:

The Geometry of Symmetric Differentials

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Introduction

The spaces we study are called complex manifolds

Locally are like open subsets of $\mathbb{C}^n$
i.e. locally with coordinates $(z_1, ..., z_n)$, with z_i complex numbers

We then know how to do analysis in these subsets

These local coordinates are related(glued) in a way

that allows us to do analysis consistently globally

Introduction

Symmetric differentials are analytic objects that exist on a complex manifold X

On open piece $U \subset X$ with a chart $(z_1, ..., z_n)$

We have the differential 1-forms dz_i (dual of vector fields)

A symmetric differential w on U is a polynomial on the dz_i's whose coefficients $a_{i_1 \dots i_n}$ are holomorphic functions on U

$$w = \sum a_{i_1 \dots i_n} dz_1^{i_1} \dots dz_n^{i_n}$$

More abstractly, to each point $x \in X$ associate a vector space

$$\Omega^1_{X,x} = \mathbb{C}dz_{1,x} + ... + \mathbb{C}dz_{n,x}$$

Together they make up the intrinsic **cotangent bundle** on X, Ω^1_X

A differential 1-form is a section of this vector bundle

Associated to the vector bundle Ω^1_X we have other intrinsic bundles, e.g:

1. **Alternating powers**: $\wedge^k \Omega^1_X$, $k = 0, ..., n$, sections are k-forms
2. **Symmetric powers**: $S^m \Omega^1_X$, $m \geq 0$, sections are symmetric m-differentials
3. **powers of the canonical line bundle**: $K^m_X := (\wedge^n \Omega^1_X)^{\otimes m}$

Differential k-forms are essential for the theory of integration on manifolds.

The **canonical bundle** is **the** intrinsic line bundle on a complex manifold. Their powers play a central role on the classification theory.

Symmetric m-differentials are algebraic differential equations of order 1 which are homogeneous of degree m

Let w be a symmetric 2-differential on $\mathbb{C}^2$:

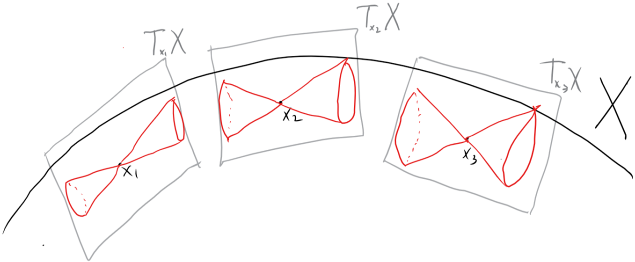
$$w = a_{2,0}(dz_1)^2 + a_{1,1}dz_1dz_2 + a_{0,2}(dz_2)^2$$

Let $\Delta \subset \mathbb{C}$, $f : \Delta \hookrightarrow \mathbb{C}^2$ a curve given by $f(t) = (f_1(t), f_2(t))$.

$$f^*w = (a_{2,0}{f_1'}^2 + a_{1,1}f_1'f_2' + a_{0,2}{f_2'}^2)(dt)^2$$

Geometrically: Individually a symmetric m-differential defines a homogeneous polynomial of degree m at each T_xX, that varies with x.

Hence, defines at each T_xX a cone with center x.



A collection of sym. m-diff $\{w_1, ..., w_k\}$ defines global system of functions on TX that tries to differentiate the tangent directions at all points in X.

- When do we have the presence of symmetric differentials?

Locally they exist with abundance, as we saw.
Globally they are constrained by the geometry, next section.

There are several forms of measuring and describing the abundance of symmetric differentials.

The symmetric m-genera: $q_m(X) := \dim H^0(X, S^m\Omega^1_X)$

The symmetric algebra: $Q(X) = \bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega^1_X)$

Asymptotic growth: the cotangent Kodaira dimension
$k_\Omega(X) := \lim_{m\to\infty} \log_m q_m(X)$

The Kodaira cotangent dimension varies:

from $-\infty$, convention for no symmetric differentials.

to $2\dim X - 1$, when for the exception of a measure zero set all other tangent directions are separated by sym. diff.

