point : $y'' + (y')^2 = 0$, $y = \log(x - C_1) + C_2$.
- Movable essential singular point : $(yy'' - (y')^2)^2 + 4y(y')^3 = 0, \quad y = C_1 e^{-1/(x - C_2)}$.

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Painlevé property

PROBLEM. Find all the algebraic differential equations free of movable branch points and movable essential singular points.

We say that such an algebraic ODE enjoys the *Painlevé property*: *its only movable singularities are poles*.

For equations of order one, only movable branch points appear. For such equations the problem was solved by L. Fuchs and H. Poincaré (mainly).

For equations of order $n \geq 3$ movable *essential singularities* can appear (and it can be even worse for $n \geq 3$...).

For equations of order $n = 2$, the problem was solved by Painlevé and his student Gambier (by a huge amount of computations...). Any equation with the Painlevé property reduces to an equation :

- which can be integrated by quadrature,
- or to a linear equation,
- or to one of the six Painlevé equations.

  For $n \geq 3$ the problem remains largely open (Chazy...).

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Riccati equations

A 1st order ODE satisfying Painlevé property

$$y' = a(x)y^2 + b(x)y + c(x) = 0, \quad a, c \in \mathbf{C}(x).$$

By the change of unknown function $y \mapsto u$,
$y = -\frac{1}{a(t)}\frac{u'}{u}$, the Riccati equation is transformed into the linear equation :

$$u'' - \left(\frac{a'(x)}{a(x)} + b(x)\right)u' + a(x)c(x)u = 0.$$

Solutions of this equation admits only fixed singularities ($a = 0$). Since a zero of u is of finite order, then the *movable* singularities of y are only poles.

By the change of unknown function $y \mapsto 1/w$, the Riccati equation is transformed into another Riccati equation :

$$w' = -c(x)w^2 - b(x)w - a(x) = 0.$$

Riccati foliation

The foliation defined by $\omega = dy - (ay^2 + by + c)\,dx = 0$ extends into an holomorphic singular foliation on $P^1(\mathbf{C}) \times P^1(\mathbf{C})$: the Riccati foliation.
There are a finite number of vertical leaves (above the singular points) and the foliation is transversal to the non singular fibres of :

$$P^1(\mathbf{C}) \times P^1(\mathbf{C}) \to P^1(\mathbf{C}), \quad (x, y) \to x.$$

The monodromy is a representation of the fundamental group of the complex plane minus the singular points (a free group of finite rank) into $PGL(2, \mathbf{C})$ (the group of Möbius transformations). The dynamics (the image of the representation), translating the “multivaluation” of the solutions, can be extremely complicated !

Elliptic differential equations

A 1st order ODE satisfying Painlevé property

Assume that $g_2, g_3 \in \mathbf{C}$, $g^3 - 27g^2 \neq 0$. We consider the elliptic curve :

$$\{y^2 = 4x^3 - g_2 x - g_3\} \cup \{\infty\} \subset P^2(\mathbf{C})$$

It is parametrized by $(x = \mathfrak{P}(t), y = \mathfrak{P}'(t))$, $t \in \mathbf{C} \mod \Lambda$, where $\Lambda \approx \mathbf{Z}^2$ is the lattice of periods.

The ODE for the Weierstrass function $\mathfrak{P}$ is :

$$(x')^2 = 4x^3 - g_2 x - g_3$$

The solutions are given by $x(t) = \mathfrak{P}(t - b)$ where $\mathfrak{P}(t)$ is the Weierstrass $\mathfrak{P}$-function. The constant b can be determined by the initial condition, so the solution $x(t) = \mathfrak{P}(t - b)$ has movable poles of order 2 at $t = b$ mod ? and no other singularity.

Classification of 1st order ODEs with Painlevé property

Theorem [L. Fuchs, H. Poincaré, J. Malmquist, M. Matsuda]

A 1st order ODE $(\star)$ has the Painlevé property if and only if it can be transformed into one of the following equations :

- a Riccati equation,
- an elliptic equation (the equation of the Weierstrass $\wp$ function)
- or if one can integrate $(\star)$ algebraically.

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P_{VI} AND ITS DYNAMICS

Following Dubrovin, Mazzocco, Iwasaki, Saito, Cantat-Loray...

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Our presentation is in elementary purely algebraic terms (no differential equations...). At the end we will introduce some topology: fundamental groups of punctured spheres.

We denote $\Gamma_3 := < u_0, u_t, u_1 >$ the free group of rank 3 generated by the letters $u_0,\ u_t,\ u_1$. It is identified with the free group $< u_0, u_t, u_1, u_\infty | u_0 u_t u_1 u_\infty = 1 >$ generated by $u_0,\ u_t,\ u_1,\ u_\infty$ up to the relation $u_0 u_t u_1 u_\infty = 1$.

Let $\rho : \Gamma_3 \to SL_2(\mathbf{C})$ be a linear representation. We set $M_l := \rho(u_l)$ $(l = 0, t, 1, \infty)$. We denote e_l and e_l^{-1} $(l = 0, t, 1, \infty)$ the eigenvalues of M_l. The representation ρ can be identified with $(M_0, M_t, M_1) \in (/SL_2(\mathbf{C}))^3$. Therefore the set of such representations $\mathrm{Hom}\,(\Gamma_3, SL_2(\mathbf{C}))$ modulo the adjoint action of $SL_2(\mathbf{C})$ can be identified with $(SL_2(\mathbf{C}))^3\,/SL_2(\mathbf{C})$ (the set of triples of matrices up to overall conjugation) :

$$\mathrm{Hom}\,(\Gamma_3, SL_2(\mathbf{C}))\,/SL_2(\mathbf{C}) = (SL_2(\mathbf{C}))^3\,/SL_2(\mathbf{C});$$

$(SL_2(\mathbf{C}))^3$ is a complex affine variety of dimension 9.

To a representation $\rho : \Gamma_3 \to SL_2(\mathbf{C})$ we associate its seven *Fricke coordinates* (or trace coordinates), the four *"parameters"* :

$$a_l := \mathrm{Tr}\, M_l = e_l + e_l^{-1}, \quad l = 0, t, 1, \infty$$

and the three *"variables"*:

$$X_0 = \mathrm{Tr}\, M_1 M_t, \quad X_t = \mathrm{Tr}\, M_1 M_0, \quad X_1 = \mathrm{Tr}\, M_t M_0.$$

These seven coordinates satisfy the *Fricke relation* $F(X, a) = 0$, where :

$$F(X,a) := F\left((X_0, X_l, X_1); (a_0, a_t, a_1, a_\infty)\right)$$
$$= X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - A_0 X_0 - A_t X_t - A_1 X_1 + A_\infty,$$

with :

$$A_i := a_i a_\infty + a_j a_k, \quad \text{for } i = 0, t, 1$$

and :

$$A_\infty := a_0 a_t a_1 a_\infty + a_0^2 + a_t^2 + a_1^2 + a_\infty^2 - 4.$$

The seven Fricke coordinates of ρ are clearly invariant by equivalence of representations. Then, using the seven Fricke coordinates, we get an algebraic map from $(Sl_2(\mathbf{C}))^3 / SL_2(\mathbf{C})$ to $\mathbf{C}^7$. The image is the six dimensional quartic hypersurface of $\mathbf{C}^7$ defined by the equation $F(X, a) = 0$.
We fix the parameter a and denote $\mathcal{S}(a)$ or $\mathcal{S}_{A_0,A_t,A_1,A_\infty}$ or $\mathcal{S}_{VI}(a)$ the cubic surface of $\mathbf{C}^3$ defined by the equation $F(X, a) = 0$. We call this surface the *the character variety* of PVI.
By a theorem of Fricke, Klein and Vogt the equivalence class of an *irreducible* representation is completely determined by its seven Fricke coordinates.

Beware : for a *reducible* representation the Fricke coordinates are not necessarily "good coordinates".

We denote $\overline{\mathcal{S}}(a)$ the projective completion[2] of $\mathcal{S}(a)$ in $\mathbf{P}^3(\mathbf{C})$. The family $\{\overline{\mathcal{S}}(a)\}_{a\in\mathbf{C}^4}$ contains all *smooth* projective cubic surfaces (up to linear transformations).

The list of projective cubic surfaces was given by Schläfli over a century ago. There are 20 families of *singular* projective cubic surfaces.

[2] As an abstract algebraic surface it is a del Pezzo surface of degree 3.

The surface $\mathcal{S}(a)$ is *simply connected*. It can be smooth or have singular points according to the values of a. The number of singular points is at most 4. Singular points of $\mathcal{S}(a)$ appear from semi-stable representations which are of two kinds :

- Either $M_l = \pm I_2$ $\big($that is $\rho(u_l)$ belongs to the center of $\mathrm{SL}_2(\mathbf{C})\big)$ for some $l = 0, t, 1, \infty$, hence $e_l = \pm 1$ and $a_l = \pm 2$. This case is called *the resonant case*.
- Or the representation is *reducible*. This condition can be translated into an algebraic condition on a : we have :

$$e_0\, e_t^{\pm 1} e_1^{\pm 1} e_\infty^{\pm 1} = 1 \qquad (2)$$

for some triple of signs.

An example of a singular cubic surface with 4 singular points is the Cayley cubic. We get it for $(A_0, A_t, A_1, A_\infty) = (0, 0, 0, -4)$ (this is true either if $a = (0, 0, 0, 0)$ or if $a = (\pm 2, \pm 2, \pm 2, \pm 2)$ with product -16):

$$X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - 4 = 0. \tag{3}$$

We denote : $F_{X_i} := \frac{\partial F(X,a)}{\partial X_i} = X_j X_k + 2X_i - A_i$. The character variety $\mathcal{S}_{VI}(a) = \mathcal{S}_{A_0,A_t,A_1,A_\infty}$ is equipped with a "natural" *algebraic symplectic form* (Poincaré residue) :

$$\omega_{VI,a} := \frac{dX_t \wedge dX_0}{2i\pi F_{X_1}} = \frac{dX_1 \wedge dX_t}{2i\pi F_{X_0}} = \frac{dX_1 \wedge dX_t}{2i\pi F_{X_t}} \tag{4}$$

We have $dF \wedge \omega_{VI,a} = -\frac{1}{2i\pi} dX_0 \wedge dX_t \wedge dX_1$. The Poisson bracket associated to $-2i\pi\, \omega_{VI,a}$ is the *Goldman bracket* defined by : $\{X_i, X_j\} = F_{X_k}$, and circular permutations.

Representations of the fundamental group of a 4-punctured sphere

Let S_4^2 be the four punctured sphere. Its fundamental group $\pi_1(S_4^2)$ is isomorphic to a free group of rank 3: we can choose as generators the homotopy classes of three simple loops turning around three punctures.
Therefore we can apply the preceding results to the study of equivalence classes of representations of $\pi_1(S_4^2)$ into $SL_2(\mathbf{C})$. It is a purely topological matter and the choice of the punctures is indifferent up to an homeomorphism. But in the following we will need the complex structure: $S^2 = P^1(\mathbf{C})$. Then, starting from 4 arbitrary punctures, up to a Möbius transformation, we can choose as punctures $0, t, 1, \infty$ for some value of t. This explains our initial notation.

We choose simple loops γ_l, $l = 0, t, 1, \infty$, based at a point $z_0 \in P^1(\mathbf{C}) \setminus \{0, t, 1, \infty\}$.

For $t \in P^1(\mathbf{C}) \setminus \{0, 1, \infty\}$ we set :

$$\widetilde{\text{Rep}}_t := \text{Hom}\left(\pi_1\left(P^1(\mathbf{C}) \setminus \{0, t, 1, \infty\}\right), SL_2(\mathbf{C})\right) / SL_2(\mathbf{C}).$$

For small changes of t, the group $\pi_1\left(P^1(\mathbf{C}) \setminus \{0, t, 1, \infty\}\right)$ *remains constant,* more precisely there exist canonical isomorphisms :

$$\pi_1\left(P^1(\mathbf{C}) \setminus \{0, t_1, 1, \infty\}\right) \to \pi_1\left(P^1(\mathbf{C}) \setminus \{0, t_2, 1, \infty\}\right).$$

Therefore there are canonical isomorphisms

$$\widetilde{\text{Rep}}_{t_2} \to \widetilde{\text{Rep}}_{t_1}.$$

Geometrically this says that the space of representations $\widetilde{\text{Rep}} := \{\widetilde{\text{Rep}}_t\}_{t \in \mathbf{P}^1(\mathbf{C}) \setminus \{0,1,\infty\}}$ can be interpreted as "a local system of varieties" parameterized by

$$t \in P^1(\mathbf{C}) \setminus \{0, 1, \infty\};$$

the fibration $\widetilde{\text{Rep}} \to P^1(\mathbf{C}) \setminus \{0, 1, \infty\}$ (whose fiber over t is $\widetilde{\text{Rep}}_t$) has a natural flat Ehresmann connection.

The homotopy classes of the loops γ_l gives an isomorphism between $\pi_1\left(P^1(\mathbf{C}) \setminus \{0,1,\infty\}; x_0\right)$ sending the class of γ_l on u_l. We write :

$$\rho(\gamma_l) = \rho(u_l) = M_l, \quad a_l = \operatorname{Tr} M_l$$

and :

$$X_0 = \operatorname{Tr} M_1 M_t = \operatorname{Tr} \rho(\gamma_1\gamma_t), \quad X_t = \operatorname{Tr}\ M_1 M_0 = \operatorname{Tr} \rho(\gamma_1\gamma_0),$$

$$X_1 = \operatorname{Tr}\ M_t M_0 = \operatorname{Tr} \rho(\gamma_t\gamma_0).$$

We can associate these trace coordinates to *pants decompositions* of the 4 punctured sphere into 3-punctured spheres ($\gamma_t\gamma_0 \sim (\gamma_1\gamma_\infty)^{-1}$) :

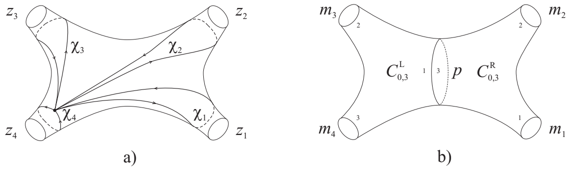

Figure 2: *Basis of loops of* $\pi_1(C_{0,4})$ *and the decomposition* $C_{0,4} = C^L_{0,3} \cup C^R_{0,3}$.

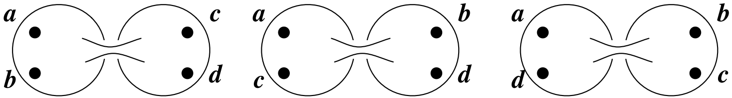

Figure: Three decompositions of S_4^2 into pairs of S_3^2

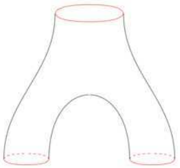

Figure: A pair of pants $\approx S_3^2$

Braids

If we move $\mathbf{a} = (a_1, a_2, a_3, a_4) \in P^1(\mathbf{C})$, the points a_i remaining distinct, the *topology* of $P^1(\mathbf{C}) \setminus \{a_1, \ldots, a_4\}$ does not change, but the *complex structure* changes, there are *moduli*.

Such a move is a path on the *the configuration space* :

$$\mathcal{B} := \mathbf{C}^4 \setminus \bigcup_{i \neq j} \Delta_{ij}, \quad \text{where} \ \ \Delta_{ij} := \{x_i = x_j\}.$$

A braid is a continuous path on the configuration space from a configuration to itself up to the ordering of the points. A pure braid is a continuous loop on the configuration space.

Using the map $P^1(\mathbf{C}) \setminus \{0, 1, \infty\} \to \mathcal{B}$ defined by $t \mapsto (0, t, 1, \infty)$, we get a map

$$\pi_1\left(P^1(\mathbf{C}) \setminus \{0, 1, \infty\}\right) \to \pi_1(\mathcal{B}) \approx PB_3.$$

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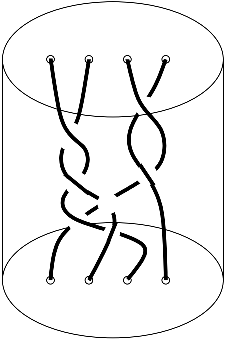


Figure: Braids and punctured disks

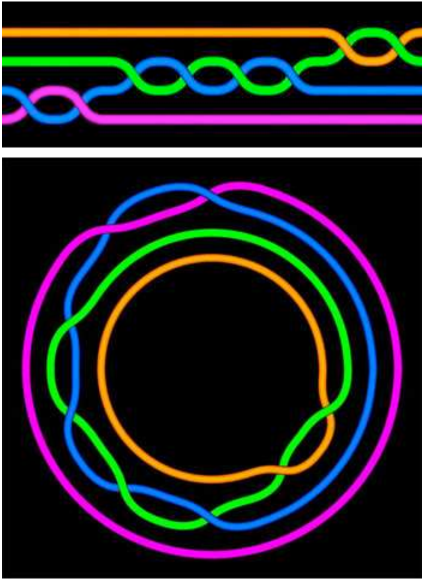


Figure: Pure braid and loop on the configuration space

The space of representations :

$$\widetilde{\mathrm{Rep}} := \{\widetilde{\mathrm{Rep}}_t\}_{t\in \mathbf{P}^1(\mathbf{C})\setminus\{0,1,\infty\}}$$

can be interpreted as “a local system of varieties” parameterized by

$$t \in P^1(\mathbf{C}) \setminus \{0, 1, \infty\},$$

or equivalently by the configuration defined by the 4 punctures $(0, t, 1, \infty)$, a point of $\mathcal{B}$.

