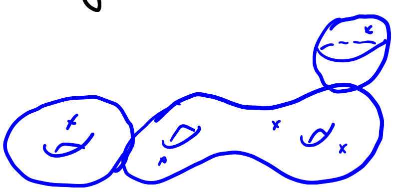
From categories to Gromov-Witten invariants

(joint work with Junwu Tu)

(I) Gromov-Witten invariants & Mirror Symmetry

(X^{2n}, ω) symplectic manifold

J = almost complex structure compatible with ω.



$$\left\{ [\Sigma_g, z_1 \ldots z_k] \xrightarrow[J\text{-holomorphic}]{u} X \quad [u] = \beta \in H_2(X) \right\}$$

$$\overline{M}_{g,k}(X,\beta) \xrightarrow{ev_i} X \qquad \text{compact, with virtual fundamental class of } \dim(g,k,n,c_1(\beta)).$$

$$[\Sigma_g, \vec{z}, u] \longrightarrow u(z_i)$$

$\overline{M}_{g,k}(X,\beta) \to \overline{M}_{g,k}$

Given $\alpha_1, \ldots, \alpha_k \in H^\bullet(X)$

$$\langle \alpha_1, \ldots, \alpha_k \rangle^X_{g,k,\beta} := \int_{\overline{M}_{g,k}(\beta)} \prod ev_i^*(\alpha_i) \in \mathbb{R}$$

Take $e_1, \ldots, e_m$ basis of $H^\bullet(X)$

$$t = \sum_{i=1}^m t_i e_i \Rightarrow \boxed{F^g_X(t) = \sum_{k,\beta} \langle t, \ldots, t \rangle^X_{g,k,\beta} \frac{T^{\omega(\beta)}}{k!}}$$

In genus = 0

$$\langle e_i *_t e_j, e_k \rangle_{PD} = \frac{\partial^3 F_X^0(t)}{\partial t_i \, \partial t_j \, \partial t_k}$$

Poincaré pairing in $H^\cdot(X)$

$\Rightarrow$ $*_t$ family of commutative, associative products on $H^\cdot(X, \Lambda)$ ← Novikov ring.

$$QH^\cdot(X) = \left(H^\cdot(X,\Lambda)[[t]], \langle , \rangle_{PD}, *_t \right)$$ Frobenius manifold

($\langle , \rangle_{PD}$ ↔ flat metric)

In higher genus: Cohomological Field Theory.

Mirror Symmetry

	symplectic: X compact ~~CY~~ Fano	mirror	complex: $\check{X}$ ~~compact~~ CY, $W: \check{X} \to \mathbb{C}$
① Enumerative (Hodge-theoretic)	Gromov-Witten inv X (genus = 0)		Period integrals $\check{X}$ of hol. vol. form; Saito's primitive forms
② Homological	$Fuk(X)$		$Coh_{dg}(\check{X})$; $MF(\check{X}, W)$

- Also works when X Fano, general type, ...

Ex: $X = \mathbb{CP}^n$ $\qquad\qquad$ $\check{X} = (\mathbb{C}^*)^n$

$$w(z_1,\ldots,z_n) = z_1 + \cdots + z_n + \frac{1}{z_1\cdots z_n}$$

Kontsevich proposal: Homological $\Rightarrow$ Enumerative Mirror Symmetry

- $Fuk(X)$, $Coh_{dg}(X)$, $MF(w)$ are examples of Calabi-Yau A_∞-categories

- genus $= 0$ GW, Period integrals, Primitive forms all determine Frobenius manifolds.

We would like to have:

$$\boxed{\text{CY } A_\infty\text{-cat.} \oplus \text{extra data}} \xrightarrow{\ \mathcal{F}_0\ } \boxed{\text{Frobenius manifold}}$$

such that:

i) A, A' derived equivalent $\Longrightarrow$ $\mathcal{F}_0(A)$, $\mathcal{F}_0(A')$ isomorphic

$\quad tw^{\pi} A \simeq tw^{\pi} A'$

ii) $A = Fuk(X) \Rightarrow \mathcal{F}_0(A) = QH^{\bullet}(X)$

$\vdots$

Or, more ambi-ious:

with analogous properties.