The extreme of abundance is when symmetric differentials distinguish all tangent directions.
We say then that Ω^1_X is **ample**.

Analytic geometry

The most well known application of symmetric differentials is to use them to **control the existence of special subvarieties**.

Example (Special subvarieties)

- Rational curves
- Elliptic curves
- entire curves, i.e. non pointwise images of holomorphic maps from $\mathbb{C}$

Symmetric differentials are also seen as **obstructions to embeddings**

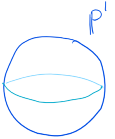
Theorem (Kobayashi, Schneider, distinct proof (Bogomolov and -))

If $X^{(n)} \subset \mathbb{P}^N$ smooth projective subvariety with $n > 1/2N$, then X has no symmetric differentials.

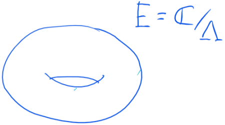
Symmetric differentials on curves

- The abundance of symmetric differentials on curves is simply determined by the degree of the line bundle, $\deg(\Omega^1_X)$.

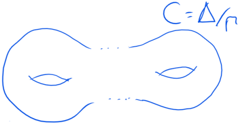
- Rational curves, genus 0 , $\deg\Omega^1_X < 0$, no symmetric differentials.



- Elliptic curves, genus 1, $\deg\Omega^1_X = 0$, only one symm. differential of each degree, they are nowhere zero.



- General type, genus $g \geq 2$, $\deg\Omega^1_X > 0$, plenty of symm. differentials.



$q_m(X) = (2g-2)m + 1 - g$ (topological)

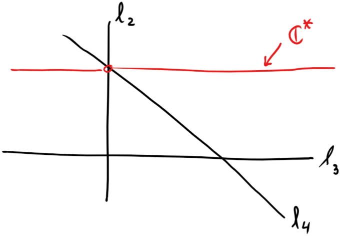
Hyperbolicity

A manifold X is said to be **analytically hyperbolic** if it has no entire curves (no non-constant holomorphic map $f : \mathbb{C} \to X$).

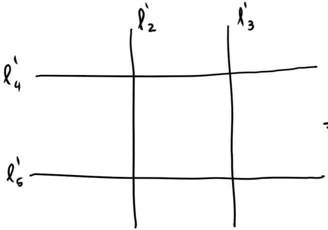
Examples:

1. Open subsets of $\mathbb{C}$: $\mathbb{C}$ and $\mathbb{C} \setminus \{p\}$ are not hyperbolic, but $\mathbb{C} \setminus S$ with $\#S > 1$ is hyperbolic (Picard's theorem).
2. Riemann surfaces: $\mathbb{P}^1$ and elliptic curves are not hyperbolic, but curves of general type are. Products of R.S of general type.
3. Complements in $\mathbb{P}^2$: The complement of any 4 lines in $\mathbb{P}^2$ is not hyp., but there are complements of 5 lines that are.

$$\mathbb{P}^2 \setminus (\ell_1 \cup \ldots \cup \ell_4) = \mathbb{C}^2 \setminus (\ell_2 \cup \ldots \cup \ell_4)$$



$$\mathbb{P}^2 \setminus (\ell'_1 \cup \ldots \cup \ell'_5) = \mathbb{C}^2 \setminus (\ell'_2 \cup \ldots \cup \ell'_5)$$



$$= \mathbb{C} \setminus \{p_1, p_2\} \times \mathbb{C} \setminus \{p_3, p_4\}$$

hyperbolic

Ample Cotangent Bundle $\implies$ Hyperbolic

X is **weakly algebraically hyperbolic** if X has no rational or elliptic curves (weaker than analytically hyperbolic).

If the cotangent bundle Ω^1_X is ample, then X is analytically hyperbolic.