We get a fibre bundle above $\mathcal{B}$ with an Ehresmann connection. Then the fundamental group of the basis $\mathcal{B}$, that is the pure braids group PB_3 acts on the fiber.

Isomonodromy and PVI

P_{VI}

The sixth Painlevé equation is :

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt}$$

$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right);$$

$\alpha, \beta, \gamma, \delta \in \mathbf{C}$ are the parameters.

The generic solution of PVI has essential singularities and/or branch points in the points $0,\ 1, \infty$, the *fixed singularities*. The other singularities, the *moving singularities* (so called because they depend on the initial conditions) are *poles*: it is the Painlevé property. A solution of P_{VI} can be *analytically continued to a meromorphic function on the universal covering of :*

$$P^1(\mathbf{C}) \setminus \{0, 1, \infty\}.$$

Transcendantes nouvelles

For generic values of the integration constants and of the parameters $\alpha, \beta, \gamma, \delta$, a solution cannot be expressed via elementary or classical transcendental functions. It is called the irreducibility property of P_{VI} (and similarly for other Painlevé equations). For this reason, Painlevé called these functions: "transcendantes nouvelles" (new transcendental functions).

The first notion of reducibility appears in Painlevé "Leçons de Stockolm". The algebrization of this notion is due to the Japanese school (at the end of nineties): K. Nishioka, H. Umemura, H. Wanabe. The proof of irreducibility follows.

Painlevé proposed to use Drach "differential Galois theory" (rationality group 1898) to get a proof. His "proof" for P_I case in this line has some gaps (and worse also Drach theory...).

Recently some proofs were obtained in this line (using Malgrange-Galois pseudogroup, Morales-Ramis-Simo theory,...): Casale 2007-2008, Casale-Weyl, Cantat-Loray, Acosta-Humanez-van der Put-Top, Horozov, Stoyanova, Christov, Morales-Ruiz...

In modern formulation, solutions of PVI parameterize *isomonodromic deformations* (in t) of rank two meromorphic connections over the Riemann sphere having simple poles at the 4 points $0,\ t, 1, \infty$.

We consider *traceless* 2×2 linear differential systems with 4 fuchsian singularities (logarithmic) on the Riemann sphere $P^1(\mathbf{C})$, parameterized by a complex variable t :

$$\frac{dY}{dz} = A(z;t)Y, \qquad A(z;t) := \frac{A_0(t)}{z} + \frac{A_t(t)}{z-t} + \frac{A_1(t)}{z-1} \quad (5)$$

with the residue matrices $A_l(t) \in sl_2(\mathbf{C})$ $(l = 0, t, 1)$ having $\pm\frac{\theta_l}{2}$ as eigenvalues (independantly of t). We set $\theta := (\theta_0, \theta_t, \theta_1, \theta_\infty)$: it encodes (through *transcendental* maps) the *local monodromy data*.

Choosing a germ of a fundamental matrix solution $\Phi(z, t)$ of the above system near some nonsingular point z_0, one has a *linear monodromy representation* (anti-homomorphism) :

$$\rho : \pi_1 \left(P^1(\mathbf{C}) \setminus \{0, t, 1, \infty\}; z_0 \right) \rightarrow SL_2(\mathbf{C})$$

such that the analytic continuation of Φ along a loop γ (based at z_0) defines another fundamental matrix solution $\Phi\, \rho(\gamma)$.

The equivalence class of ρ in $SL_2(\mathbf{C})$ is independant of the choice of the fundamental solution Φ. The system (5) is said *isomonodromic* if this conjugation class is locally constant with respect to t, or equivalently if the matrices A_l $(l = 0, t, 1)$ depends on t in such a way that the monodromy of a fundamental solution $\Phi(z : t)$ does not change for small deformations of t.

A meromorphic connection ∇ can be interpreted as an equivalence class of systems modulo rational equivalence (gauge transformation). If two systems :

$$\frac{dY}{dz} = A(z;t)Y \quad \text{and} \quad \frac{dY}{dz} = B(z;t)Y,$$

satisfying the conditions (5), are *rationally equivalent* on $P^1(\mathbf{C})$, that is if there exists a rational matrix P such that :

$$B = P^{-1}AP - P^{-1}\frac{dP}{dz},$$

then the two corresponding monodromy representation are equivalent. The isomonodromy property is invariant by a rational equivalence. We can speak of :

isomonodromic deformations of connections.

Schlesinger (??) found that the isomonodromy condition is equivalent to having the linear differential equation :

$$\frac{dY}{dt} = B(z,t)Y, \quad \text{with} \quad B(z,t) := -\frac{A_t(t)}{z-t}Y. \tag{6}$$

We define the *Schlesinger system* as the system (5) and (6) :

$$\frac{dY}{dz} = A(z,t)Y, \quad \frac{dY}{dt} = B(z,t)Y,$$

Then the isomonodromy of the system (5) is equivalent to the *complete integrability condition* (also called *zero curvature condition*) of the Schlesinger system :

$$\frac{\partial B}{\partial z} - \frac{\partial A}{\partial t} = [A, B]. \tag{7}$$

Expliciting this condition, we see that the isomonodromicity of the system (5) is expressed by the following equations (called the Schlesinger equations).

Schlesinger equations

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \quad \frac{dA_t}{dt} = \frac{[A_0, A_t]}{t} + \frac{[A_1, A_t]}{t-1}, \quad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1}.$$

This is a *non-linear* differential system with the unknown function (A_0, A_t, A_1) (9 scalar unknown functions).

General version: $t_1, \ldots, t_m$ logarithmic singular points. If :

$$\frac{\partial A_i}{\partial t_j} = \frac{[A_i, A_j]}{t_i - t_j}, \quad \frac{\partial A_i}{\partial t_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{t_i - t_j},$$

then we have an isomonodromic deformation.

Conversely, in the generic case, an isomonodromic deformation satisfies the Schlesinger equations.

Figure: Ludwig Schlesinger 1864-1933, follower and son-in-law of Lazarus Fuchs

We suppose now that the Schlesinger equations are satisfied by the matrix A of the system (5) and we will derive P_{VI} for some values of the parameter (under some genericity condition on the local monodromy exponents $\pm\theta_l/2$).

We set $A_\infty := -A_0 - A_t - A_1$ and we suppose that the matrices A_l ($l = 0, t, 1, \infty$) are *semi-simple*. The eigenvalues of the A_l ($l = 0, t, 1, \infty$) are independant of t and we denote them by e_l, e_l^{-1}. We suppose $e_l \neq \pm 1$ or equivalently $\theta_l \notin \mathbf{Z}$; *non-resonance conditions*.

From Schlesinger equations we get $\dfrac{dA_\infty}{dt} = 0$, therefore, up to a constant gauge transformation, we can suppose

$$A_\infty = \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}.$$

We denote $[A]_{ij}$ the (i,j) entry of the matrix of the differential system (5). We suppose that the system is *irreducible*. Then $[A]_{12}$ is not identically 0. We have $A_0 + A_t + A_1 = -A_\infty$, therefore $[A_0 + A_t + A_1]_{12} = 0$. Hence $z(z-t)(z-1)[A]_{12}$ is linear in z and it admits a unique zero at the point $z = q(t)$, where :

$$q(t) = -\frac{t[A_0]_{12}}{t[A_t]_{12} + [A_1]_{12}}$$

The point $q(t)$ is *an apparent singularity* of the second order linear ODE satisfied by the first component y of any solution Y of the system (5).

We denote :

$$p(t) := [A(q(t),t))]_{11} + \frac{\theta_0}{2q} + \frac{\theta_t}{2(q-t)} + \frac{\theta_1}{2(q-1)}.$$

Then the Schlesinger system is equivalent to the Hamiltonian system of PVI whose (non autonomous) Hamiltonian is H_{VI} :

$$t(t-1)H_{VI}(q,p,t) := q(q-1)(q-t)p2$$

$$-\left(\theta_0(q-1)(q-t) + \theta_1 q(q-t) + (\theta_t - 1)q(q-1)\right)p$$

$$+\frac{1}{4}\left((\theta_0+\theta_1+\theta_t-1)^2 - \theta_\infty^2)(q-t)\right).$$

Now we can write the Hamiltonian system in PVI form with the following values for the parameters :

$$\alpha = (\theta_\infty - 1)^2/2 \quad \beta = -\theta_0^2/2, \quad \gamma = \theta_1^2/2, \quad \delta = 1 - \theta_t^2/2.$$

P_{VI} as an Hamiltonian system

P_{VI} is equivalent to a Hamiltonian system :

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{\partial H_{VI}}{\partial p} \\ \dfrac{dp}{dt} = -\dfrac{\partial H_{VI}}{\partial q} \end{cases},$$

$$H_{VI} \in \mathbf{C}(t)[q, p, \theta].$$

The corresponding Hamiltonian (rational) vector field is :

$$\mathcal{X}_{VI} = \frac{\partial}{\partial t} + \frac{\partial H_{VI}}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H_{VI}}{\partial q}\frac{\partial}{\partial p}.$$

The field is regular on $(\mathbf{C} \setminus \{0, \infty\}) \times \mathbf{C}^2$. We set

$$L := P^2(\mathbf{C}) \setminus \mathbf{C}^2 \approx P^1(\mathbf{C});$$

$\mathcal{X}_{VI}$ has a pole along $(\mathbf{C} \setminus \{0, \infty\}) \times L$.

The Riemann-Hilbert correspondence RH is the map between the space of linear systems (5) with prescribed poles and local exponents $\pm\theta_l/2$, modulo $SL_2(\mathbf{C})$-gauge transformations, on one side (the source or “left hand side”), and the space of monodromy representations with prescribed local exponents modulo conjugation in $SL_2(\mathbf{C})$ on the other side (the target or “right hand side”).

Parameters:

$$e_l = e^{i\pi\theta_l}, \quad a_l = \mathrm{Tr}\ M_l = 2\cos 2\pi\theta_l.$$

We will see that the Riemann-Hilbert correspondence can be translated into a correspondence between solutions of PVI and equivalence classes of monodromy representations.

Okamoto space of initial conditions

The naïve phase space of the non-linear system P_{VI} is $\big(P^1(\mathbf{C}) \setminus \{0,1,\infty\}\big) \times \mathbf{C}^2$.

It is not a good phase space because the solutions have poles: the Painlevé flow is not complete.

Using a series of blowing-ups K. Okamoto introduced a good space of initial conditions $\mathcal{M}_{t_0}(\theta)$ at any point $t_0 \in \mathbf{C} \setminus \{0,1\}$ (1979). It is a convenient semi compactification of the naïve phase space $\mathbf{C}^2$, an open rational surface. (A 8 points blow-up of the Hirzebruch surface Σ_2 minus an anti-canonical divisor.)

The Okamoto variety is endowed with an algebraic symplectic structure given by the extension of the standard symplectic form $dp \wedge dq$.

The pole divisor of this extension is the anticanonical divisor of a compactification of the Okamoto variety: the vertical leaves. The vertical leaves configuration is described by *an extended Dynkin diagram*. Today a “good list” of the Painlevé equations is labelled by such diagrams.

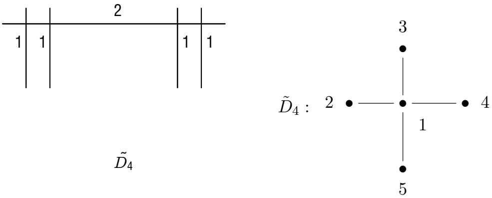


Figure: P_{VI}. Divisor: $2E_0 + E_1 + E_2 + E_3 + E_4$

$\tilde{D}_4$

The Okamoto variety of initial conditions at t_0 can be identified with the moduli space of meromorphic connections over the Riemann sphere[3] having simple poles at the four points $0,\ t_0, 1, \infty$ with local exponents $\{\pm\theta_l\}_{l=0,t,1,\infty}$.

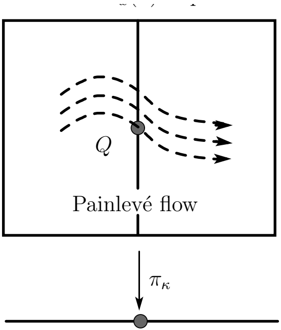
[3] In the non resonant case. In the resonant case, that is if one of the θ_l is an integer, then $\mathcal{M}_{t_0}(\theta)$ is the moduli space of *parabolic* connections.

For θ fixed, we have a fiber bundle :

$$\pi_\theta : \mathcal{M}(\theta) \to P^1(\mathbf{C}) \setminus \{0, 1, \infty\};$$

the fiber above t_0 is $\mathcal{M}_{t_0}(\theta)$.

The naïve Painlevé foliation extends to this fiber bundle. This extension is transverse to the fibers and we get a complete (symplectic) flow, the Painlevé flow. For all $t_0, t_1 \neq 0, 1, \infty$ this flow induces an analytic symplectic diffeomorphism : $\mathcal{M}_{t_0}(\theta) \to \mathcal{M}_{t_1}(\theta)$.



We get also *analytic* maps (Riemann-Hilbert maps) :

$$\mathrm{RH} : \mathcal{M}_t(\theta) \to \mathcal{S}_{P_{VI}}.$$

Such a map can be interpreted as an analytic map:

$$\mathrm{RH} : \mathcal{M}_t(\theta) \to \mathcal{S}_{A_0 A_t A_1 A_\infty},$$

where :

$$A_i = 4\left(\cos\theta_i \cos\theta_\infty + \cos\theta_j \cos\theta_k\right),$$

((i, j, k) is a permutation of $(0, t, 1)$), and

$$\begin{aligned} A_\infty &= 16(\cos\theta_0 \cos\theta_t \cos\theta_1 \cos\theta_\infty) \\ &+ 4(\cos^2\theta_0 + \cos^2\theta_t + \cos^2\theta_1 + \cos^2\theta_\infty - 1). \end{aligned}$$

This map is allways *proper*. If the cubic surface $\mathcal{S}_{A_0 A_t A_1 A_\infty}$ is smooth, then this map is an analytic symplectic isomorphism.

In the singular case, the proper map RH realizes an *analytic minimal resolution of singularities* of the cubic surface $\mathcal{S}_{A_0 A_t A_1 A_\infty}$.

Along the irreducible components of the exceptional divisor, PVI restricts to a *Riccati equation*.

The singular points of type A_1 , A_2 , A_3 and D_4 on the cubic surface yield 1, 2, 3 and 4 exceptional Ricatti curves.

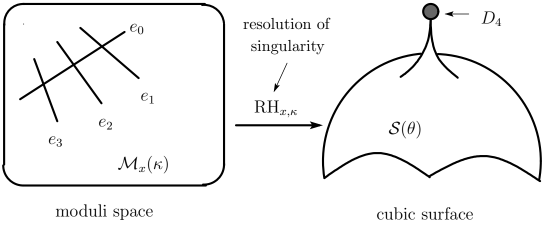


Figure 8: Resolution of singularities by Riemann-Hilbert correspondence

Figure: From Iwasaki-Uehara

"Pulling back" the fiber bundle $\widetilde{\text{Rep}} \to P^1(\mathbf{C}) \setminus \{0, 1, \infty\}$ and its connection by the Riemann-Hilbert map (i. e. keeping the base and changing the fibers through RH) yields the fiber bundle $\mathcal{M} \to P^1(\mathbf{C}) \setminus \{0, 1, \infty\}$ with its PVI connection. This allows one to give an important interpretation of the non-linear monodromy of PVI using a braid group.

An algebraic dynamics on the character variety induced by a braids group. Comparison with the dynamics of P_{VI}

We interpret the isomonodromic deformations as an Ehresmann connection on a bundle above the "space of configurations" $\mathcal{B}$ (the space of complex planes minus 4 points) whose fiber is $S_{(A,B,C,D)}$.

The fundamental group of the basis $\mathcal{B}$ is the pure braids group PB_3 and it acts *algebraically* and symplectically on $S_{(A,B,C,D)}$. Moreover it is "easy" to compute explicitely the action. The computation of the conjugate by RH of the monodromy of P_{VI} follows :

$$\pi_1(P^1(\mathbf{C}) \setminus \{0, 1, \infty\}, \star) \to \mathrm{Aut}(S_{(A,B,C,D)}).$$

The *algebraic dynamics* on $S_{(A,B,C,D)}$ depends *algebraically* on (A, B, C, D).

Application: Computation of the Galois-Malgrange groupoid of P_{VI} (Cantat-Loray 2007).

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Roughly speaking this program was to extend the results obtained for P_{VI} to the others Painlevé équations :

$$P_V, \; P_{IV}, \; P_{III}, \; , P_{II}, \; P_I.$$

We will explain the problems and afterwards we will present the state of the arts for P_{II} and P_V, a joint work (in progress...) with Martin Klimes and Emmanuel Paul for P_{II} and a work of Klimes for P_V.

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The Wuhan Program (2012)

Figure: Sakuras blossom in Wuhan Campus, China

STOKES PHENOMENON
and
DYNAMICS ON WILD CHARACTER VARIETIES
of
PAINLEVÉ EQUATIONS

Jean-Pierre Ramis

Académie des Sciences
and
Institut de Mathématiques de Toulouse

Minicourse (I.1)

9th IST lectures on Algebraic Geometry and Physics,
Lisbon, february 2020

FROM P_{VI} TO THE OTHERS PAINLEVÉ EQUATIONS

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The *wild dynamics* of a linear meromorphic ODE

One can associate to a linear meromorphic differential equation on $P^1(\mathbf{C})$ (or more generally a complete Riemann surface) a *wild monodromy representation*. It is in general a *groupoid* (not a group) representation. It is built using the *classical monodromy*, the *Stokes multipliers* and some *links*.

