Facultad de Ciencias
MATEMÁTICAS

Hitchin systems

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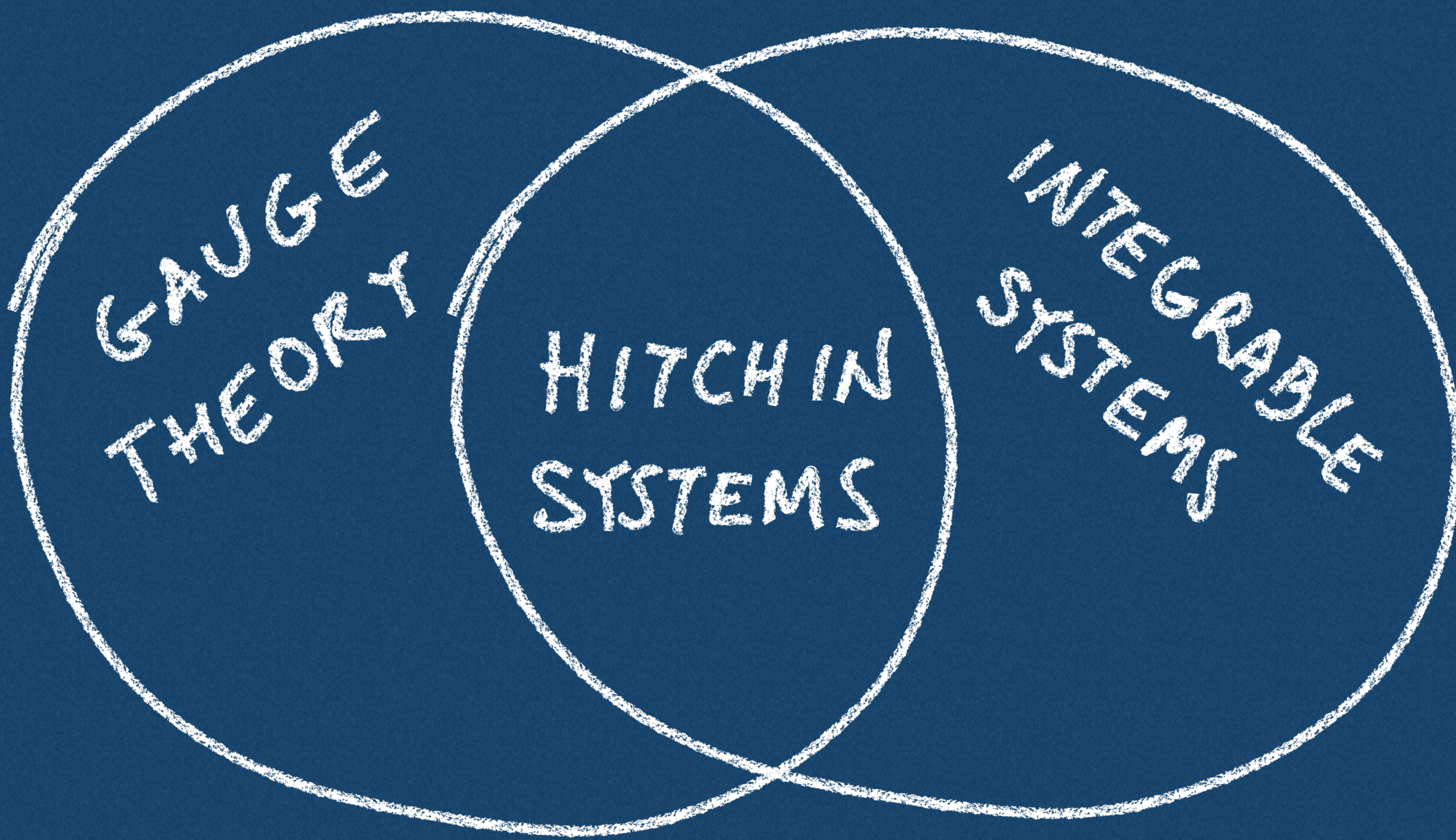