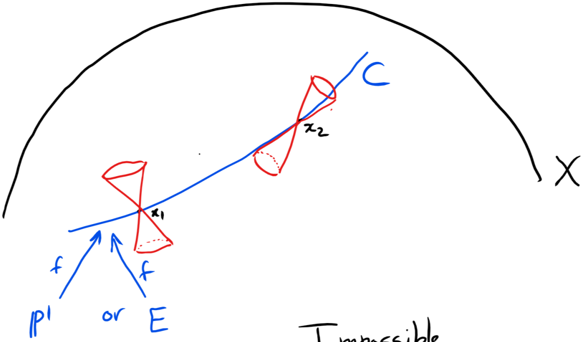
Ω^1_X ample

$\Downarrow$

$\exists\ \omega$ sym. diff such that

1) $\omega(x_1)|_{T_{x_1}C} \neq 0$

2) $\omega(x_2)|_{T_{x_2}C} = 0$



Impossible

Hence: $f^*\omega$ is a sym. diff on $\mathbb{P}^1$ or E nonvanishing but vanishing at some points

Conjectures

Conjecture (Kobayashi):

A general smooth hypersurface in $\mathbb{P}^3$ of degree ≥ 5 is analytically hyperbolic.

Problem: we saw before that there are no symmetric differentials on hypersurfaces hence one can not use them.

Using jet-differentials which are algebraic differential equations of order ≥ 1, the work of Demailly, Siu, Rousseau, Merker, Diverio, Paun and many others dealt with $d > 18$.

Conjecture (Green-Griffiths):

A variety of general type X has a proper subvariety $Y \subset X$ such that all entire curves are contained in Y.

Surfaces with $c_1^2(X) - c_2(X) > 0$

If X is a surface, then Riemann-Roch gives:
$h^0(X, S^m\Omega) - h^1(X, S^m\Omega) + h^2(X, S^m\Omega) = 1/6(c_1^2(X) - c_2(X))m^3 + O(m^2)$

[Bogomolov 78] showed that if a surface X of general type satisfies $c_1^2(X) - c_2(X) > 0$, then:

1. $k_\Omega(X) = 3$ (maximal, we say Ω_X^1 is big).
2. Enough symmetric differentials to force that there are only finitely many rational and elliptic curves on X.

But for hypersurfaces $X \subset \mathbb{P}^3$ of $d \geq 5$, $c_1^2(X) - c_2(X) < 0$.

Symmetric Differentials on Deformations of Hypersurfaces

Idea: consider nodal hypersurfaces $X \subset \mathbb{P}^3$ and their minimal resolutions $\tilde{X}$.

$\tilde{X}$ can be view as deformations of smooth hypersurfaces of the same degree as X (Atiyah).

Hence $1/6(c_1^2(\tilde{X}) - c_2(\tilde{X})) = 1/6(c_1^2(X) - c_2(X)) < 0$ still.

But one can show that the presence of sufficiently many nodes can make:

$$h^1(X, S^m\Omega) + 1/6(c_1^2(X) - c_2(X))m^3 > km^3, \quad k > 0$$

Symmetric Differentials on Deformations of Hypersurfaces

Theorem (Bogomolv,-, 06)

There are nodal hypersurfaces $X \subset \mathbb{P}^3$ of degree $d \geq 13$ whose minimal resolution $\tilde{X}$ is such that:

1. $\Omega^1_{\tilde{X}}$ *is big.*
2. $\tilde{X}$ *satisfies the Green-Griffths conjecture.*
3. $\tilde{X}$ *is a deformation of smoth hypersurfaces of degree d.*

Corollary: The symmetric pluri-genera is not a deformation invariant.

Interplay with Topology

Question:

To what extent does $Q(X) = \bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega^1_X)$ determine or is determined by the topology of X?

In the case of curves ($\dim X = 1$) we saw that $Q(X)$ determines the topology.

Are the symmetric pluri-genera, $q_m(X) = \dim H^0(X, S^m\Omega^1_X)$, topological?

$q_1(X) = 1/2 b_1(X)$, by Hodge theory, hence is topological.

$q_m(X)$, with $m > 1$ are not necessary deformation invariants (our and Brotbek's examples) hence not necessarily topological.

(they depend on the complex struture on the underlying topological space).