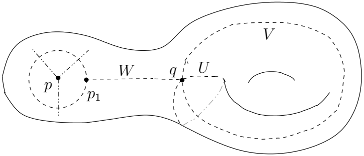

FIGURE 2. A Riemann surface C, here taken to be of genus $g_C = 1$, with an irregular singularity at a point p. A basepoint is taken at q. Show are the Stokes rays near p and the important paths in defining the generalized monodromy data.

We can "put" also in this representation the *exponential tori action* (derived from the formal exponential exponents). This allows to reformulate the things in a "resurgent style" (Martinet-Ramis).

One can replace the notion of *isomonodromic deformation* of a linear system by a notion of *iso-irregular deformation*. This was discovered by R. Garnier (1919) in relation with the Painlevé equations: these equations are derived from iso-irregular deformations (Lax pairs).

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The *wild dynamics* of some meromorphic ODEs

For a linear equation with irregular singularities, one must replace the classical notion of *monodromy representation* (Poincaré dynamics) by the notion of *wild monodromy representation* (Jimbo-Miwa-Ueno, Martinet-Ramis, van der Put-Saito, Boalch ...).

Similarly for P_V,...,P_I it is necessary to replace the *non linear monodromy representations* :

$$\pi_1(P^1(\mathbf{C}) \setminus \{0, 1, \infty\}, \star) \to \mathrm{Aut}(\bullet).$$

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At the Wuhan conference I *conjectured* the existence and the description of these objects. Today it is possible to give rigorous definitions due to the work of Amaury Bittmann (2016) on the *doubly resonant saddle nodes of* $\mathbf{C}^3$ (extending the works of Setsuji Yoshida and Kyoichi Takano around 1980).

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Wild Riemann-Hilbert map: (generically...) symplectic analytic isomorphisms between the Okamoto spaces of initial conditions and the wild character varieties.

The wild dynamics of the Painlevé equations act *locally* on the Okamoto varieties of initial conditions. Conjugating by the wild RH isomorphisms, we get symplectic *local dynamics on the wild character varieties*. We put the *exponential tori actions* in this dynamic. It is an important ingredient !

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(New transcendental functions...)

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"What algebra sees fom the dynamics" (Malgrange).
More formally: Zariski closure of the dynamics (it is difficult to formalize !).
Conjecture. The wild dynamics of the Painlevé equations are made of solutions of the Malgrange groupoid.

Main observation: If the wild dynamics is "rich" (for example chaotic...), then the Malgrange groupoid is "big".

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The Malgrange groupoids of the Painlevé equations are the biggest possible: "Vol" (except for some exceptionnal cases).

There are many results in this direction obtained by Guy Casale, Jacques-Arthur Weil and Damien Davy using different methods (P_I, $P_{II,0}$ and for generic values for the others).

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The Riemann Hilbert maps

Picture in the generic case...

The map:

regular singular connection $\longrightarrow$ monodromy representation up to equivalence

is, by definition, *the Riemann Hilbert map* (RH).

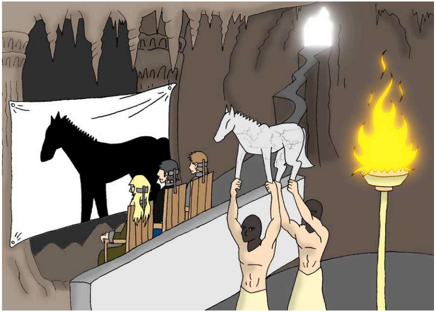
The map:

connection $\longrightarrow$ wild monodromy representation up to equivalence

is, by definition, *the generalized (or wild) Riemann Hilbert map* (RH_w). These maps are *transcendental*.

The maps RH and RH_w induce *equivalences of categories*.

De Rham side $\longrightarrow$ Betti side



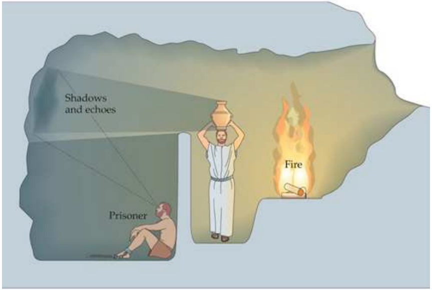


Figure: Plato's cave

Actual world	$\longrightarrow$	Algebraic (ideal...) world
Complex analysis	*RH*	Algebraic geometry
Complicated (transcendental computations)		Simple (algebraic computations)

For the Painlevé equations:

Actual world. Okamoto varieties of initial conditions (iso-irregular families) and *analytic* dynamics on these varieties: actual non-linear monodromy, Stokes dynamics, actions of exponential tori, reflecting the "fundamental group" of a "configuration space" (??).

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Actual world. Okamoto varieties of initial conditions (iso-irregular families) and *analytic* dynamics on these varieties: actual non-linear monodromy, Stokes dynamics, actions of exponential tori, reflecting the “fundamental group” of a “configuration space” (??).

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Actual world	$\longrightarrow$	Algebraic (ideal...) world
Complex analysis	*RH*	Algebraic geometry
Complicated (transcendental computations)		Simple (algebraic computations)

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For P_{II}, in the *Plato ideal word* (that is on the character varieties), the wild dynamics is rational and explicit and the formulas are very simple and very nice, but unfortunately the proofs are difficult and technical (and remain today uncomplete...).

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Van der Put-Saito classification

Wild monodromy representations of the linearized equations

Dynkin	Painlevé equation	$r(0)$	$r(1)$	$r(\infty)$	$r(t)$	$\dim \mathcal{P}$
$\tilde{D}_4$	PVI	0	0	0	0	4
$\tilde{D}_5$	PV	0	0	1	-	3
$\tilde{D}_6$	$\mathrm{PV}_{\mathrm{deg}}=$ PIII(D6)	0	0	1/2	-	2
$\tilde{D}_6$	PIII(D6)	1	-	1	-	2
$\tilde{D}_7$	PIII(D7)	1/2	-	1	-	1
$\tilde{D}_8$	PIII(D8)	1/2	-	1/2	-	0
$\tilde{E}_6$	PIV	0	-	2	-	2
$\tilde{E}_7$	PII	0	-	3/2	-	1
$\tilde{E}_7$	PII	-	-	3	-	1
$\tilde{E}_8$	PI	-	-	5/2	-	0

Figure: Van der Put-Saito Table

- $r(\bullet)$ is the Katz-rank (the slope of the Newton-Ramis polygon) at the singular point $\bullet$ of the linearized equation.
- $\mathcal{P}$ is the parameter space.
- P_{II}, $r(\infty) = 3/2$ is P_{II}^{FN} (our model) and P_{II}, $r(\infty) = 3$ is P_{II}^{JM}.

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(Wild) character varieties of the Painlevé equations (Van der Put-Saito; Chekhov-Mazzocco-Rubtsov version)

P-eqs	Polynomials
PVI	$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 + \omega_4$
PV	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 + 1 + \omega_3^2 - \frac{\omega_3(\omega_2+\omega_1\omega_3)(\omega_1+\omega_2\omega_3)}{(\omega_3^2-1)^2}$
PV_{deg}	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_1 - 1$
PIV	$x_1x_2x_3 + x_1^2 + \omega_1x_1 + \omega_2(x_2 + x_3) + \omega_2(1 + \omega_1 - \omega_2)$
$PIII$	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_1 - 1$
$PIII^{D_7}$	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 - x_2$
$PIII^{D_8}$	$x_1x_2x_3 + x_1^2 + x_2^2 - x_2$
PII^{JM}	$x_1x_2x_3 - x_1 + \omega_2x_2 - x_3 - \omega_2 + 1$
PII^{FN}	$x_1x_2x_3 + x_1^2 + \omega_1x_1 - x_2 - 1$
PI	$x_1x_2x_3 - x_1 - x_2 + 1$

Correction: permute P_{II}^{JM} and P_{II}^{FN}.

Affine cubic surface: $\{F_\star = 0\} \subset \mathbf{C}^3$; polynomial $F_\star$:

$$F_\star = x_1x_2x_3 + \epsilon_1^\star x_1^2 + \epsilon_2^\star x_2^2 + \epsilon_3^\star x_2^2 + \omega_1^\star x_1 + \omega_2^\star x_2 + \omega_3^\star x_3 + \omega_4^\star,$$

$\star = VI,\ V,\ V\mathrm{deg}, \ldots, I$, $\epsilon_i^\star = 1$ or 0, $\omega_3^\star \in \mathbf{C}$.

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PV_{deg}	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_1 - 1$
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$\star = VI,\ V,\ V\text{deg}, \dots, I$, $\epsilon_i^\star = 1$ or 0, $\omega_3^\star \in \mathbf{C}$.

A fundamental heuristic idea

In order to understand the dynamics and the wild dynamics on the character varieties and the confluence mechanisms, it is essential to look at THE LINES ON THESE SURFACES and their configuration, the “surface skeleton”.

The generic P_{VI} case, $S_{(A,B,C,D)}$: $27 = 24 + 3$ lines

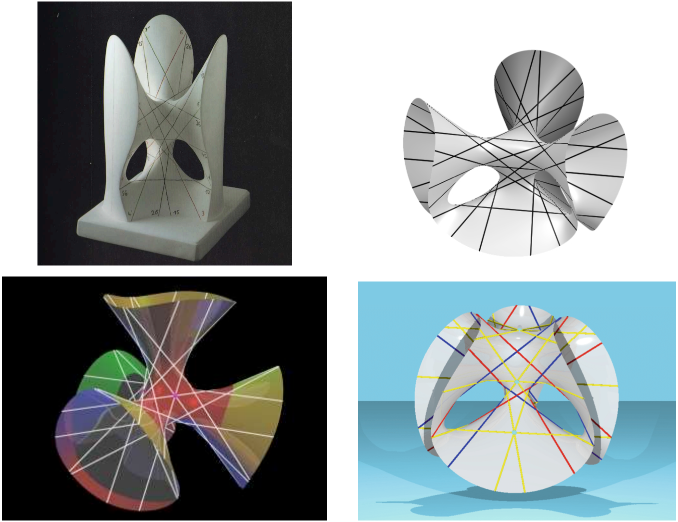

Figure: Clebsch surface: 27 lines

Figure: A mikado game !

CONFLUENCE = REMOVE STICKS

Figure: A mikado game !

CONFLUENCE = REMOVE STICKS

Figure: Conical (or nodal) singularity: A_1 type

The Cayley surface (a P_{VI} case and some P_{II} cases) :

$$S_{(0,0,0,1)} = \{xyz + x^2 + y^2 + z^2 - 1 = 0\} \subset \mathbf{C}^3.$$

For the complete surface (in $P_3(\mathbf{C})$), there are 9 lines (6+3 at infinity) and 4 nodal singular points.

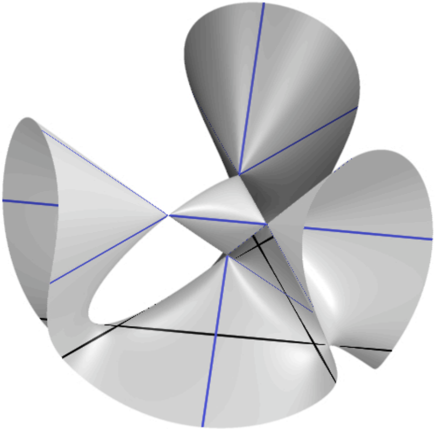

Figure: The Cayley surface.

Confluences of Painlevé equations

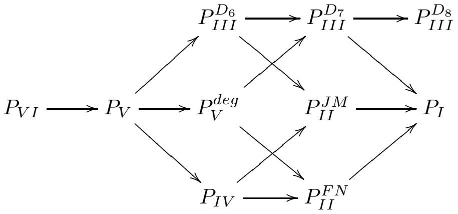

Figure: Confluence scheme of Painlevé equations according to Ohyama-Okumura

The remarkable idea of Martin Klimes is to interpret the confluences on the (wild) character varieties as a pure process of algebraic geometry involving birational transformations (a mikado game !).

In the *Plato ideal world*, all the (difficult) analysis disappears !

STOKES PHENOMENON
and
DYNAMICS ON WILD CHARACTER VARIETIES
of
PAINLEVÉ EQUATIONS

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Académie des Sciences
and
Institut de Mathématiques de Toulouse

Minicourse (I.3)

9th IST lectures on Algebraic Geometry and Physics,
Lisbon, february 2020

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FIRSTLY WE WILL DETAIL THE P_{II} CASE

More precisely the $P_{II,\alpha}^{FN}$ case

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THE CHARACTER VARIETY OF P_{II}

and

ITS GEOMETRY

The character variety of P_{II}

The second Painlevé equation :

$$(P_{II,\alpha}) \quad y'' = 2y^3 + xy - \alpha, \quad \alpha \in \mathbf{C}.$$

The second Painlevé equation can be interpreted as an *iso-irregular deformation* equation of a linear equation. There are two models (Lax pairs) Flaschka-Newell (FN) and Jimbo-Miwa (JM).

In the *first model*, the Flaschka-Newell model, the deformed linear equation admits 6 Stokes matrices at infinity (defined by 3 parameters s_1, s_2, s_3) and the character variety χ_α of P_{II} is defined by the cubic equation :

$$s_1 s_2 s_3 + s_1 - s_2 + s_3 = -2 \sin \pi\alpha$$

in $\mathbf{C}^3$ (α corresponds to the monodromy exponent of the linear equation at the origin).

We denote: $X := s_1, \quad Y := -s_2, \quad Z := s_3$ and $G_\alpha(X, Y, Z) := XYZ - X - Y - Z - 2\sin \pi\alpha$.

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We denote:

$X := s_1, \quad Y := -s_2, \quad Z := s_3$ and :

$$G_\alpha(X,Y,Z) := XYZ - X - Y - Z - 2\sin\pi\alpha.$$

The complex gradient of G_α is :

$$\frac{\partial G_\alpha}{\partial X} = YZ - 1, \quad \frac{\partial G_\alpha}{\partial Y} = XZ - 1, \quad \frac{\partial G_\alpha}{\partial Z} = XY - 1.$$

One equation of χ_α in $\mathbf{C}^3$ is : $G_\alpha = 0$.

A (quadratic) rational parametrization of χ_α is:

$$(X,Y) \mapsto Z := \frac{X + Y + 2\sin\pi\alpha}{XY - 1}; \quad (X,Y) \in \mathbf{C}^2 \setminus \{XY = 1\},$$

We have also the two other parametrizations obtained by circular permutations.

If $\alpha \notin \frac{1}{2} + \mathbf{Z}$, then χ_α is *smooth* and the images of the three rational parametrizations cover the surface χ_α:

$X = Y = Z = \pm 1$ if and only if $\sin\alpha = \mp 1$, or equivalently, $\alpha \in \frac{1}{2} + \mathbf{Z}$.

If $\alpha \in \frac{1}{2} + \mathbf{Z}$, then χ_α is singular (Cayley cubic surface) and "it admits another component" isomorphic to $P_1(\mathbf{C})$.

Contents

The $\mathbf{Z}/3\mathbf{Z}$ symmetries

Let $\omega \in \mathbf{C}$ be a third root of unity : $\omega^3 = 1$.

Let $\tilde{t} \mapsto \tilde{y}(\tilde{t})$ be a solution of the differential equation:

$$\frac{d^2\tilde{y}}{d\tilde{t}^2} = 2\tilde{y}^3 + \tilde{t}\tilde{y} - \alpha.$$

If:

$$t := \omega^{-1}\tilde{t} \quad \text{and} \quad y(t) := \omega\tilde{y}(\tilde{t}) = \omega\tilde{y}(\omega t),$$

then $t \mapsto y(t)$ is a solution of the differential equation:

$$y'' = \frac{d^2 y}{dt^2} = 2y^3 + ty - \alpha.$$

Therefore *the cyclic group* $\mathbf{Z}/3\mathbf{Z}$ *acts on the set of solutions of* P_{II}.

This action extends to the Okamoto semi-compactification. On the character variety, the corresponding action is the cyclic permutation of the coordinates :

$$(X, Y, Z) \mapsto (Y, Z, X) \mapsto (Z, X, Y).$$

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The symplectic structure on the character variety

There exists a rational symplectic structure on χ_α:

$$\omega = \frac{dX \wedge dY}{\frac{\partial G_\alpha}{\partial Z}} = \frac{dY \wedge dZ}{\frac{\partial G_\alpha}{\partial X}} = \frac{dZ \wedge dX}{\frac{\partial G_\alpha}{\partial Y}},$$

$$\omega = \frac{dX \wedge dY}{XY - 1} = \frac{dY \wedge dZ}{YZ - 1} = \frac{dZ \wedge dX}{ZX - 1}.$$

It is the Poincaré residue of the volume form :

$$dX \wedge dY \wedge dZ.$$

The *RH* map gives analytic *symplectic isomorphisms* between each Okamoto variety of initial conditions and χ_α (up to a choice of constant).

The symplectic structure is clearly invariant by the $\mathbf{Z}/3\mathbf{Z}$ action.

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Interpretation of P_{II} using *iso-irregular deformations*.

Lax pairs:

structure group $G := SL_2(\mathbf{C})$,

$$\frac{\partial \Psi}{\partial \xi} = \mathcal{A}\Psi, \qquad \frac{\partial \Psi}{\partial t} = \mathcal{U}\Psi,$$

$\mathcal{A}$, $\mathcal{U}$ "meromorphic" matrices (rational in ξ, ramified in $t \in \mathbf{C}^*$) with values in $sl_2(\mathbf{C})$, Ψ unknown matrice with values in $sl_2(\mathbf{C})$.