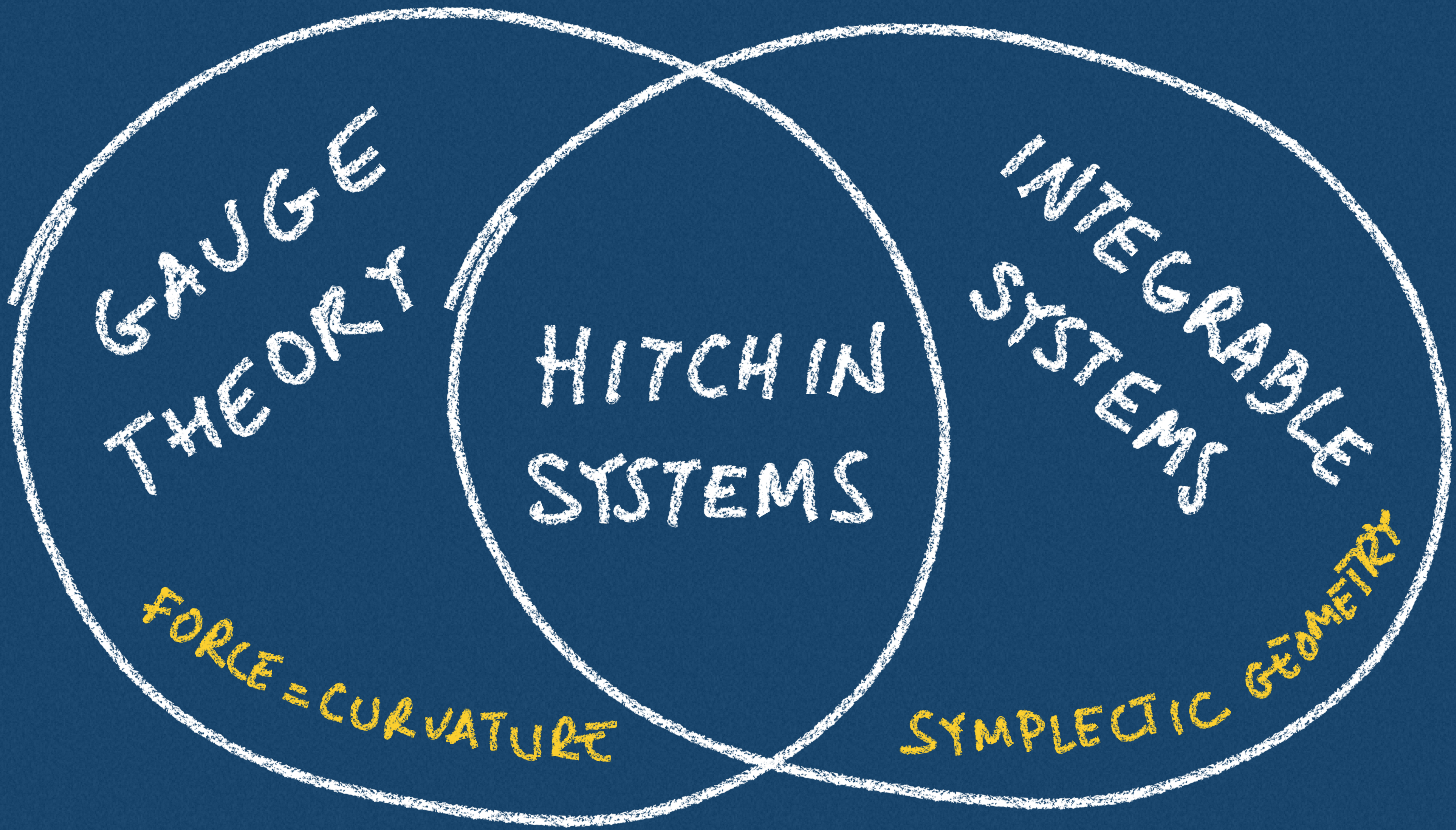
Plan

- Motivation
- Integrable systems
- What are Hitchin systems?
- Why?