Rmk: A_∞ categories - unital, proper, smooth

+ Calabi-Yau $\left[\hom(X,Y)^\vee \cong \hom(Y,X) \right]$

+ splitting of non-commutative Hodge filtration.

II Hochschild invariants of A_∞-algebras

$\mathcal{A}$ A_∞-category $\left(\begin{smallmatrix}\text{smooth}\\ \text{proper}\end{smallmatrix}\right)$

Hochschild chains: $(CC_\bullet(\mathcal{A}), b)$

" cochains: $(CC^\bullet(A), b^*)$

Negative cyclic complex: $\underbrace{(CC_\bullet(A)[[u]], b+uB)}_{CC^-_\bullet(A)}$

X = (compact) complex manifold

$(\Omega^{\bullet,\bullet}(X), \bar\partial)$

$\Lambda^\bullet T_X$ polyvector fields

$(\Omega^{\bullet,\bullet}(X), \bar\partial + \partial)$

Idea: LHS has all operations that exist on RHS:

- Lie derivatives, inner products, Lie brackets, integration, ...

defn: (weak) Calabi-Yau structure on A is an element $\omega \in HH_\bullet(A)$ satisfying a non-degeneracy condition.

Analogy: A holomorphic volume form.

Rmk: ω induces $HH_\bullet(A) \overset{\tilde{\omega}}{\cong} HH^\bullet(A) \leftarrow$ finite dimensional.

defn: Splitting of Hodge filtration $s : HH_\bullet(A) \longrightarrow HC_\bullet^-(A)$ splitting the map $\pi : HC_\bullet^-(A) \xrightarrow[u=0]{} HH_\bullet(A)$.

Rmk: $HC_\bullet^-(A) \cong HH_\bullet(A)[[u]]$.

Analogy: X Kähler $\Rightarrow$ $H^\bullet_{dR}(X) \cong \bigoplus_{p+q=\bullet} H^{p,q}(X)$

Conjecture (Kontsevich-Soibelman): A smooth proper $\Rightarrow$ splittings exist.

thm (Kaledin): Conjecture holds when A is $\mathbb{Z}$-graded.

Upshot: F and F^0 are invariants of the triple

$$(A, \omega, s)$$

A: smooth, proper A_∞-cat

ω: CY str

s: splitting of Hodge filtration.

III F°: Categorical primitive forms

Step 1: $R = \mathbb{K}[[HH^\bullet(A)^\vee]]$

$\mathbb{A} :=$ versal deformation of A (R-linear A_∞-cat)

[Requires only existence of ω, s.] $\Rightarrow$ $HC^-_\bullet(\mathbb{A})$ finite dimensional free $R[[u]]$-module.

equipped with · flat connection ∇^{GGM}

· pairing $\langle , \rangle_{hres}$: $HC^-_\bullet(\mathbb{A})^{\otimes 2} \longrightarrow R[[u]]$ (non-deg)

→ Variational Semi-infinite Hodge structure

Step 2: Need to choose $\zeta \in HC^-_\bullet(\mathbb{A})$ categorical primitive form! (Saito)

Claim: ζ determines Frobenius manifold on $\operatorname{Spec}(R)$.

thm (A-Tu): $\left\{ \begin{array}{l} \text{primitive forms} \\ \zeta|_{t=0=u} = \tilde{\omega}(1) \end{array} \right\} \xleftrightarrow{\text{bijection}} \{\text{good splittings}\}$

$\updownarrow$ bijection

$\{\text{gradings on } HH_\bullet(A)\}$

if $HH^\bullet(A)$ is a semi-simple ring.

Question: If $\mathcal{A} = Fuk(X)$ how do we pick "correct" CY str + splitting?