Compatibility condition:

$$\frac{d\mathcal{A}}{dt} - \frac{d\mathcal{U}}{d\xi} + [\mathcal{A}, \mathcal{U}] = 0.$$

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$$\frac{d\mathcal{A}}{dt} - \frac{d\mathcal{U}}{d\xi} + [\mathcal{A}, \mathcal{U}] = 0.$$

More precisely $\mathcal{A}$, $\mathcal{U}$ are rational in $(\xi, y, z := \frac{dy}{dt})$. The compatibility condition is equivalent to P_{II}. If y is a solution of P_{II}, then the compatiblity condition is satisfied. Conversely if, for a unknown function y (of t), the compatibility condition is satisfied, then y is a solution of P_{II}.

Each solution y of P_{II} gives an iso-irregular family of *linear* differential systems (in ξ): $\frac{\partial Y}{\partial \xi} = \mathcal{A}(\xi, t)Y$. The deformation parameter of an iso-irregular family is t.

From the analytic classification to the Lax pairs

Our approach is to start from the point of view of the *classification* (as in van der Put-Saito). Then the P_{II} case corresponds to a *unipotent matrix* A_0 and to the case $(0, 3/2)$. We get the FN system – Its-Kapaev version – (in ξ) after a ramified shearing of order two of the initial system (in x): $\xi^2 = x^{-1}$. Therefore there are in fact *two* systems corresponding to $\xi \mapsto -\xi$ (Bäcklund transformation):

$$\frac{\partial \Psi}{\partial \xi} = \mathcal{A}^* \Psi, \qquad \frac{\partial \Psi}{\partial t} = \mathcal{U}^* \Psi,$$

$$\mathcal{A}^*(\xi, t) := -\mathcal{A}(-\xi, t), \qquad \mathcal{U}^*(\xi, t) := \mathcal{U}(-\xi, t)$$

Compatibility condition for the second system:

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$$\frac{d\mathcal{A}^*}{dt} - \frac{d\mathcal{U}^*}{d\xi} + [\mathcal{A}^*, \mathcal{U}^*] = 0.$$

It is equivalent to the P_{II} equation with the parameter α changed in $-\alpha$:

$$P_{II,-\alpha} : \qquad y'' = \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha.$$

Explicit formulae

Newell-Flashka, Its, Kapaev, Kitaev, Novokshenov...

Pauli spin matrices:

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{cases} \mathcal{A}(\xi, t) & = -i(4\xi^2 + t + 2y^2)\sigma_3 - \left(4y\xi + \frac{\alpha}{\xi}\right)\sigma_2 - 2z\sigma_1 \\ \mathcal{U}(\xi, t) & = -i\xi\sigma_3 - y\sigma_2 \end{cases}$$

$$\begin{cases} \mathcal{A}^*(\xi, t) & = -\mathcal{A}(-\xi, t) \\ & = i(4\xi^2 + t + 2y^2)\sigma_3 - \left(4y\xi + \frac{\alpha}{\xi}\right)\sigma_2 + 2z\sigma_1 \\ \mathcal{U}^*(\xi, t) & = \mathcal{U}(-\xi, t) = i\xi\sigma_3 - y\sigma_2 \end{cases}$$

$$\sigma_2\sigma_1\sigma_2 = -\sigma_1, \qquad \sigma_2\sigma_3\sigma_2 = -\sigma_3$$

$$\mathcal{A}^* = \sigma_2\mathcal{A}\sigma_2$$

Irregular types:

$$q_t(\xi) := -i\left(\frac{4}{3}\xi^3 + t\xi\right), \; q_t^*(\xi) := q_t(-\xi) = i\left(\frac{4}{3}\xi^3 + t\xi\right)$$

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To each Stokes line Γ_k is associated an unipotent Stokes-matrix S_k, defined by $S_k := \Psi_k^{-1}\Psi_{k+1}$. We have:

$$S_{2p-1} = \begin{pmatrix} 1 & 0 \\ s_{2p-1} & 1 \end{pmatrix}, \quad S_{2p} = \begin{pmatrix} 1 & s_{2p} \\ 0 & 1 \end{pmatrix}, \quad p = 1,\ 2,\ 3.$$

From $\mathcal{A}(-\xi, t) = -\sigma_2 \mathcal{A}(\xi, t)\sigma_2$, we get $S_{k+3} = \sigma_2 S_k \sigma_2$, therefore $s_{k+3} = -s_k$.

We have:

$$S_1 S_2 \dots S_6 = M,$$

therefore:

$$s_1 - s_2 + s_3 + s_1 s_2 s_2 = -2\sin \pi\alpha.$$

We get the equation of an *affine cubic surface* of $\mathbf{C}^3$:
$X := s_1,\ Y := -s_2,\ Z := s_3,$

$$G_\alpha(X, Y, Z) = XYZ - X - Y - Z - 2\sin \pi\alpha = 0.$$

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Stokes lines and Stokes multipliers of $\frac{\partial \Psi}{\partial \xi} = \mathcal{A}\Psi$

Γ_k: $\frac{\pi}{6} + (k-1)\frac{\pi}{3}$, $k = 1, \ldots, 6$

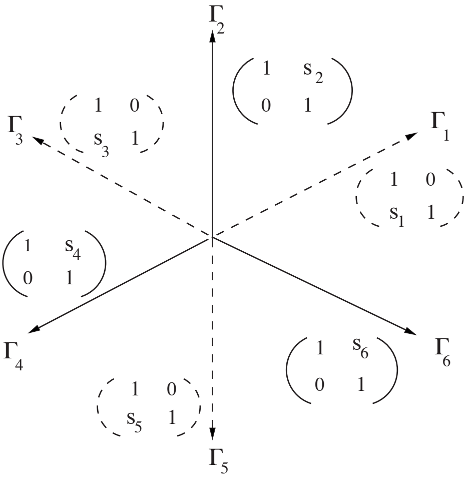

Figure: Following A. Its

Contents

The lines on the character variety

We suppose $\alpha \notin 1/2 + \mathbf{Z}$ and $\alpha \notin \mathbf{Z}$.

There are $9 = 6 + 3$ lines on χ_α plus 3 lines at infinity:

$$YZ = 1, \quad ZX = 1, \quad XY = 1, \quad X = 0, \quad Y = 0, \quad Z = 0.$$

There are 3 nodal singularities at infinity.

The 6 lines form a non planar *hexagon*: 6 edges; $XY = 1$ (resp. ...) defines two parallel lines.

The 3 lines form a *triangle*: 3 edges.

The 9 lines and the 9 points correspond to "special solutions" of P_{II} that one can identify using *asymptotic properties* at infinity $t \to \infty$ (non trivial...) : Gevrey asymptotics, summability, resurgence, Boutroux tronquées and tritronquées solutions...

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The lines on the character variety

We suppose $\alpha \notin 1/2 + \mathbf{Z}$ and $\alpha \notin \mathbf{Z}$.

There are $9 = 6 + 3$ lines on χ_α plus 3 lines at infinity:

$$YZ = 1, \quad ZX = 1, \quad XY = 1, \qquad X = 0, \quad Y = 0, \quad Z = 0.$$

There are 3 nodal singularities at infinity.

The 6 lines form a non planar *hexagon*: 6 edges;
$XY = 1$ (resp. ...) defines two parallel lines.

The 3 lines form a *triangle*: 3 edges.

The 9 lines and the 9 points correspond to "special solutions" of P_{II} that one can identify using *asymptotic properties* at infinity $t \to \infty$ (non trivial...) :
Gevrey asymptotics, summability, resurgence, Boutroux tronquées and tritronquées solutions...

The 9 lines are the "skeleton" of 6 *symplectic rational dynamics* on χ_α ; in particular one parameter groups, corresponding via *RH* to actions of exponential tori.

The 12 lines picture

Each pair of parallel lines of the hexagon cut one of the lines of the triangle at two points and one of the lines of the triangle at infinity at one point.

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Singularities

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SOLUTIONS OF P_{II}

NON LINEAR STOKES PHENOMENA

Asymptotics solutions of P_{II} at infinity

In the complex t plane, there are 6 sectors (at infinity) separated by the half lines :

$$\arg t = k\pi/3 \ (k = 0, 1, 2, 3, 4, 5).$$

Our presentation is based upon some works of :

– P. Boutroux (1913):
In general the asymptotics in each sector of a solution are *elliptic* ; for some exceptional solutions (tronquées, tri-tronquées, bi-tronquées) the asymptotics are *"classical"* in some sectors (in 2 or 4 sectors).

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All the asymptotics are explicitly parametrized by the Stokes data $(s_1, s_2, s_3) = (X, -Y, Z)$ (Andrei Kapaiev).
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Boutroux tronquées and tritronquées solutions of P_I

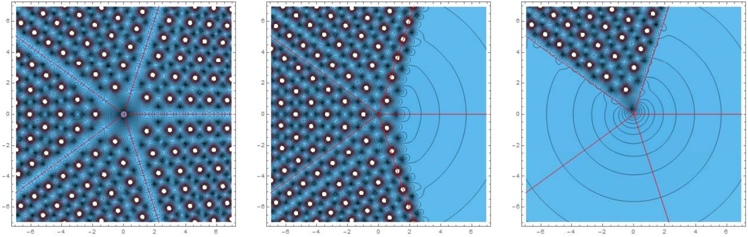

Figure: Poles of solutions near infinity. From Marcel Wonk

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Boutroux tronquées and tritronquées solutions of P_{II}

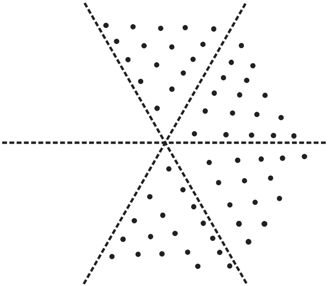

Figure: Poles of a truncated solution.

For P_{II}: 4 pole free sectors (among 6) for the tritronquées and 2 pole free sectors for the tronquées (4 pole free sectors for the bitronquées).

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Truncated solutions
"The Russian Orientation Table"

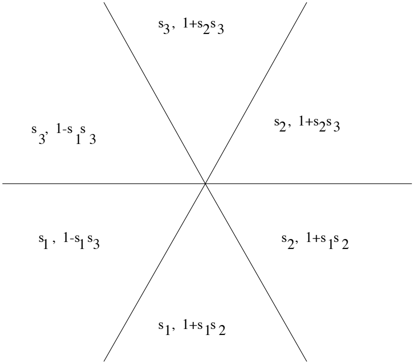

Figure 2: Combinations of the Stokes multipliers whose non-triviality yields the elliptic asymptotic behavior of Painlevé function in the corresponding sector

Figure: Following Its-Kapaiev; $X = s_1$, $Y = -s_2$, $Z = s_3$

Contents

Asymptotics solutions of P_{II} at infinity

Puiseux asymptotics: perturbative solutions (tritruncated, weak separatrices at ∞)

It is easy to check that, for the solutions of P_{II}, the only possible classical asymptotics at the first order are :

$$\sigma\sqrt{-t/2} \ \ (\sigma = \pm 1) \text{ and } \ -\alpha/t \ \ (\text{if } \ \alpha \neq 0).$$

Then it is possible to compute recursively *formal* Puiseux solutions: perturbative solutions $\hat{y}_{pert}$:

$$\hat{y}_{pert} = i\,\frac{t^{1/2}}{\sqrt{2}} \sum_{n\geq 0} \frac{a_n}{t^{3n/2}}, \quad a_0 = 1, \ \ a_1 = -i\,\frac{\alpha}{\sqrt{2}},$$

$$\hat{y}_{pert} = -\frac{\alpha}{t} \sum_{n\geq 0} \frac{b_n}{t^{3n}}, \quad b_0 = 1.$$

These solutions are (in general) *divergent*, but *summable*, and their sums are actual solutions of P_{II}: using *RH*, we get the *hexagon edges* in the first case and the *triangle edges* in the second case. (Tritronquées, 0-instantons, weak separatrices.)

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One parameter solutions of P_{II}
(Tronquées, proper transseries, 1-instantons)

Boutroux discovered that the Painlevé equations P_I and P_{II} admit a finite number of *1-parameter truncated solutions*) "formally represented" by a sum:

$$\hat{y} = \text{(Puiseux power series)} + \text{(exponential terms)}$$
$$\hat{y} = \hat{y}_{pert} + \hat{y}_{nonpert} \,.$$

In the terminology of resurgence (Ecalle) such an expression is named a (proper) *transseries*. In the physicists terminology (quantum field theory) it is named a (formal) 1*-instanton solution* (the perturbative part y_{pert} is a formal 0*-instanton solution*).

One parameter ($C \in \mathbf{C}$) formal solutions:

$$\hat{y}_{pert} + \hat{y}_{nonpert} = \hat{y}_{pert} + (-t)^{-\frac{3}{2}\sigma\alpha - \frac{1}{4}} \sum_{m \geq 1} C^m e^{\pm \frac{2\sqrt{2}\,m}{3}(-t)^{3/2}} \hat{y}^{(m)}$$

or

$$\hat{y}_{pert} + \hat{y}_{nonpert} = \hat{y}_{pert} + t^{-1/4} \sum_{m \geq 1} C^m e^{\pm \frac{2m}{3} t^{3/2}} \hat{y}^{(m)}.$$

The $\hat{y}^{(m)}$ ($m \in \mathbf{N}^*$) are (ramified) summable power series. With the good sign (according to the sectors) we get proper convergent transseries whose sums are actual solutions (C small).

The map : $\quad C \mapsto \Phi(C) = (X(C), Y(C), Z(C)) \in \mathbf{C}^3,$

induced by RH, extends analytically to $C \in \mathbf{C}$ and, in all cases, it is an *affine* map.

It is an important and difficult result coming from the russian works (explicit formulas).

The Stokes maps transform a sum of (Puiseux) power series solution to another, that is, via RH, an edge of the hexagon or the triangle to a contiguous one.

Comparing the sums of the $\hat{y}^{(m)}$, we can extend the Stokes maps along the lines of χ_α (cf. the russian works). We get *translations* on each line, obtained, via *RH* from :

$$C \mapsto C + a.$$

The exponential tori actions on the formal one parameter solutions :

$$C \mapsto \tau C \ (\tau \in \mathbf{C}^*)$$

give, via RH, *affine transforms* on the lines (each one fixing an edge).

Hexagon : $a = -\frac{2\pi\sigma i}{\Gamma(\frac{1}{2}-\sigma\alpha)} 2^{-\frac{5}{2}\sigma\alpha - \frac{7}{4}}$. Triangle: $a = i\,\frac{\sin \pi\alpha}{\sqrt{\pi}}$.

(Examples of russian computations.)

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(Examples of russian computations.)

Elliptic asymptotics: the Boutroux ansatz
(using Jacobi elliptic functions)

Large $|t|$ asymptotics of the generic solutions in the 6 open basic sectors (cf. the russian orientation table):

$$y(t) \approx \frac{i}{\sqrt{1+\kappa^2}} \operatorname{sn}\left(\frac{2}{3} c\, t^{3/2} + K \log Z - iK' \log(1 - ZX); \kappa\right);$$

κ, c transcendental functions of arg t.

Shift (automorphism) : $K \log Z - iK' \log(1 - ZX)$.

There is a mysterious relation with the exponential tori dynamics on χ_α.

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WILD DYNAMICS ON THE CHARACTER VARIETY OF P_{II}

The heart of the matter

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The heart of the matter

Painlevé equations at infinity
(Orbifolds, following Hayato Chiba)

H. Chiba introduced *orbifold* compactifications of the Painlevé equations (2014). For P_{II} he get the weighted projective space $\mathbf{C}P^3(2,1,2,3)$. After some weighted blow-ups, he can recover the Okamoto picture and *at infinity (the essential singularity)* he gets an autonomous Hamiltonian vector field on $\mathbf{C}^2$ (in fact $\mathbf{C}^2/\mathbf{Z}_2$) :

$$2X_3\frac{\partial}{\partial Y_3} + (4Y_3^3 + 2Y_3)\frac{\partial}{\partial X_3};$$

$$\mathcal{H}_{II} = X_3^2 - Y_3^4 - Y_3^2.$$

There are $3 = 2 + 1$ singular points :

$(X_3, Y_3) = (0, \pm i/\sqrt{2})$ and $(X_3, Y_3) = (0, 0)$, There is a corresponding *non linear Stokes-phenomena* at these points. We will describe it using recent technics introduced by Amaury Bittmann : *transversally symplectic doubly-resonant saddle nodes in* $\mathbf{C}^3$.

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Analytic classification of doubly-resonant saddle nodes in $\mathbf{C}^3$ (Following Amaury Bittmann)

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Notation: $(\varepsilon, \mathbf{y}) = (\varepsilon, y_1, y_2) \in \mathbf{C}^3$.

Doubly-resonant analytic saddle node:

$$\mathcal{X} := \varepsilon^2 \frac{\partial}{\partial \varepsilon} + (-\lambda y_1 + F_1(\varepsilon, \mathbf{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + F_2(\varepsilon, \mathbf{y})) \frac{\partial}{\partial y_2},$$

with $\lambda \in \mathbf{C}^*$, $F_1, \ F_2$ germs of analytic functions at the origin vanishing at order at least two.

Eigenvalues of the linear part $(0, -\lambda, \lambda)$: *two resonances*.

A. Bittmann classified the doubly-resonant analytic saddle-nodes up to analytic fibered diffeomorphisms using the cohomological method of Martinet-Ramis for the two dimensional saddle-nodes. It is more difficult...