Motivation



Motivation


GAUGE THEORY
HITCHIN SYSTEMS
INTEGRABLE SYSTEMS
FORCE = CURVATURE
SYMPLECTIC GEOMETRY

Motivation

- Mathematically rigorous QFT for the quantised version of the standard model (still an open question)
- Geometric quantisation
- Donagi "Seiberg-Witten Integrable systems" 1997
- ...

Motivation

HIGGS BUNDLES

GEOMETRY $\rightsquigarrow$ I.S.

TOPOLOGY $\rightsquigarrow$ TQFT

Motivation

- Maxwell equations

$$\star d \star F_A = 0 \qquad \begin{array}{c} M \times U(1) \\ \downarrow \\ M \end{array} \qquad A \in \Omega^1_M \qquad F_A = dA$$

- Yang-Mills equations

GAUGE

$$\star d_A \star F_A = 0 \qquad \begin{array}{c} M \times G \\ \downarrow \\ M \end{array} \qquad A \in \Omega^1_M \qquad F_A = dA + [A, A]$$

$A = \sum A_i dx_i$

- Hitchin's equations

$$\begin{cases} F_A + [\phi, \phi^*] = 0 \\ \bar{\partial}_A \phi = 0 \end{cases}$$

"RY" R.S.

$$M \rightsquigarrow X \qquad \begin{array}{c} P \\ \downarrow \\ X \end{array} G_{\mathbb{C}} \qquad \phi \in H^0(ad(P) \otimes K_X)$$

$\mathbb{R}^2 \cong \mathbb{C}$

Higgs

- Higgs bundles

$$G_{\mathbb{C}} = GL(r, \mathbb{C}) \qquad \boxed{\begin{array}{c} E \\ \downarrow \\ X \end{array} \quad \phi \in H^0(X, End(E) \otimes K_X)}$$

Integrable systems

- Hamiltonian systems

(M,ω) $\omega\in\Omega^2_M$ $d\omega$ $\omega^n\neq 0$ $n=\dim M$

$H\in\mathcal{C}^\infty(M)$ $dH=\omega(X_H,\cdot)$ $\{f,g\}:=\omega(X_f,X_g)$

$M=\mathbb{R}^{2n}\ni(q_1,p_1,\dots,q_n,p_n)$

$\{f,g\}:=\sum_{i=1}^n \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i}-\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$

$\dot{q}_i=p_i=\{q_i,H\}$

$\dot{p}_i=\frac{\partial V(q)}{\partial q_i}=\{p_i,H\}$

$H=\sum_{i=1}^n\frac{p_i^2}{2}+V(\vec{q})$

- Liouville theorem

$M^{2n},(f_1,\dots f_s)$ $\{f_i,f_j\}=0$ $\mathrm{rk}\{\cdot,\cdot\}=n$

- $s=2n-r$ $df_1,\dots df_s$ l. ind.

$\Rightarrow$ X_{f_i} can be integrated (locally)

$s=n$ completely int. $s<n$ partially $s>n$ super

A.C.I.S (Hitchin)

hol.

(M^{2n}, ω) symplectic $\quad F : M \longrightarrow B$

$F = (f_1, \text{---} f_n) \quad \underline{f_i \in \mathcal{O}_M} \quad \underline{\{f_i, f_j\} = 0}$

→ $X_{f_1}, \text{———} X_{f_n}$ l. ind.

→ $F^{-1}(b) \underset{\text{open}}{\subset}$ abelian variety

A.C.I.S

$\mathrm{rk}\ \# \, T^*M \to TM = 2n$

$(M^{2n+k}, \{\cdot,\cdot\})$ Poisson $\quad F : M \longrightarrow B$

$F = (f_1, \ldots, f_{n+k}) \quad f_i \in \mathcal{O}_M \quad \{f_i, f_j\} = 0$

$X_{f_1}, \ldots, X_{f_{n+k}}$ l. ind.