$HC^-_\cdot(Fuk\,X) \xrightarrow{OC_{cyc}} H^\cdot(X)[[u]]$

Open-closed map: $OC : HH_\cdot(Fuk(X)) \longrightarrow H^\cdot(X)$

(often an isomorphism)

if OC isomorphism we can
1) pull-back integer grading on $H^\cdot(X)$ to obtain a grading μ on $HH_\cdot$
2) pull-back the volume form to obtain CY str.

thm (A-Tu): Assume OC is isomorphism and $HH^\cdot(Fuk\,X)$ is semi-simple.

Then $F^\circ(Fuk\,X, \omega^{oc}, \mu^{oc}) \cong QH^\cdot(X)$.

Application: X = toric Fano manifold w/ generic symplectic form.

(or Hori-Vafa potential = Morse function).

IV F : Categorical enumerative invariants

Idea: Need spaces of curves from the get go.

$\mathcal{M}^{fr}_{g,k,\ell}$ = moduli of smooth Riemann surfaces of genus $= g$, k incoming and ℓ outgoing framed marked pts.

thm (Kontsevich-Soibelman, Costello) A = cyclic A_∞-algebra

$C_\bullet(\mathcal{M}^{fr}_{g,k,\ell})$ acts on $CC_\bullet(A_{cyc})$, that is

$$\exists \quad \rho : C_\bullet(\mathcal{M}^{fr}_{g,k,\ell}) \longrightarrow \mathrm{Hom}\left(CC_\bullet(A_{cyc})^{\otimes k}, CC_\bullet(A_{cyc})^{\otimes \ell}\right)$$

compatible with sewing $(k \geq 1)$.

thm (Costello + Caldararu-Tu)

① $\hat{g} := C_\bullet\left(\bigsqcup_{g,k,\ell} \mathcal{M}^{fr}_{g,k,\ell}\right)_h$ has DGLA structure

② $\exists!$ ν Maurer-Cartan element in $\hat{g}$ = "String vertices"

Picture: $\mathcal{M}_{0,3,1}$:

$\nu = \overline{\mathcal{M}}_{g,k,\ell} \setminus \text{nbhd}\left(\begin{smallmatrix}\text{nodal}\\ \text{curves}\end{smallmatrix}\right)$

Long story short: Using ρ and assuming existence of a splitting of Hodge filt. can transfer ν. to Maurer-Cartan element $\tilde{\beta}$ on

dgla $\longrightarrow$ $$h_A := \left[\mathrm{Sym}\left(CC(A_{cyc})[u^{-1}] \right), b + uB + \Delta, \{-,-\} \right]$$

$\big\downarrow$ K^s = quasi-isomorphism of dgla

determined by splitting.

$$h_A^{triv} := \left[\mathrm{Sym}\left(CC_\cdot(A_{cyc})[u^{-1}] \right), b, 0 \right]$$

$$\Rightarrow \quad \beta_{A_{cyc},s} = \sum_{g,n} F^{g,n}_{A_{cyc}} \in \mathrm{Sym}\left(HH_\cdot(A_{cyc})[u^{-1}] \right)$$

↳ "Gronov-Witten potential"

Rmk: This was computed for

- $\mathcal{A} = \mathbb{K}$ (T_v) matches $GW(pt)$
- $\mathcal{A} = \mathrm{Coh}(\text{elliptic curve})$ for $g = n = 1$.

Problem: A being cyclic is not a derived notion.

$(A, \omega, s) \rightsquigarrow s(\omega) \in HC_\bullet^-(A)$ is a strong CY str.

thm (Kontsevich-Soibelman)
$s(\omega)$ determines a cyclic A_∞-cat A_{cyc} and isomorphism $A \longrightarrow A_{cyc}$.

defn: $F(A, \omega, s) := \beta_{A_{cyc}, s}$

thm (A-Tu)
$f : tw^{\pi} A \longrightarrow tw^{\pi} B$ derived equivalence
$\Rightarrow \quad f_* F(A, \omega, s) = F(B, f_*\omega, f_* s)$. //