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Bittmann formal normal forms

I skip some technical conditions... Normal forms :

$\mathcal{X}_{norm} =$
$\varepsilon^2 \frac{\partial}{\partial \varepsilon} + (-\lambda y_1 + a_1\varepsilon + c_1(v)))\, y_1 \frac{\partial}{\partial y_1} + (\lambda y_2 + a_2\varepsilon + c_2(v))\, y_2 \frac{\partial}{\partial y_2},$
where $v := y_1 y_2$, $c_1, c_2 \in \varepsilon\mathbf{C}[[\varepsilon]]$, $a_1, a_2 \in \mathbf{C}$ with $a_1 + a_2 = res(\mathcal{X})$.

Even if $\mathcal{X}$ is convergent, c_1 and c_2 can be divergent.

In the case of the *transversally Hamiltonian* saddle-nodes (the case of the Painlevé equations at the irregular singularity at infinity), $a_1 + a_2 = 1$ and $c_1 + c_2 = 0$.

Moreover, if $\mathcal{X}$ is convergent, then $c_1 = -c_2$ is also *convergent*.

This is related to the classication of germs of (generic) analytic Hamiltonians in the one degree of freedom case in a neighborhood of a stationary point (elimination of the non-resonant terms): resonant monomial $y_1 y_2$. (Cf. $\mathcal{H}_{II}$.)

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Integrability of the normal form

Notation:

$$c(v) = c_1(v) = \sum_{k\in\mathbf{N}^*} c_k v^k; \quad \tilde{c}(v) := \sum_{k\geq 2} \frac{c_k}{k-1} v^k.$$

The Bittmann normal form is *integrable in closed form*:

$$\begin{cases} y_1(\varepsilon) = C_1 \exp\left(\lambda/\varepsilon - \tilde{c}(C_1 C_2 \varepsilon)\right)\varepsilon^{a - C_1 C_2} \\ y_2(\varepsilon) = C_2 \exp\left(-\lambda/\varepsilon + \tilde{c}(C_1 C_2 \varepsilon)\right)\varepsilon^{1 - a + C_1 C_2} \end{cases}$$

We have $C_1 C_2 = \frac{y_1 y_2}{\varepsilon}$: $\frac{y_1 y_2}{\varepsilon}$ is a first integral.

This is a parametrization of the space of leaves of $\mathcal{X}_{norm}$ which are not contained in $\{\varepsilon = 0\}$.

Exponential torus action : $\tau \in \mathbf{C}^*$ acts on the leaves by $C_1 \mapsto \tau C_1$ and $C_2 \mapsto \tau^{-1} C_2$ ($e^{\lambda/\varepsilon} \mapsto \tau e^{\lambda/\varepsilon}$).

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Summability of the normalizing transormation and non-linear Stokes phenomena

Bittmann proves that the normalizing transformation is 1-summable onto two “big sectors”. By comparison we get two Stokes diffeomorphisms. There are also two actions of the exponential torus.

Be careful: the sums exists only on *“effilated domains”*

For the Painlevé equations (except of course P_{VI}), one obtains a rigourous definition of the wild dynamics. It we translate these dynamics on the corresponding character varieties (using RH), we get a priori *local* dynamics: 9 in the P_{II} case.

In the P_{II} case, we know already the dynamics on the lines (the skeleton). Then *the local dynamics extend in some neighborhoods of the lines*. We have sewed some flesh on the 9 lines skeleton. Now, we will do better !

Summability of the normalizing transormation and non-linear Stokes phenomena

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The 3 hexagonal tori dynamics

For $\tau \in \mathbf{C}^*$, we define $T_{3,\tau} : \mathbf{C}^3 \setminus \{Z=0\} \mapsto \mathbf{C}^3 \setminus \{Z=0\}$ by:

$$X' = T_{3,\tau}(X) := \frac{1-\tau}{Z} + \tau X,$$
$$Y' = T_{3,\tau}(Y) := \frac{1-\tau^{-1}}{Z} + \tau^{-1} Y,$$
$$Z' = T_{3,\tau}(Z) := Z.$$

Similarly we define $T_{2,\tau}$ and $T_{1,\tau}$. For $j = 1,2,3$, we have : $T_{j,\tau_2} \circ T_{j,\tau_1} = T_{j,\tau_2\tau_1}$.
Therefore each $\{T_{j,\tau}\}_{\tau\in\mathbf{C}^*}$ is a one parameter subgroup of rational transformations.

Proposition

We suppose $\alpha \notin \frac{1}{2} + \mathbf{Z}$. For $j = 1,2,3$, $\{T_{j,\tau}\}_{\tau\in\mathbf{C}^*}$ is a one parameter subgroup of rational transformations on χ_α. This group fixes (globally) 4 lines and two points.

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The 3 triangular tori dynamics

For $\tau \in \mathbf{C}^*$, we define $T'_{3,\tau} : \mathbf{C}^3 \setminus \{Z=0\} \mapsto \mathbf{C}^3 \setminus \{XY=1\}$ by :

$$X' = T'_{3,\tau}(X) := \tau X, \quad Y' = T_{3,\tau}(Y) := \tau^{-1} Y,$$

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The 6 Exponential Tori Dynamics on χ_α

"Theorem"

We suppose $\alpha \notin \frac{1}{2} + \mathbf{Z}$ and $\alpha \notin \frac{1}{2} + \mathbf{Z}$. The 3 groups $\{T_{j,\tau}\}_{\tau \in \mathbf{C}^*}$ and the 3 groups $\{T'_{j,\tau}\}_{\tau \in \mathbf{C}^*}$, $j = 1, 2, 3$, corresponds, via RH, to exponential tori actions on the Okamoto varieties of initial conditions.

Up our (strong) rationality conjecture, this theorem follows from simple unicity results for the rational one-parameter groups acting on χ_α and from the knowledge of the exponential tori dynamics on the lines (cf. the results of the russian school).

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The one parameter group $\{T_{3,\tau}\}_{\tau\in\mathbf{C}^*}$ is generated by the following field:

$$\frac{XZ-1}{Z}\frac{\partial}{\partial X}-\frac{YZ-1}{Z}\frac{\partial}{\partial Y}=\frac{\partial G_\alpha}{\partial Y}\frac{\partial}{\partial X}-\frac{\partial G_\alpha}{\partial X}\frac{\partial}{\partial Y}.$$

It is the hamiltonian field associated to the Hamiltonian :

$$H(X,Y,Z)=\log Z,$$

using the symplectic form ω_α.

The one parameter group $\{T'_{3,\tau}\}_{\tau\in\mathbf{C}^*}$ is generated by the following field:

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Stokes maps on the character variety χ_α

We know the existence of the Stokes maps on a neighborhood of the skeleton.

We conjecture that the Stokes maps belongs to the dynamics generated by the 6 exponential tori dynamics. More precisely it is possible to guess explicit (simple) formulas looking at what we know on the skeleton. The rationality of the Stokes maps would follow.

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RESURGENCE(S) of P_{II}

Resurgence of P_{II}

We recall the Bittmann formal normal form :

$$\begin{cases} y_1(\varepsilon) = C_1 \exp\left(\lambda/\varepsilon - \tilde{c}(C_1 C_2 \varepsilon)\right)\varepsilon^{a - C_1 C_2} \\ y_2(\varepsilon) = C_2 \exp\left(-\lambda/\varepsilon + \tilde{c}(C_1 C_2 \varepsilon)\right)\varepsilon^{1 - a + C_1 C_2} \end{cases}$$

We use :

$$\varepsilon^{C_1 C_2} = e^{C_1 C_2 \log \varepsilon}.$$

Then, using Bittmann results, we get two parameters (C_1 and C_2) series solutions in y_1, y_2 with summable power series coefficients (2-instantons solutions), therefore transasymptotic series solutions of some type. They give actual solutions in convenient domains.

The apparition of $\log \epsilon$ in the transasymptotic series is related to the *symplectic resonance*.

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In the case $\alpha = 0$ (only the hexagon Stokes phenomena exists), Ricardo Schiappa and Ricardo Vaz 2015 obtained formally (ansätz and recursion relations) two parameters transasymptotic series solutions.

By comparison it would be possible to prove the summability of the power series coefficients in Schiappa-Vaz formulas. Their resurgence is another story !

CONFLUENCES

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The character varieties of the Painlevé equations
(Following M. van der Put-M.H. Saito, Chekhov-Mazzocco-Roubtsov...)

PVI	$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$
PV	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 = \omega_4$
PIV	$x_1x_2x_3 + x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_2 x_3 + 1 = \omega_4$
PIII	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1 x_1 + \omega_2 x_2 = \omega_1 - 1$
PII	$x_1x_2x_3 + x_1 + x_2 + x_3 = \omega_4$
PI	$x_1x_2x_3 + x_1 + x_2 + 1 = 0$

Figure: Affine cubic surfaces

Confluence scheme of the Painlevé equations

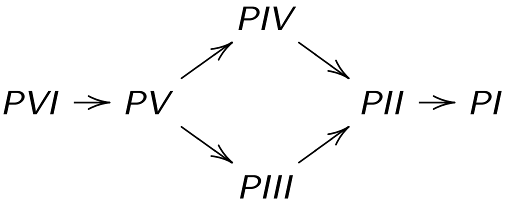

Figure: Painlevé-Okamoto confluence scheme

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Confluence scheme (or coalescent diagram)
of the Painlevé equations

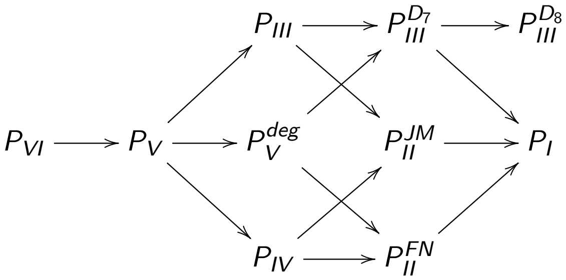

Figure: From Y. Ohyama, S. Okumura 2006

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Confluence and unfolding of linear systems

According to: Garnier, Ramis, Duval, Zhang, Schäfke,...

The model is the confluence of the *hypergeometric equations*. We rescale : $(0, 1, \infty) \to (0, 1/\varepsilon, \infty)$ and afterwards : $\varepsilon \to 0$ ("$1 \to \infty$");

$$\xi(\xi - \varepsilon)d/d\xi \to \xi^2 d/d\xi.$$

The semi-simple part of the monodromy of a vanishing loop (in between $1/\varepsilon$ and ∞) *swirls* : NO LIMIT ! ($e^{2i\pi/\varepsilon}$). The unipotent part of this monodromy admits in some sense a limit, giving birth to the Stokes maps.

MAIN IDEA. It is not so good to consider a *continuous* confluence. It is better to consider *discrete* confluences: $\left(\frac{1}{\varepsilon_n} = \frac{1}{\varepsilon_0} + n\right)$. Then it is possible to *fix* the monodromy of a vanishing loop. A change of discretization corresponds to an exponential torus action ($\tau = e^{2i\pi/\varepsilon_0} \in \mathbf{C}^*$).

This idea was firstly developed by Zhang 1996.

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Confluence of dynamics on character varieties

The affine character varieties χ_{VI} and χ_V of P_{VI} and P_V are *birationnally equivalent*. The Fricke coordinates of P_{VI} *do not pass to the limit*. The idea of M. Klimes is to replace these coordinates by some rational functions *passing to the limit*. Then he gets “good coordinates” on the (wild) character varieties χ_V.

Afterwards he translates the (algebraic) braids action on χ_{VI} (half-monodromies) into a rational dynamics on χ_V.

The wild dynamics on χ_V

(Tout est pour le mieux dans le meilleur des mondes possibles)

The wild dynamics on χ_V (Stokes, exponential tori actions...) is built using Bittmann work on doubly resonant saddle nodes as we explained before.

In order to translate this wild dynamics on χ_V (by RH), M. Klimes uses unfoldings of the Bittmann saddle nodes. This allows him to prove the rationality conjecture and to compute explicitely the rational wild dynamics

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Our project is to extend Klimes method to the confluences :

$$P_V \to P_V^{deg} \to P_{II}^{FN}$$

in order to prove the conjectures presented before in the P_{II} case.

Next step : the differential Galois groupoid of P_{II} is "big" for all the values of α.

Heuristics:

If $\alpha \notin 1/2 + \mathbf{Z}$ and $\alpha \notin \mathbf{Z}$, the hexagonal AND the triangular dynamics are "rich".

If $\alpha \in 1/2 + \mathbf{Z}$, the triangular dynamics is "rich" (the other is "lost" : Ricatti-Airy solutions, singular point of type A_1).

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STOKES PHENOMENON
and
DYNAMICS ON WILD CHARACTER VARIETIES
of
PAINLEVÉ EQUATIONS

Jean-Pierre Ramis

Académie des Sciences
and
Institut de Mathématiques de Toulouse

Minicourse (II-1)

9th IST lectures on Algebraic Geometry and Physics,
Lisbon, february 2020

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- The character variety of the free group of rank 3
- Partial reducibility of representations and lines on character varieties
- The (wild) character varieties of P_{VI}, P_{V}, P_{II}

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THE CUBIC SURFACE $X \subset \mathbf{P}^3$

Crashcourse...

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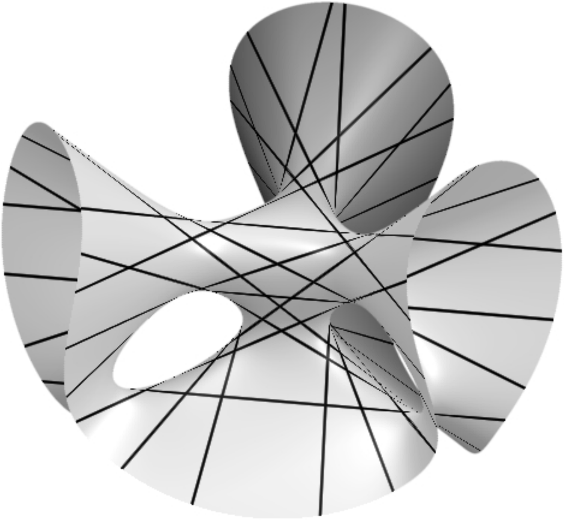

Figure: 27 lines on a cubic surface

Motivations

Lines on cubic surfaces and Painlevé equations

The (wild) character varieties, that is the variety of (wild) monodromy representations (up to equivalence) of the linearized equations of the Painlevé equations are *affine cubic surfaces* $\mathcal{S}$:

$$\mathcal{S} = X \setminus \{3 \text{ lines at infinity}\}.$$

The *lines* on $\mathcal{S}$ correspond to some *special representations* : they are *locally reducible* (classically or wildly).

On the Painlevé equations side (the left side of RH), the lines correspond to classical (or less classical...) one parameter families of solutions: *truncated solutions* (or less classical variants...)

The intersections of two lines correspond to some "special solutions" : *tri-truncated solutions* or *bi-truncated solutions* (or variants...).

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are new :

- E. Paul, J.P. Ramis for P_{II}, 2017;
- M. Klimes, E. Paul, J.P. Ramis for P_{VI}, P_V and degenerated P_V, 2019.

1. is proved for all cases;
2. is a work in progress.

Singularities on the character variety $\mathcal{S}$

In some cases $\mathcal{S}$ has some singularities (the maximal number is 4). They are isolated *rational singularities*.

Generically the RH map is *an analytic diffeomorphism*. In all cases it is *a proper map*. It is conjectured (proved for P_{VI}) that RH is *a minimal desingularization*. The inverse image of a singular point of $\mathcal{S}$ is a family of *Riccati solutions*.

Figure: Conical (or nodal) singularity: A_1 type

In local coordinates : $z^2 = xy$.

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The Cayley surface (a P_{VI} case and some P_{II} cases) :
4 nodal singular points.

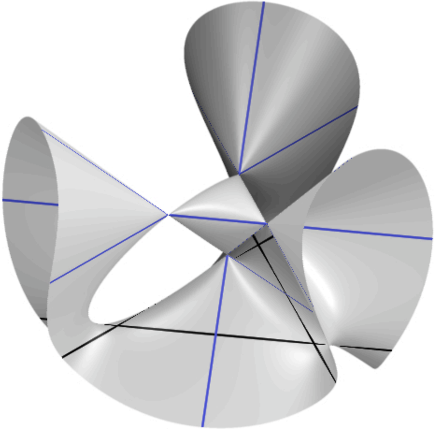

Figure: The Cayley surface.

There are several ways of studying the *smooth* cubic surface $\mathcal{S} \subset \mathbf{P}^3$ and its 27 lines :

- using elementary coordinate geometry in $\mathbf{P}^3$;
- as blowup of $\mathbf{P}^2$ in 6 points, or of $\mathbf{P}^1 \times \mathbf{P}^1$ in 5 points;
- ....

Approaches in this style do not give *the full symmetry of the configuration of lines*. The symmetry group of the configuration has order $51\,840 = 2^7.3^4.5 = 2 \times 25920$. It is the Weyl group $W(E_6)$ of the complex linear algebraic group E_6 (of dimension 78): the automorphism group of the unique simple group of order 25920.

The configuration of lines does have an entirely symmetric description in terms of a certain lattice :

$$A(\mathcal{S}) = \operatorname{Pic} \mathcal{S} = H^2(\mathcal{S}; \mathbf{Z}) \approx H_2(\mathcal{S}; \mathbf{Z}).$$

The lines $\ell \subset \mathcal{S}$ are the solutions in this lattice of the equations $h\ell = 1$, $\ell^2 = -1$, where h is the class of the hyperplane section.