$\underline{F^{-1}}(b) \underset{\text{open}}{\subset}$ abelian variety

$f_1, \ldots, f_k$ casimir i.e $\{f_i, g\} = 0 \ \forall g \in \mathcal{O}_M$

$(f_1, \ldots, f_k)^{-1}(b) = \overline{\text{symplectic leave}}$

$\uparrow$ top dim

Motivation

- Many integrable systems can also be complexified
- Examples for integrable systems come from: celestial mechanics, water waves etc
- Krichever'93: Elliptic solitons (solutions to KP equations for which there is a geometric construction in terms of spectral curves and vector bundles)
- Treibich and Verdier '93: solutions to KP subset of $\mathcal{M}_{K(D)}(r,0)$
- Kim'11: Generalised construction of such solutions.

Motivation

- Kontsevich and Soibelman'13: Semi-polarised integrable systems

  (M^{2n}, ω) $\omega \in \Omega^2_M$ $d\omega = 0$ $\omega^n \neq 0$ (hol.)

  $F : M^{2n} \longrightarrow B$

  $F^{-1}(b)$ semiabelian variety $b \in B$ (smooth part)

- Diaconescu, Donagi and Pantev'07 some G-Hitchin systems isomorphic to Calabi-Yau systems ($F : J^2(Y) \longrightarrow B$, $Y \longrightarrow B$ family of CY 3-folds)

- Beck'19: Langlands duality related to Poincaré Verdier duality for Calabi Yau 3-folds

- Lee and Lee'20: extend DDP correspondence to meromorphic Higgs bundles

What are Hitchin systems?

- X compact Riemann surface of genus g ($g \geq 2$)
- $E \longrightarrow X$ holomorphic vector bundle of rank r
- $L \longrightarrow X$ line bundle
- $E \xrightarrow{\phi} \underbrace{E \otimes L}_{\text{Hitchin pairs}}$ Higgs field $\phi \in H^0(X, \mathrm{End}(E) \otimes K)$

Higgs bundles

What are Hitchin systems?

- $\mathscr{M}_L(r, L)$ moduli of Hitchin pairs (Nitsure)
- $L = K$: $\mathscr{M}_H(r, d)$ moduli space of Higgs bundles (Hitchin)
- $D = \sum_{i=1}^{\ell} x_i$; $L = K(D)$:

D reduced

meromorphic Higgs bundles

$\mathscr{M}_{K(D)}(r, d)$ moduli of K(D)-pairs

- + With a weighted filtration on E: $\mathscr{M}_{ParH}(r, d, \alpha)$ moduli of parabolic Higgs bundles (Simpson)

$E = E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_r$

$0 \leq \alpha_1 < \alpha_2 \quad < \alpha_r < 1$

Moduli spaces

- Stability condition   GIT

E stable

$$\forall F \subset E \text{ subbundle} \qquad \underline{\mu(F) < \mu(E)} = \frac{\deg(E)}{\mathrm{rk}(E)}$$

$$\rightsquigarrow (E,\phi) \qquad \forall F \subset E \quad \underline{\phi(F)} \subset \underline{F \otimes L} \quad \mu(F) < \mu(E)$$

- Symplectic structure

moduli of vector bundles

$$H^1(X, \mathrm{End}(E)) \cong T_E \mathcal{N}$$

Moduli spaces

- Stability condition

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- Symplectic structure

moduli of vector bundles

$$\left(H^1(X, \operatorname{End}(E))\right)^* \cong T_E^* N$$

Serre du. $\rightarrow$ $\wr\wr$

$$H^0(X, \operatorname{End}(E) \otimes K_X) \rightsquigarrow \mathcal{M}_H(r,d) \supset T_E^* N$$

Morse fn.

open dense

$\exists$ E, ϕ stable but E unst.