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Lines on cubic surfaces of $\mathbf{P}^3$

Important points

Let X be a nonsingular cubic surface of $\mathbf{P}^3$. A *triangle* is a set of 3 distinct coplanar lines $\ell_1,\ \ell_2,\ \ell_3 \subset X$ such that $\ell_1 + \ell_2 + \ell_3 = X \cap H$ is a hyperplane section. We will prove :

- X contains at least one line;
- any 2 intersecting lines determine a triangle;
- if $\ell_1,\ \ell_2,\ \ell_3$ is a triangle and ℓ' a fourth line of X, then ℓ' meets exactly one ℓ_i;
- if ℓ is a line of X, then there is exactly 10 lines meeting ℓ, falling into 5 coplanar pairs; the pairs are disjoint;
- there exist 2 disjoint lines (a pair of skew lines).

Some points are evident but some others not at all !

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The magic number

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We admit the following (non trivial...) result. We will prove it later.
In what follows k is an algebraically closed field of characteristic 0.

Theorem

A cubic suface $X \subset \mathbf{P}^3(k)$ (smooth or not) allways contains a line.

Pencil of planes defined by $\ell \subset \mathcal{S}$

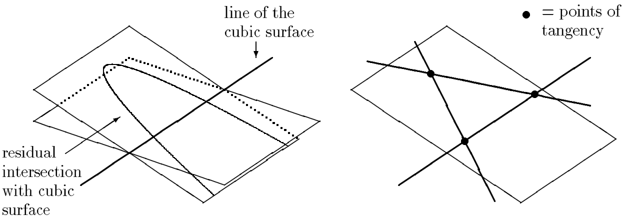

Figure: Generic section and tritangent section

The section is a *cubic curve*: $\ell \cup \mathcal{C}$, $\mathcal{C}$: conic curve.
If $\mathcal{C}$ degenerates into 2 lines, then we have a *tritangent plane*.

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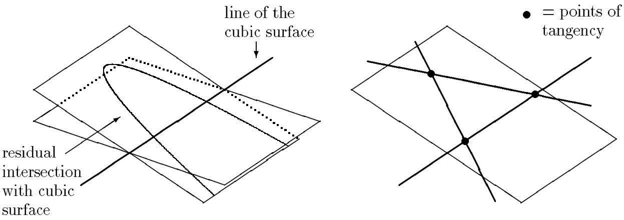

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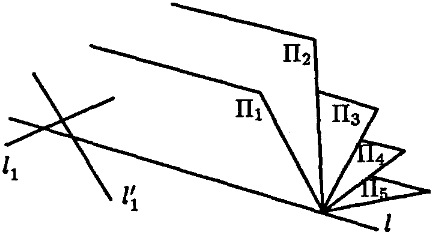
There are 5 tritangent planes in a pencil

Proposition

We suppose $X \subset \mathbf{P}^3$ smooth. Given a line $\ell \subset X$:

(i) there exists exactly 5 planes containing ℓ such that the conic $\mathcal{C}$ is *decomposed* into 2 lines ℓ_i, ℓ_i';

(ii) moreover, for $i \neq j$, $(\ell_i \cup \ell_i') \cap (\ell_j \cup \ell_j') = \varnothing$



Proof

Homogeneous coordinates $(u, v, w, t) \in (k^4)^*$. Suppose that $\ell = \{w = t = 0\}$. Then $X = V(f)$, with :

$$f(u, v, w, t) = Au^2 + Buv + Cv^2 + Du + Ev + F,$$

where $A, B, C, D, E, F \in k[w, t]$, A, B, C linear forms, D, E quadratic forms, F a cubic form.
If we consider $f = 0$ as a conic in (u, v) then it is singular if and only if :

$$\Delta(w, t) := \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 4ACF + BDE - AE^2 - B^2F - CD^2 = 0.$$

$\Pi = \{\mu w = \lambda t\}$. If $\mu \neq 0$, we can assume $\mu = 1$, then $\Pi = \{w = \lambda t\}$ and $\Delta(w, t)$ has 5 roots in λ counted with multiplicities. Using the smoothness of X it is easy to prove that the roots are simple.

The magic number 27

We start from a given tritangent plane (3 lines: ℓ_1, ℓ_2, ℓ_3) and we count. It meets $3.4 = 12$ other tritangent planes in each of which there are 2 other lines which gives :

$$3 + 12.2 = 27$$

lines on X.

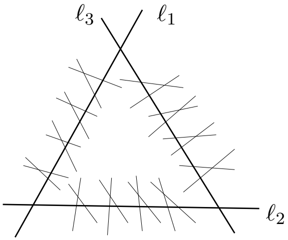
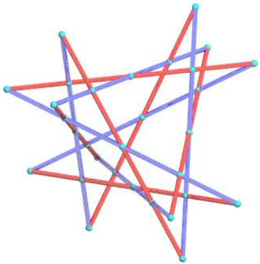


Figure: 27=3+8+8+8 lines

More on lines configuration

- 45 tritangent planes
- 135 points
- 216 pairs of skew-lines
- 36 doublesix



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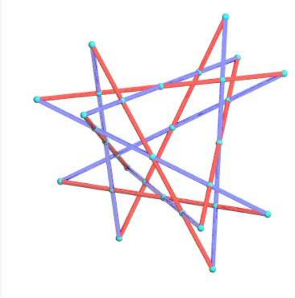

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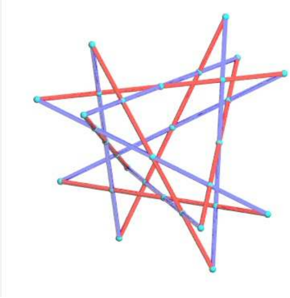

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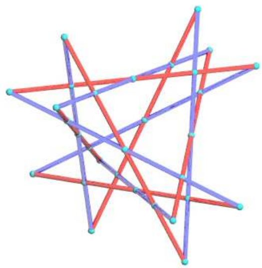

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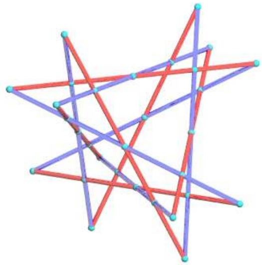

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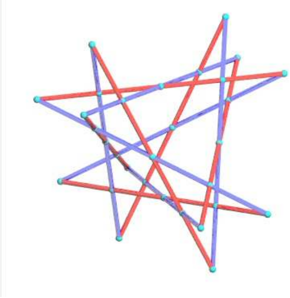

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Proposition

A smooth cubic surface $X \subset \mathbf{P}^3(k)$ is a *rational surface*.

Using a pair of skew lines (ℓ_1, ℓ_2) on X, we can define a birational map :

$$\varphi : X \to \ell_1 \times \ell_2 \approx \mathbf{P}^1 \times \mathbf{P}^1.$$

If $M \in X \setminus \ell_1 \cup \ell_2$, then there exists a unique line $\ell \subset \mathbf{P}^3$ through M which meets both ℓ_1 and ℓ_2. Then we set :

$$\varphi(M) := (\ell \cap \ell_1, \ell \cap \ell_2).$$

If $P \in \ell_1$, $Q \in \ell_2$, we set $\ell := PQ$. Generically this line cut X into 3 points P, Q, M. We set $\psi(P, Q) := M$.
The map ψ is rational. The maps φ and ψ are mutual inverses.

The lattice $A(\mathcal{S})$ and Pic $\mathcal{S}$

A claim

We choose a pair of skew lines ℓ, m. The line ℓ takes part in exactly 5 triangles ℓ, ℓ_i, ℓ'_i and m meets exactly one ℓ_i, ℓ'_i for each $i = 1, 2, 3, 4, 5$. By renumbering let these be ℓ_i. Then the 10 lines meeting m form 5 triangles m, ℓ_i, ℓ''_i. We have $17 = 2 + 5 + 5$ lines.
The following result is easy. We denote (i, j, k, l, m) a permutation of $(1, 2, 3, 4, 5)$.

Claim

- There are 10 more lines ℓ_{klm} which meet ℓ_k, ℓ_l, ℓ_m and not ℓ_i, ℓ_j;
- $\ell'_i, \ell''_j, \ell_{klm}$ is a triangle.

The lattice $A(X)$
(X nonsingular)

The lattice $A(X)$ is a **Z**-module defined by generators and relations. The generators are the 27 lines and the relations are roughly speaking "triangle=constant". More precisely $A(X)$ is the free abelian group on the 27 lines modulo the relations $\ell + \ell' + \ell'' = m + m' + m''$ (for each pair of triangles).

Lemma

The lattice $A(X)$ is generated by the 7 elements $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_5', \ell_5''$.

The proof is easy : $\ell + \ell_i + \ell_i' = \ell + \ell_5 + \ell_5' \dots$. Then :

$$\ell_i' = \ell_5 + \ell_5' - \ell_i \quad \ell_i'' = \ell_5 + \ell_5'' - \ell_i \quad \ell_{klm} = \ell + \ell_i - \ell_j''$$

$$\ell = 2(\ell_5 + \ell_5' + \ell_5'') - \ell_5' - \ell_1 - \ell_2 - \ell_3 - \ell_4.$$

Scalar product and $A(X) \approx \mathbf{Z}^7$
(X nonsingular)

One verifies easily that there existes a unique scalar product $A(X) \times A(X) \to \mathbf{Z}$ such that :

– if ℓ, ℓ' are distinct, then $\ell.\ \ell' = 0$ or 1 according as they are disjoint or intersect;

– for any line ℓ, $\ell^2 = -1$;

– for any line ℓ and any triangle $m,\ m',\ m''$, $\ell(m + m' + m'') = 1$.

Let :

$$e_0 := \ell_5 + \ell'_5 + \ell''_5,\ e_i := \ell_i,\ i = 1, 2, 3, 4,\ e_5 := \ell'_5,\ e_6 := \ell''_5.$$

Proposition

(i)

$$e_0^2 = 1, \quad e_i^2 = -1, \quad i = 1, 2, 3, 4, \quad e_i e_j = 0, \quad i \neq j.$$

(i) $(e_i)_{i=0,\dots,6}$ is a $\mathbf{Z}$-basis of $A(X)$.

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The scalar product can be diagonalized to

$$\mathrm{Diag}(-1, 1, \dots, 1).$$

Let $h \in A(X)$ be the class of a triangle. Then $h^2 = 3$ and, for all $x \in A(X)$, $hx = x^2 \mod 2$.

The 27 lines are the solutions of the equations $h\ell = 1$ and $\ell^2 = -1$.

It is a *symmetric* characterization of the lines.

Interpretations: $A(X)$, $H^2(X;\mathbf{Z})$, $H_2(X;\mathbf{Z})$, $\operatorname{Pic} X$ $(k := \mathbf{C})$

If $\ell \subset X$ is a line then its homology class $[\ell]$ belongs to $H_2(X;\mathbf{Z})$. The map $\ell \mapsto [\ell]$ induces an isomorphism $A(X) \to H_2(X;\mathbf{Z})$.

The hyperplane section defines a cohomology class $h \in H^2(X;\mathbf{Z})$ such that $h^2 = 3$ and $hx = x^2 \mod 2$ for all $x \in H^2(X;\mathbf{Z})$.

Let $\alpha,\ \alpha'$ be linear forms on $\mathbf{P}^3$ such that the planes $\{\alpha = 0\}$ and $\{\alpha' = 0\}$ cut out respectively two triangles T and T' on X. Then $\alpha/\alpha' \in k(X)$ and $\operatorname{div}(\alpha/\alpha') = T - T'$.

Two triangles are *linearly equivalent*. One gets a map :

$$A(X) \to \operatorname{Pic} X = \operatorname{Div} X/\sim .$$

It is a bijection.

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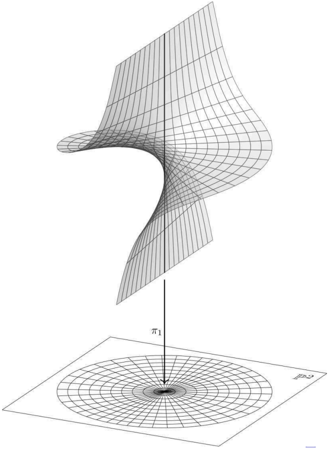
Blow-up of a point in a plane

$$\{X := ((x,y),[z:w]) |\, xz - yw = 0\} \subset A^2 \times \mathbf{P}^1$$

($w = 1, \;\; y = xz$, saddle surface);
$\pi : X \to A^2$ is an isomorphism away from $(0,0)$; the exceptionnal divisor is $\pi^{-1}(0,0) \approx \mathbf{P}^1$



The cubic surface as a blow up of $\mathbf{P}^2$

General position for 6 points in the plane:
the points are distinct, no 3 collinear points, not all 6 points on a conic.

Theorem

Every nonsingular cubic surface is the blow up of 6 points in general position in the projective plane $\mathbf{P}^2$.

We can describe the 27 lines from this point of vue.

- 6 skew lines are the inverse images of the 6 points;
- there is a single conic in the plane for each choice of 5 points, the strict inverse images give 6 skew lines;
- each line through 2 points of the plane gives a line (15 lines).

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The cubic surface as a blow up of $\mathbf{P}^2$

(A del Pezzo surface)

Let $\Sigma := \{P_i\}_{i=1,2\ldots,6}$ be a set of six points in general position in the plane. Let X be the corresponding blow-up at the 6 points.
The vector space V of 3 forms on P^2 vanishing at Σ is 4 dimensional. If $(F_j)_{j=1,2,3,4}$ is a basis of V, then the rational map $\mathbf{P}^2 \to \mathbf{P}^3$ defined by $P \mapsto (F_j(P))$ induces an isomorphism of X with a cubic surface of $\mathbf{P}^3$.

This construction can be extended to special configurations of 6 points. We get singular cubic surfaces.

If 3 points are collinear, then the corresponding line is contracted into a singular point of X.

The canonical divisor class of a cubic surface

Let X be a non-singular (projective) variety of dimension n. Let $f_1, f_2, \dots, f_n \in k(X)$ such that $k(f_1, f_2, \dots, f_n) \subset k(X)$ is a finite algebraic extension. A rational n-form on X is by definition $\omega := g df_1 \wedge df_2 \wedge \dots \wedge df_n$, with $g \in k(X)$. In local coordinates $z_1, z_2, \dots, z_n$, $s = Jg\, df_1 \wedge df_2$, with $J := \frac{D(f_1, f_2, \dots, f_n)}{D(z_1, z_2, \dots, z_n)}$. If Γ is a prime divisor on X, then we set $v_\Gamma(s) := v_\Gamma(Jg)$ and $\operatorname{div} s = \sum_\Gamma v_\Gamma(s)\Gamma$; the class of $\operatorname{div} s$ (modulo linear equivalence) is independent of the choice of s, it is the *canonical divisor class* of X:

$$K_X := (\text{class of}) \operatorname{div} s.$$

The canonical divisor class of $\mathbf{P}^n$ is $K_{\mathbf{P}^n} = -(n+1)H$, where H is the class of the hyperplane $\mathbf{P}^{n-1} \subset \mathbf{P}^n$.

Adjunction formula : if $X \subset Y$, then $K_X = (K_Y + X)_X$.

If $X \subset \mathbf{P}^3$ is a non-singular cubic surface, then $K_X = (K_{\mathbf{P}^3} + X)_X = (-4H + 3H)_X = -H_X = -h$.

More generally, let X be a non-singular hypersurface of degree d of $\mathbf{P}^m$. Let $f = 0$ be an equation of degree d of $X \cap \mathbf{C}^m$. Then the Poincaré residue on X of the meromorphic form $f^{-1} dz_1 \wedge \ldots \wedge dz_m$ vanishes of order $d - m - 1$ at infinity, and so $\omega_X \approx \mathcal{O}_X(d - n - 1)$.
If $d = n = 3$, then $\omega_X \approx \mathcal{O}_X(-1)$, i. e. $K_X = -H_X$.

Let Y be a compact connexed complex manifold of dimension n. Its canonical sheaf ω_Y is the sheaf of holomorphic n-forms, that is of the holomorphic sections of holomorphic line bundle defined by the n-th exterior power of the holomorphic cotangent bundle of Y.

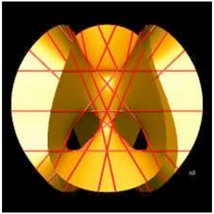
If the *anticanonical sheaf* ω_Y^{-1} is ample, then, for some $k \in \mathbf{N}^*$, the linear system $|\omega_Y^{-k}|$ embeds Y in a projective space. By definition Y is a *Fano variety*.

The hypersurface X is a Fano variety if and only if $d \leq m$. The Fano varieties of dimension 2 are called *del Pezzo surfaces*. A smooth cubic surface is a del Pezzo surface.

Monodromy, star points

We consider a family $\{X_t\}_{t\in T}$ of nonsingular cubic surfaces. The 27 lines give a 27-covering $\Lambda \to T$. Moving along a closed loop in T permutes the 27 lines. We get an homomorphism $\pi_1(T, t_0) \to W(E_6)$.

A *star point* (also called Eckardt point) on a nonsingular cubic surface is the intersection point of 3 lines on the surface. A non-singular cubic surface does not have more than 18 star points.
One can move a nonsingular cubic surface with star points in such a way that these points disappear.



Existence of a line on a cubic surface

Theorem

A cubic surface $X \subset \mathbf{P}^3(k)$ (smooth or not) allways contains a line.

We recall the definition of the Plücker coordinates of a line in P^3. Let $V \subset k^4$ be a vector plane. To $x = (x_0, x_1, x_2, x_3),\ y = (y_0, y_1, y_2, y_3) \in V$, we associate the six numbers $p_{ij} := x_i y_j - x_j y_i$ $(i, j = 0, 1, 2, 3,\ i < j)$. We suppose that $x,\ y$ generates V. Then $p := (p_{ij}) \in k^6 \setminus \{0\}$ is independant of the choice of x, y up to a scaling. We can associate to V an element of $\mathbf{P}^5$. We have the Plücker relation :

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.$$

Then $G(2, 4)$, the Grassmanian of 2 planes in k^4, or equivalently the variety of lines in P^3, is identified with an hypersurface $G_4 \subset \mathbf{P}^5$.

For $r, k \in \mathbf{N}^*$, we have dim $H^0(\mathbf{P}^r; \mathcal{O}_{\mathbf{P}^r}(k) = \binom{r+k}{r}$. If $r = k = 3$, then $\binom{r+k}{r} = 20$.

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Therefore we can identify the variety of cubic surfaces in $\mathbf{P}^3$ with $\mathbf{P}^{19}$.
We can define the incidence variety $Z \subset G_4 \times \mathbf{P}^{19}$:

$$Z := \{(\ell, X) | \, \ell \subset X\}.$$

We have two maps :

$$p : Z \to G_4, \quad q : Z \to \mathbf{P}^{19}$$

induced by the natural projections.

Any line ℓ is contained in a (say reducible) cubic surface, therefore p is surjective. If $\ell = \{x_2 = x_3 = 0\}$, then $\ell \subset X$ if and only if the 4 coefficients of $x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3$ in an homogeneous form of degree 3 defining X vanishes. Therefore the dimension of the fibres of p is $19 - 4 = 15$ and the dimension of Z is $15 + \dim G_4 = 15 + 4 = 19$.

There exists a cubic surface X_0 with a finite set of line. Consider $X_0 := \{x_1x_2x_3 - x_0^3 = 0\}$: in the part in k^3 the surface X_0 does ot contain any line, whereas in the plane at infinity it contains 3 lines $(x_1x_2x_3 = 0)$.

The fiber $q^{-1}(X_0)$ is finite. The image $q(Z) \subset \mathbf{P}^{19}$ is closed and we have $0 = \dim q^{-1}(X_0) \geq \dim Z - \dim q(Z)$. Therefore $\dim q(Z) = 19 = \dim \mathbf{P}^{19}$ and $q(Z) = \mathbf{P}^{19}$: the map q is surjective.

Remark

We consider the quartic surfaces in $\mathbf{P}^3$. The dimension of the variety of quartics is $35 - 1 = 34$. The dimension of the incidence variety Z' is 33, therefore the map q' cannot be surjective. *There exists a quartic surface in $\mathbf{P}^3$ on which there are no line at all*.

Contents

SINGULAR CUBIC SURFACES

Type	Nonsing.	A_1	$2A_1$	A_2	$3A_1$	A_1A_2	A_3	$4A_1$	A_22A_1	A_3A_1
Codimension	0	1	2	2	3	3	3	4	4	4
Class	12	10	8	9	6	7	8	4	5	6
No. of lines	27	21	16	15	12	11	10	9	8	7

Type	$2A_2$	A_4	D_4	A_32A_1	$2A_2A_1$	A_4A_1	A_5	D_5	$3A_2$	A_5A_1	E_6
Codimension	4	4	4	5	5	5	5	5	6	6	6
Class	6	7	6	4	4	5	6	5	3	4	4
No. of lines	7	6	6	5	5	4	3	3	3	2	1

Figure: Bruce-Wall table

(Wild) character varieties of some Painlevé equations
(Upper Bruce-Wall table)

- *Column 1:* generic P_{VI}, $27 = 24 + 3$ lines, smooth;
- *Column 2:* generic P_V, $21 = 18 + 3$ lines, 1 singularity A_1 (at infinity);
- *Column 4:* generic $P_V^{\deg}$, $15 = 12 + 3$ lines, 1 singularity A_2 (at infinity);
- *Column 5:* generic P_{II}, $12 = 9 + 3$ lines, 3 singularities A_1 (at infinity);
- *Column 7:* $P_{II,\alpha}$ with $\alpha \in \mathbf{Z}$, $10 = 7 + 3$ lines, 1 singularity A_3;
- *Column 8:* $P_{II,\alpha}$ with $\alpha \in \frac{1}{2} + \mathbf{Z}$, $9 = 6 + 3$ lines, 4 singularities A_1;
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Verification "by hand" on the equations (elementary but a little bit boring...).

P-eqs	Polynomials
PVI	$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 + \omega_4$
PV	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 + 1 + \omega_3^2 - \frac{\omega_3(\omega_2+\omega_1\omega_3)(\omega_1+\omega_2\omega_3)}{(\omega_3^2-1)^2}$
PV_{deg}	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_1 - 1$
PIV	$x_1x_2x_3 + x_1^2 + \omega_1x_1 + \omega_2(x_2 + x_3) + \omega_2(1 + \omega_1 - \omega_2)$
$PIII$	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_1 - 1$
$PIII^{D_7}$	$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 - x_2$
$PIII^{D_8}$	$x_1x_2x_3 + x_1^2 + x_2^2 - x_2$
PII^{JM}	$x_1x_2x_3 - x_1 + \omega_2x_2 - x_3 - \omega_2 + 1$
PII^{FN}	$x_1x_2x_3 + x_1^2 + \omega_1x_1 - x_2 - 1$
PI	$x_1x_2x_3 - x_1 - x_2 + 1$

Correction: permute P_{II}^{JM} and P_{II}^{FN}.

CONJECTURE. (Wild) character varieties of Painlevé equations cannot appear in the columns 12, 16, 17, 18, 20, 21 of the Bruce-Wall table.

STOKES PHENOMENON
and
DYNAMICS ON WILD CHARACTER VARIETIES
of
PAINLEVÉ EQUATIONS

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Minicourse (II-2)

9th IST lectures on Algebraic Geometry and Physics, Lisbon, february 2020

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- The character variety of the free group of rank 3
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The geometry of the cubic surfaces $\mathcal{S}(a)$

Contents

We recall the equation of the Fricke cubic $\mathcal{S}(a)$:

$$F(X_0, X_t, X_1, a) = X_0X_tX_1 + X_0^2 + X_t^2 + X_1^2 - A_0X_0 - A_tX_t - A_1X_1 + A_\infty.$$

The equation of the projective surface $\overline{\mathcal{S}(a)} \subset \mathbf{P}^3(\mathbf{C})$ in projective coordinates $(\tilde{X}_0, \tilde{X}_t, \tilde{X}_1, \tilde{T})$ is :

$$\tilde{X}_0\tilde{X}_t\tilde{X}_1 + \tilde{X}_0^2\tilde{T} + \tilde{X}_t^2\tilde{T} + \tilde{X}_1^2\tilde{T} - A_0\tilde{X}_0\tilde{T}^2 - A_t\tilde{X}_t\tilde{T}^2 - A_1\tilde{X}_1\tilde{T}^2 + A_\infty\tilde{T}^3 = 0.$$

The plane at infinity $\tilde{T} = 0$ is a tri-tangent plane and its intersection with the surface is the triangle $\{\tilde{X}_0\tilde{X}_t\tilde{X}_1 = 0\}$. Therefore the affine cubic surface $\mathcal{S}(a)$ contains exactly 24 affine lines. We will describe these lines.

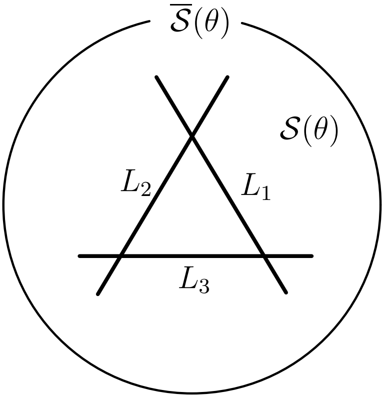

Figure: The triangle at infinity

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Each line at infinity is contained in 4 tri-tangent planes different from the plane at infinity. The intersection of such a tri-tangent plane and $\overline{\mathcal{S}(a)}$ is a triangle, therefore the intersection with $\mathcal{S}(a)$ is the union of 2 affine lines with a common point. Therefore for each line at infinity we get 8 affine lines on $\mathcal{S}(a)$. Using the coordinates X_0, X_t, X_1 we see that for each $l = 0, t, 1$ there exists 4 exceptional values of X_l such that $\{X_l = 0\} \cap \mathcal{S}(a)$ is the union of 2 affine lines.

We will interpret the 24 lines on $\mathcal{S}(a)$ using some properties of representations.

Contents

24 lines into $\mathcal{S}(a)$
(distinct or not)

Let $a_0, a_t, a_1, a_\infty \in \mathbf{C}$ *arbitrary*. The 24 lines *distinct or not* defined in $\mathbf{C}^3$ by the following equations are contained in the cubic surface $\mathcal{S}(a) \subset \mathbf{C}^3$:

$$X_k = e_i e_j^{-1} + e_j e_i^{-1}, \quad e_i X_i + e_j X_j = a_\infty + e_i e_j a_k,$$
$$X_k = e_i e_j^{-1} + e_j e_i^{-1}, \quad e_i X_j + e_j X_i = a_k + e_i e_j a_\infty,$$
$$X_k = e_i e_j + e_i^{-1} e_j^{-1}, \quad X_i + e_i e_j X_j = e_j a_k + e_i a_\infty,$$
$$X_k = e_i e_j + e_i^{-1} e_j^{-1}, \quad X_j + e_i e_j X_i = e_j a_\infty + e_i a_k,$$
$$X_k = e_k e_\infty^{-1} + e_\infty e_k^{-1}, \quad e_\infty X_i + e_k X_j = a_i + e_k e_\infty a_j,$$
$$X_k = e_k e_\infty^{-1} + e_\infty e_k^{-1}, \quad e_k X_i + e_\infty X_j = a_j + e_k e_\infty a_i,$$
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Contents

There are strong relations between the classical geometry of a smooth complex cubic surface (27 lines, 45 tritangent planes...) and some properties of the representations into $SL_2(\mathbf{C})$ of a free group of rank 3. As far as we know these simple but important relations remained unnoticed until recently.

Let :

$$\omega : \ \Gamma_2 =< u, v > \to SL_2(\mathbf{C})$$

be a linear representation. We set :

$$M' := \omega(u) \ \text{ and } \ M'' := \omega(v).$$

We denote e' and $(e')^{-1}$ (resp. e'' and $(e'')^{-1}$) the eigenvalues of M' (resp. M''). We denote e and e^{-1} the eigenvalues of $M := M'M''$.

The following assertions are equivalent

(i) The representation ω is reducible.

(ii) The pair (M', M'') is reducible.

(iii) We have : $e = e'e''$ or $e = e'(e'')^{-1}$ or $e = (e')^{-1}e''$ or $e = (e')^{-1}(e'')^{-1}$.

(iv) We have Tr $M = e'e'' + (e'e'')^{-1}$ or Tr $M = e'(e'')^{-1} + (e')^{-1}e''$.

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Reducibility and mixed basis

Let ω be a representation and M', M'' as above.

We suppose that Tr $M' \neq \pm 2$ and Tr $M'' \neq \pm 2$. Then M' and M'' are diagonalisable. There allways exists a *mixed basis* $\{v', v''\}$ of $\mathbf{C}^2$ formed by an eigenvector of M' and an eigenvector of M''. In general there are (up to rescaling of the eigenvectors) 4 ways one can form such a basis.

The 4 cases of reducibility of ω correspond to the degeneracy of one of these 4 basis.

We recall that if ω is irreducible, then it is determined, up to equivalence, by the traces of M', M'' and M.

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We will describe a relation between a notion of *partial reducibility of a representation of* Γ_3 and *the lines on the cubic surface* $\mathcal{S}(a)$. This relation is apparently new.

If we replace representations by *wild representations*, then this relation can be extended to all the Painlevé equations.

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We denote $\Gamma_3 :=< u_0, u_t, u_1 >$ the free group of rank 3 generated by the letters $u_0,\ u_t,\ u_1$ and we set $u_\infty = u_1^{-1} u_t^{-1} u_0^{-1}$.

Let $\rho : \Gamma_3 \to SL_2(\mathbf{C})$ be a linear representation. We set $M_l := \rho(u_i)$ $(i = 0, t, 1, \infty)$ and we denote e_l and e_l^{-1} the eigenvalues of M_l.

We will say that the representation ρ is *partially reducible* if there exists $i, j \in \{0, t, 1, \infty\}$, $i \neq j$, such that the pair of matrices $\left(\rho(u_i), \rho(u_j)\right)$ is reducible.

We have the following characterizations of smoothness (some are classical and some are apparently new).

Let $a \in \mathbf{C}^4$. We suppose $a_l \neq \pm 2$ ($l = 0, t, 1, \infty$) (non resonance). The following conditions are equivalent :

(i) The affine cubic surface $\mathcal{S}(a)$ is smooth.

(ii) The projective cubic surface $\overline{\mathcal{S}(a)}$ is smooth.

(iii) The 24 lines described before are pairwise distinct.

Contents

(iv) The 3 following conditions are satisfied:

- the 4 numbers built from the e_l ($l = 0, t, 1, \infty$)

$$e_te_1^{-1} + e_1e_t^{-1},\quad e_te_1 + e_t^{-1}e_1^{-1},$$
$$e_0e_\infty^{-1} + e_\infty e_0^{-1},\quad e_0e_\infty + e_0^{-1}e_\infty^{-1}$$

  are pairwise distinct;
- the 4 numbers :

$$e_1e_\infty^{-1} + e_\infty e_1^{-1},\quad e_1e_\infty + e_1^{-1}e_\infty^{-1},$$
$$e_te_0^{-1} + e_0e_t^{-1},\quad e_te_0 + e_t^{-1}e_0^{-1}$$

  are pairwise distinct,
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$$e_te_0^{-1} + e_0e_t^{-1},\quad e_te_0 + e_t^{-1}e_0^{-1}$$

  are pairwise distinct.

Contents

(v) We have the 8 conditions: $e_0\, e_t^{\pm 1} e_1^{\pm 1} e_\infty^{\pm 1} \neq 1$ (the 3 signs are chosen independantly).

(vi) If ρ is a representation such that Tr $\rho(u_l) = a_l$ for all $l = 0, t, 1, \infty$, then it is irreducible.

If we use the parameters θ_l, then the conditions (v) are translated into:

$$\theta_0 \pm \theta_l \pm \theta_1 \pm \theta_\infty \in \mathbf{Z}$$

These conditions already appeared in several papers: Jimbo, Guzzetti, Lisovyy-Gavrilenko....

We suppose that the surface $\mathcal{S}(a)$ is smooth. For each pair (l, m) of elements of $\{0, t, 1, \infty\}$, each of the 2 planes :

$$\begin{cases} X_n = e_l e_m + e_l^{-1} e_m^{-1} \\ X_n = e_l e_m^{-1} + e_l^{-1} e_m \end{cases} \quad \text{with} \quad (l, m, n) = \begin{cases} (i, j, k) \\ (k, \infty, k) \end{cases} \tag{1}$$

intersects $\mathcal{S}(a)$ at 2 lines. The resulting 4 lines correspond to the reducibility of the pair of matrices (M_l, M_m).

Contents

More precisely if two matrices M_l and M_m are diagonalizable, for each of them there exists a pair of invariant subspaces giving rise to a basis. Then there are in general 4 possibilities of pairing of invariant subspaces out of which one can form a *mixed* basis. The cases of reducibility of the pair (M_l, M_m) corresponds to the degeneracy of (at least) one of these mixed bases. Each of the 4 lines corresponds to such a case of degeneracy.

Partial reducibility of representations and lines

Proposition

Let $a \in \mathbf{C}^4$ arbitrary and $\rho : \Gamma_3 \to SL_2(\mathbf{C})$ a representation such that Tr $\rho(u_l) = a_l$ $(l = 0, t, 1, \infty)$.

(i) If the representation ρ is *partially reducible*, then its equivalence class belongs to one of the 24 lines (distinct or not) defined before.

(ii) We suppose that $\mathcal{S}(a)$ is smooth. Then ρ is partially reducible if and only if its equivalence class belongs to one of the 24 lines of $\mathcal{S}(a)$.

Fibration by coordinates

We suppose that we are in the "generic case" (i. e. $\mathcal{S}_{VI}(a)$ is smooth).

Let $\Pi_0 : \mathcal{S}_{VI}(a) \to \mathbf{C}$, $\Pi_0 : (X_0, X_t, X_\infty) \mapsto X_0$. We recall :

$$\mathcal{S}_{VI}(a) = \left\{(X_0, X_t, X_1) \in \mathbf{C}^3 | \, F(X_0, X_t, X_1) = 0\right\}$$

$$F(X_0, X_t, X_1) = X_0 X_t X_1 + X_0^2 + X_t^2 + X_1^2 - A_0 X_0 - A_t X_t - A_1 X_1 + A_\infty .$$

For $c \in \mathbf{C}$, $\Pi_0^{-1}(c)$ is interpreted as an affine conic in the (X_t, X_1)-plane:

$$X_t^2 + X_1^2 + cX_t X_1 - A_t X_t - A_1 X_1 - cA_0 + A_\infty = 0.$$

The generic fiber of Π_0 is isomorphic to $\mathbf{C}^*$. The exceptional fibers are of of two types :

- either $X_0 = \pm 2$, then $X_t^2 + X_1^2 \pm 2X_tX_1 = (X_t \pm X_1)^2$, the fiber is a *parabola* and it is isomorphic to $\mathbf{C}$;
- either we are in a partially reductible case, that is in one of the 4 cases :

$$X_0 = e_t e_1^{-1} + e_1 e_t^{-1} \ \text{ or } \ X_0 = e_t e_1 + e_t^{-1} e_1^{-1} \ \text{ or}$$

$$X_0 = e_0 e_\infty^{-1} + e_\infty e_0^{-1} \ \text{ or } \ X_0 = e_0 e_\infty + e_0^{-1} e_\infty^{-1},$$

then the fiber is *degenerated into two lines*. The intersection of these two lines is a critical point of Π_0. Its image is a critical value of Π_0.