Moduli spaces

- Stability condition

$\forall F \subset E$ subbundle $\quad \mu(F) < \mu(E) = \dfrac{\deg(E)}{\mathrm{rk}(E)}$

$\rightsquigarrow (E,\phi) \quad \forall F \subset E \quad \phi(F) \subset F \otimes L \quad \mu(F) < \mu(E)$

- Symplectic structure

moduli of vector bundles

$$\big(H^1(X, \mathrm{End}(E))\big)^* \cong T_E^* N$$

$$\cong$$

$$H^0(X, \mathrm{End}(E) \otimes K_X)$$

$\leftarrow L = K_X$

what about L ???

- **Hitchin map:** (Hitchin) A.C.I.S.

$$h : \mathscr{M}_H \longrightarrow B := \bigoplus_{i}^{r} H^0(X, K^i) \qquad F : M \longrightarrow B$$

$$(E, \phi) \longmapsto \det(\lambda I - \phi) = k^r + a_1(\phi) k^{r-1} + \dots + a_r(\phi)$$

$$\begin{tikzcd} \pi^* K_X \arrow[r] \arrow[d] & K_X \arrow[d] \\ {|K_X|} \arrow[r, "\pi"] \arrow[u, bend left, "\lambda"] & X \arrow[u, bend right, "k"'] \end{tikzcd}$$

- **Spectral correspondence:**

$b \in B \qquad \boxed{h^{-1}(b) \cong Jac(X_b)}$

$|K| \supset X_b$ zero set of $\lambda^r + b_1(\phi)\lambda^{r-1} + \dots + b_r(\theta)$

$$b_i(\phi) = \pi^* a_i(\phi)$$

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d] & p_* \mathcal{L} = E \arrow[d] \\ X_b \arrow[r, "r:1", "p"'] & X \end{tikzcd}$$

ϕ = given by spectral data (Beauville Narasimhan Ramanan)

Hitchin systems

- $\mathcal{M}_{K(D)}(r,d)$ moduli space of K(D) pairs
- dimension $\mathcal{M}_{K(D)}(r,d) = 2r^2(g-1) + r(r+1)n$
- Poisson manifold
- (Markman, Botaccin) A.C.I.S.
- $$h : \mathcal{M}_{K(D)}(r,d) \longrightarrow \bigoplus_{i=1}^{r} H^0(X, K^i D^i)$$
- dimension $\displaystyle\bigoplus_{i=1}^{r} H^0(X, K^i D^i) = r^2(g-1) + \frac{r(r+1)n}{2}$ $= \frac{1}{2} \dim \mathcal{M}_{K(D)}$

Hitchin systems

- $\mathcal{M}_{K(D)}(r,d)$ moduli space of K(D) pairs
- dimension $\mathcal{M}_{K(D)}(r,d) = 2r^2(g-1) + r(r+1)n$
- Poisson manifold rk # $= r(r-1)\deg(L) + 2r(g-1) + 2$

  $L = K(D) \longrightarrow$ ||

  $2r^2(g-1) + r(r-1)n + 2$
- (Markman, Botaccin) A.C.I.S.
- $h : \mathcal{M}_{K(D)}(r,d) \longrightarrow \bigoplus_{i=1}^{r} H^0(X, K^iD^i)$
- dimension $\bigoplus_{i=1}^{r} H^0(X, K^iD^i) = r^2(g-1) + \frac{r(r+1)n}{2}$ $= \frac{1}{2}\dim \mathcal{M}_{K(D)}$

$\dim \bigoplus_{i=1}^{r} H^0(X, L^i) = \frac{1}{2} r(r-1)\deg(L) - r(g-1)$

Hitchin systems

Martens
non-strongly

- $\mathcal{M}_{ParH}(r, d, \alpha)$ moduli space of parabolic Higgs bundles

$(E, \phi) \quad \phi \in H^0(X, P\mathrm{End}(E) \otimes K(D))$

Flag $\quad E = E^0 \supset E^1 \supset E^2 \supset \cdots \supset E^r \supset 0$

$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r < 1$

$\phi(E_i) \subset \begin{cases} E_i & P\mathrm{End}(E) \\ E_{i+1} & SP\mathrm{End}(E) \end{cases}$

- $h : \mathcal{M}_{ParH}(r, d, \alpha) \longrightarrow \bigoplus_{i=1}^{r} H^0(X, K^i D^i)$

$$\begin{tikzcd} \mathcal{M}_{ParH}(r, d, \alpha) \arrow[r, "h"] \arrow[dr, "\tilde{h}"'] & \bigoplus_{i=1}^{r} H^0(X, K^i D^i) \arrow[d] \\ & B/B^0 \end{tikzcd}$$

b

$h^{-1}(b) = \coprod \mathrm{Jac}(X_b)$

for full flags:

$\tilde{h}^{-1}(b)$ symplectic leaves

$B^0 = H^0(X, K^i D^{i-1}) \longleftarrow$ base for strongly Par

Frames

- $D = \sum_{i=1}^{\ell} n_i x_i \quad n = \sum_{i=1}^{\ell} n_i$
- framing $\underline{\delta : E_D \xrightarrow{\cong} \mathcal{O}_D^{\oplus r}}$, framed bundle (E, δ)
- Framed Higgs bundle $\underline{(E, \phi, \delta)}$

Markman's $\neq$

- $\mathcal{M}_{FH}(r, d)$ moduli space of framed Higgs bundles

stability $\mu(F) < \mu(E) \quad \forall F \subset E \mid \phi(F) \subset F \otimes K(D)$

- $\underline{T_{(E,\delta)}\mathcal{M}_F \cong H^1(X, End(E) \otimes \mathcal{O}(-D))}$

indeed $T^*_{(E,\delta)}\mathcal{M}_F \cong \underline{H^0(X, End(E) \otimes K(D))}$

- $\mathscr{C}_\bullet : C_0 := End(E)(-D) \longrightarrow C_1 := End(E) \otimes K(D)$
- $T_{(E,\phi,\delta)}\mathscr{M}_{FH}(r,d) \cong \mathbb{H}^1(\mathscr{C}_\bullet)$ ← origin in: Biswas Ramanan.
- dimension $\mathscr{M}_{FH}(r,d) = 2r^2(g+n-1)$

$$\Psi : \mathbb{H}^1(\mathscr{C}_\bullet) \times \mathbb{H}^1(\mathscr{C}_\bullet) \longrightarrow \mathbb{C}$$

$$\Psi \in \Omega^2_{\mathcal{M}_{FH}} \quad d\Psi = 0 \quad \Psi^n \neq 0$$

actually it is the Liouville symplectic form

- $Q : \mathscr{M}_{FH}(r,d) \longrightarrow \mathscr{M}_{K(D)}(r,d)$ forgetful map is Poisson and if D reduced it is a $GL(r,\mathbb{C})^{\ell}/\mathbb{C}^\star$-torsor

- On-going work: Irregular Parabolic Higgs

$$\mathbb{D} = \sum_{i=1}^{\ell} n_i x_i$$

$$\mathcal{M}_{ParH}(r,d,\alpha) = \{ (E,\phi) \quad \phi \in H^0(X, PEnd(E) \otimes K(\mathbb{D})) \}$$

BLP-NS: Poisson Structure

The spectral data adapts from $D = \mathbb{D}$

$$
\begin{tikzcd}
 & T^*\mathcal{M}_P \arrow[d, hook] & T^*\mathcal{M}_F \arrow[l, hook'] \arrow[r, hook] \arrow[d] & \mathcal{M}_{FH} \arrow[dl, "Q"] \arrow[dll, bend left] \\
\mathcal{M}_H \arrow[d] & \mathcal{M}_{PH} \arrow[l, "\text{forget}"'] \arrow[r, "\text{forget.}"] \arrow[d] & \mathcal{M}_{K(D)} \arrow[dl] & \\
\bigoplus_i H^0(X, K^i) \arrow[r] & \bigoplus H^0(X, K^i D^i) & &
\end{tikzcd}
$$

? ?

KP

elliptic (g=1)

solitons