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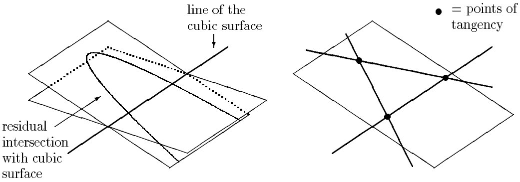

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If we remove the 6 exceptional fibers (that is 8 lines and two curves) from $\mathcal{S}_{VI}(a)$ then we can parameterize the remaining set by a Zariski open set of $\mathbf{C} \times \mathbf{C}^*$. Such "pants parameterizations" already appeared in many papers, following Jimbo (Lisovyy ?) :

$$(X_1^2 - 4)X_0 = D_{0,+}s + D_{0,-}s^{-1} + D_{0,0}$$
$$(X_1^2 - 4)X_t = D_{t,+}s + D_{t,-}s^{-1} + D_{t,0},$$

with coefficients given by (using $X_1 = 2\cos 2\pi\sigma_1$) :

$$D_{0,0} := X_1 A_t - 2A_0, \quad D_{t,0} = X_1 A_0 - 2A_t,$$
$$D_{0,\pm} := 16 \prod_{\epsilon=\pm 1} \sin\pi(\theta_t \mp \sigma_1 + \epsilon\theta_0) + \sin\pi(\theta_1 \mp \sigma_1 + \epsilon\theta_\infty),$$
$$D_{t,\pm} := -D_{0,\pm}\, e^{\mp 2i\pi\sigma_1}.$$

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Contents

LINES ON THE CHARACTER VARIETIES AND SPECIAL SOLUTIONS

The P_{VI} case

LINES ON THE CHARACTER VARIETIES

AND

SPECIAL SOLUTIONS

The P_{VI} case

Some special solutions of P_{VI} (Riccati solutions, rational solutions, algebraic solutions) are known for *special values* of the parameters. For generic values of the parameters, as far I know, the only known special solutions are the 12 Kaneko solutions.

I will return to this question using the dictionnary between the 24 lines and the partial reducibility of the representations and the RH map.

Contents

The Kaneko solutions
(Interpreted as central solutions)

Kaneko searched for formal solutions of P_{VI} at the fixed singularities $t = 0$, 1 or ∞. He got 4 solutions for each singularity and these formal power series are *convergent* (Briot and Bouquet).

The (extended) Painlevé Hamiltonian vector field $\mathcal{X}_{VI}$ has $12 = 3 \times 4$ singularities over $t = 0$, 1 or ∞. We can perform a local study of the field nearby these singularities. It is transversally symplectic and the eigenvalues are $(\lambda, -\lambda, \mu)$. We normalize $\mu := 1$. The singular points satisfy the hypothesis of Hadamard-Perron theorem, therefore there exists two local *analytic* invariant dimension 2 manifolds which intersect on an *analytic* central solution, a Kaneko solution.

We have already used this method for P_{II}. The formal Puiseux solutions correspond (generically) to 9 singular points of the Hamiltonian vector field $\mathcal{X}_{II}$ extended using Chiba orbifold technic.

We can do the same for P_V. Over $t = 0$ we get 3 singular points corresponding to the 3 Kaneko-solutions. The situation is similar to the P_{VI} case and the central solutions (intersection of 2 analytic invariant manifolds) are convergent.

Over $t = \infty$ it is more difficult. There are 5 formal solutions and there are (in general) divergent. It it not so easy to get them (one can use Bruno power geometry[1] or Chiba methods). Afterwards we can use the Bittmann results as for P_{II}.

[1]Newton polygons, cf. A. Bruno, I.V. Goryuchkina, around 2010.

Special solutions of P_{VI}

(In the generic case)

We return to P_{VI}. We will see that there are some special solutions corresponding by RH to some points on the 24 lines or at the intersections of two of these lines. By analogy with the case pf P_{II} and with Boutroux terminology we call them[2] *truncated*, *tri-truncated* and *bi-truncated*.

There are (generically) :

- 24 *one-parameter* families of *truncated* solutions,
- 12 *tri-truncated* solutions, the Kaneko solutions,
- 96 *bi-truncated* solutions.

[2] It is only a terminology; for these solutions,the asymptotic picture is different !

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We recall that the reducible locus of each pair (M_i, M_j) defines 4 lines into 2 parallel planes $\{X_k = \text{const}\}$. We return to this question in more details.

We choose a *mixed basis* taking an eigenvector of M_i and an eigenvector of M_j. In such a basis, we have :

$$M_i = \begin{pmatrix} e_i & 0 \\ f_i & e_i^{-1} \end{pmatrix} \quad M_j = \begin{pmatrix} e_j & f_j \\ 0 & e_j^{-1} \end{pmatrix}$$

Let $D_i := diag(e_i, e_i^{-1})$, $D_j := diag(e_j, e_j^{-1})$. We have :

$$\text{Tr}\, M_i M_j = \text{Tr}\, D_i D_j + f_i f_j$$

Such a pair is reducible if and only if $f_i = 0$ or $f_j = 0$, i. e. if and only if :

$$\text{Tr}\, M_i M_j = \text{Tr}\, D_i D_j.$$

Then the pair (M_i, M_j) is reducible if and only if $X_k = e_ie_j + e_i^{-1}e_j^{-1}$ or $X_k = e_ie_j^{-1} + e_i^{-1}e_j$.

Let $c_{i,j}^- := e_ie_j + e_i^{-1}e_j^{-1}$ and $c_{i,j}^+ := e_ie_j^{-1} + e_i^{-1}e_j$;
$\ell_{i,j}^+ := e_iX_i + e_jX_j - a_ke_ie_j - a_\infty$,
$\overline{\ell_{i,j}^+} = e_i^{-1}X_j + e_j^{-1}X_i - a_\infty e_i^{-1}e_j^{-1} - a_k$.

We recall $(F = F(X, a))$:

$$F = (X_k - c_{i,j}^+)(F_{X_k} - X_k + c_{i,j}^+) - \ell_{i,j}^+\overline{\ell_{i,j}^+}$$

Therefore the plane $\{X_k = c_{i,j}^+\}$ intersects the cubic surface on the union of two lines :

$$L_{i,j}^+ := \{X_k = c_{i,j}^+, \ell_{i,j}^+ = 0\} \quad \overline{L_{i,j}^+} := \{X_k = c_{i,j}^+, \overline{\ell_{i,j}^+} = 0\}.$$

We denote $p_{i,j}^+ := L_{i,j}^+ \cap \overline{L_{i,j}^+}$. We call it *a central point*.

Let :

$\ell_{i,j}^- := e_i X_i + e_j^{-1} X_j - a_k e_i e_j^{-1} - a_\infty,$

$\overline{\ell_{i,j}^-} = e_i^{-1} X_j + e_j X_i - a_\infty e_i^{-1} e_j - a_k.$

The plane $\{X_k = c_{i,j}^-\}$ intersects the cubic surface on the union of two lines :

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We denote $p_{i,j}^- := L_{i,j}^- \cap \overline{L_{i,j}^-}$. We call it *a central point*.

From the relation $M_0 M_t M_1 M_\infty = I$, we have :

$$X_k = \operatorname{Tr} M_i M_j = \operatorname{Tr} M_k M_\infty,$$

therefore the reducible locus of the pair (M_k, M_∞) appears in the two other planes :

$X_k = e_k e_\infty + e_k^{-1} e_\infty^{-1} = c_{k,\infty}^-$ and $e_k e_\infty^{-1} + e_k^{-1} e_\infty = c_{k,\infty}^+$.

The 8 lines meet the same line L_k at infinity.

The 12 central points are $p^{\pm}_{i,j}$ for any pair $\{i,j\}$ in $\{0, t, 1, \infty\}$. These points correspond to a representation such that M_i and M_j have 2 common eigenvectors ($p^{+}_{i,j}$ is defined by $f_i = f_j = 0$).

Using RH we can see what happens on the solutions side.

Proposition

(i) RH sens the 12 Kaneko solutions (central solutions) to the 12 central points.

(ii) The solutions in the 2 invariant planes at a Kaneko point correspond by RH to the points on the 2 lines intersecting at the corresponding central point.

Proofs: (i) follows from Kaneko work (Jimbo method); (i) and (ii) can be proved using the dynamics on each side of RH.

Bi-truncated solutions of P_{VI}

The bi-truncated solutions of P_{VI} correspond to the intersection of two invariant surfaces relative to two *different* Kaneko points. RH send them to intersections of lines which are not central points.

Such a solution is simply characterized by the fact that it is bounded at both of the singular points when approached along certain logarithmic spiral.

Contents

THE DYNAMICS OF P_{VI}

Contents

An algebraic symplectic dynamics on $\mathcal{S}$

Antisymplectic involutions on $\mathcal{S}(a)$

We denote L_h and q_h the 3 edges and the 3 vertices of the triangle at infinity Δ of $\mathcal{S}(a)$ ($h = 0, t, \infty$).

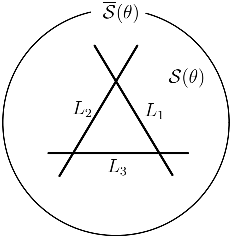

Figure: The triangle Δ

The affine cubic $\mathcal{S}(a)$ is a $(2, 2, 2)$ surface: if we fix 2 coordinates X_i, X_j, then $F(X, a)$ is a polynomial of degree 2 in X_k. The exchange of the two roots, X_i, X_j remaining fixed, defines an *involution* σ_k on $\mathcal{S}(a)$.

The 3 involutions σ_h are antisymplectic. We have a geometric interpretation of these involutions. A line in $P^3(\mathbf{C})$ passing by the vertice q_h (and not contained into the plane at infinity) cut $\mathcal{S}(a)$ in 2 points (distinct or not). The bijection of $\mathcal{S}(a)$ exchanging these 2 points is σ_h.

We denote $\mathcal{A} :=< s_0, s_t, s_1 >$ the subgroup of Aut$[\mathcal{S}(a)]$ generated by the 3 involutions.

$$\sigma_0 = \begin{cases} X_0 \mapsto X_0 \\ X_t \mapsto -X_1 - X_0X_t + A_1 \\ X_1 \mapsto X_t \end{cases} \qquad \sigma_t = \begin{cases} X_0 \mapsto X_1 \\ X_t \mapsto X_t \\ X_1 \mapsto -X_0 - X_tX_1 + A_0. \end{cases}$$

$$\sigma_1 = \begin{cases} X_0 \mapsto -X_t - X_1X_0 + A_t \\ X_t \mapsto X_0 \\ X_1 \mapsto X_1. \end{cases}$$

Quadratic transformations.

Theorem [Èl'-Huti 1974, Cantat-Loray 2009]

(i) The group $\mathcal{A}$ is isomorphic to the *free product* :

$$\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} =< s_0 > * < s_t > * < s_1 > .$$

(ii) The group $\mathcal{A}$ is of finite index (≤ 24) into Aut$[\mathcal{S}(a)]$. Generically $\mathcal{A} =$ Aut$[\mathcal{S}(a)]$.

(iii) The group Aut$[\mathcal{S}(a)]$ is generated by $\mathcal{A}$ and the group of affine transformations of $\mathbf{C}^3$ which preserves $\mathcal{S}(a)$.

We set : $\underline{g}_1 := \sigma_t \circ \sigma_0, \quad \underline{g}_0 := \sigma_1 \circ \sigma_t, \quad \underline{g}_t := \sigma_0 \circ \sigma_1$.
We have :

$$\underline{g}_1(X_0, X_t, X_1) =$$
$$\left(A_0 - X_0 - X_t X_1, A_t - A_0 X_1 + X_1 X_0 + (X_1^2 - 1)X_t, X_1\right)$$

A quadratic diffeomorphism of $\mathbf{C}^3$ which preserves the coordinate fibration by Π_1. On each fiber (a conic) it is induced by an affine transformation of $\mathbf{C}^2$.

The fixed points of $\underline{g}_1$ are 4 central points and the corresponding 8 lines (meeting L_1 at infinity) are (globally) invariant by $\underline{g}_1$.

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From the dynamics on $\mathcal{S}$ to the non-linear monodromy of P_{VI}

Contents

PARTIAL REDUCTIBILITY OF WILD REPRESENTATIONS

AND

LINES ON CHARACTER VARIETIES

Contents

For *all* Painlevé equations there exists a characterization of the lines on the (wild) character varieties by a property of (wild) reducibility of the (wild) representation of the linearized equation.

By definition, the (wild) reducibilty corresponds to the degeneracy of certain *“mixed bases” of privileged solutions*, that is to (wild) monodromy representations for which the corresponding pair of solutions is linearly dependent.

The way how these locally defined privileged solutions are continued is of essential importance.

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The way how these locally defined privileged solutions are continued is of essential importance.

The one-dimensional subspaces of the solution space from which come these bases are the following:

1. for each regular singularity there is a pair of *eigenspaces for the monodromy* if diagonalizable (Levelt basis) or a single eigenspace if nonlinearizable,
2. for each irregular singularity and each Stokes direction there is a unique associated space of *subdominant solutions*.

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The 18 lines on $\mathcal{S}_V$ from wild reducibility
(Klimes-Paul-Ramis)

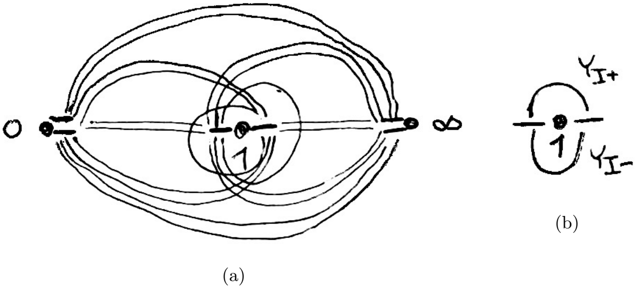

Figure 2: (a) Pairing on the distinguished subspaces of the solution space corresponding to the 18 lines on $\mathcal{S}_V$.
(b) The two pairs of subspaces of the solution space (each attached to one anti-Stokes direction) corresponding to the sectoral bases $Y_{I\pm}$.

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The 12 lines on $\mathcal{S}_V^{deg}$ from wild reducibility
(Klimes-Paul-Ramis)

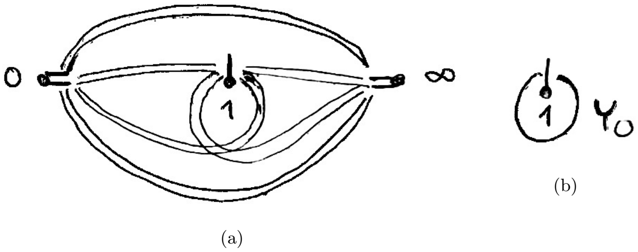

Figure 3: (a) Pairing on the distinguished subspaces of the solution space corresponding to the 12 lines on $\mathcal{S}_V^{deg}$.
(b) The pair of subspaces of the solution space corresponding to the sectoral basis Y_O.

Contents

Type	Nonsing.	A_1	$2A_1$	A_2	$3A_1$	A_1A_2	A_3	$4A_1$	A_22A_1	A_3A_1
Codi-mension	0	1	2	2	3	3	3	4	4	4
Class	12	10	8	9	6	7	8	4	5	6
No. of lines	27	21	16	15	12	11	10	9	8	7

$2A_2$	A_4	D_4	A_32A_1	$2A_2A_1$	A_4A_1	A_5	D_5	$3A_2$	A_5A_1	E_6
4	4	4	5	5	5	5	5	6	6	6
6	7	6	4	4	5	6	5	3	4	4
7	6	6	5	5	4	3	3	3	2	1

Figure: Bruce-Wall table

(Wild) character varieties of some Painlevé equations (Upper Bruce-Wall table)

- *Column 1:* generic P_{VI}, $27 = 24 + 3$ lines, smooth;
- *Column 2:* generic P_V, $21 = 18 + 3$ lines, 1 singularity A_1 (at infinity);
- *Column 4:* generic P_V^{deg}, $15 = 12 + 3$ lines, 1 singularity A_2 (at infinity);
- *Column 5:* generic P_{II}, $12 = 9 + 3$ lines, 3 singularities A_1 (at infinity);
- *Column 7:* $P_{II,\alpha}$ with $\alpha \in \mathbf{Z}$, $10 = 7 + 3$ lines, 1 singularity A_3;
- *Column 8:* $P_{II,\alpha}$ with $\alpha \in \frac{1}{2} + \mathbf{Z}$, $9 = 6 + 3$ lines, 4 singularities A_1;
- *Column 9:* P_I, $8 = 5 + 3$ lines, 1 singularity A_2 and 2 singularities A_1.

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Properties of (wild) reducibility of the (wild) monodromy of a linear equation are also strongly related to *spectral problems*.

- In 1975 Y. Sibuya studied an eigenvalues problem for the Schrödinger equations $y'' - P(x)y = \lambda y$, with $P(x) := x^m + a_1 x^{m-1} + \cdots + a_{n-1}x + a_m$, in relation with Stokes multipliers.

  He proposed to define eigenvectors as the solutions which are subdominant in two different sectors (a natural generalisation of the classical scattering[3]).

[3] Not in the L^2 spirit but in the pure spirit of Schrödinger original paper on the hydrogen atom !

- In a joint work (in progress...) with F. Richard-Young and J. Thomann we give a relation between the spectrums of the spheroidal differential equations and the reducibility of their monodromy (which can be considered as the wild reducibility of the wild monodromy...). This gives in particular a nice picture for the 4 Floquet spectrums of the Mathieu equation. The relation gives also a quite good method of numerical computation of the eigenvalues.

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