$\pi_1(X) \neq 0 \implies \exists$ Symmetric Differentials?

There are projective manifolds with $\pi_1(X) \neq 0$ but finite with no sym. diff.

Simple case: if the $H_1(X, \mathbb{C}) \neq 0$, then there many sym. diff., in particular products of diff. 1-forms (Hodge Theory).

Conjecture (Esnault) A projective manifold with $\pi_1(X)$ infinite has symmetric differentials.

Theorem (Brunebarbe,Klingler,Totaro 13) If the $\pi_1(X)$ has representation in a linear group with infinite image, then X has symmetric differentials.

$\exists$ Symmetric Differentials $\implies \pi_1(X) \neq 0$?

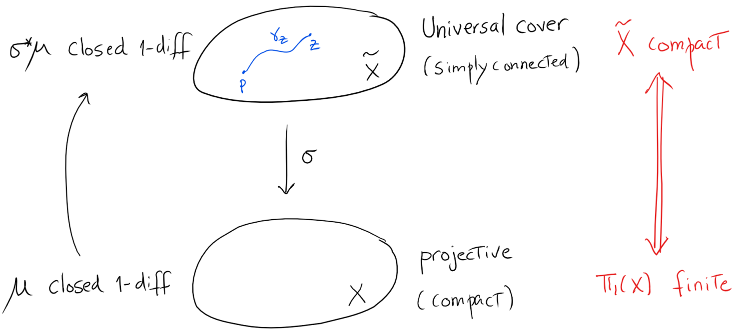
Our example shows that X can have abundance of symm. diff. but $\pi_1(X) = 0$.

But if $q_1(X) \neq 0$, then $b_1(X) \neq 0$ hence $\pi_1(X) \neq 0$ (Hodge Theory).

So the existence of certain type of symm. diff. $\implies$ large $\pi_1(X)$.

Essential: 1-diff on projective manifolds (always compact) must be closed, i.e. $d\mu = 0$.

$\exists$ closed 1-diff $\implies$ $\pi_1(X)$ infinite

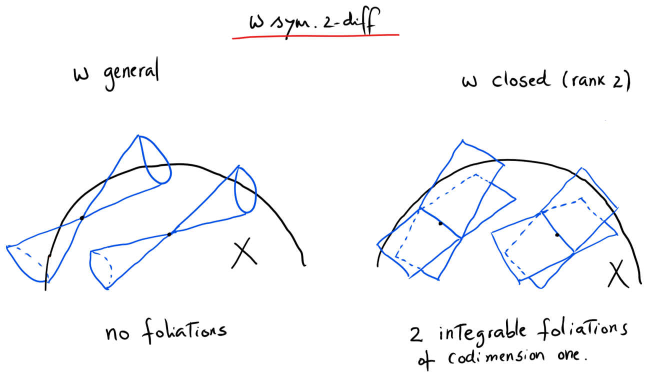


- Set $f(z) = \int_p^z \sigma^* \mu$ (fixed → p)
  - 1) Well defined (ind. of γ_z)
  - 2) non-constant (if $\mu \neq 0$)
  - 3) holomorphic
- Contradiction, $\tilde{X}$ compact $\Rightarrow \exists$ max, violates Max. Principle

Closed Symmetric Differentials

A symmetric m-diff. w is **closed** if near a general point $x \in X$ it is: locally the product of closed homorphic 1-forms.

A closed sym. m-diff defines a global k-web (k foliations), $k \leq m$ is called the **rank**



When X is a Surface

The fundamental groups of curves are special among those of projective varieties but by Lefschetz those of surfaces encompass all.

Near a general point on a surface any symmetric m-differential, w, has the form:

$$w = F df_1 ... df_m$$

w is closed iff $F = F_1(f_1)...F_m(f_m)$.

Question: Find the characterizing (non-linear) differential operators.

Description of these operators has been done for degree 2 and 3 for surfaces [Bogomolov, - 13],[Buonerba, Zakharov 17]

Examples

0) Basic: $w = \mu_1 ... \mu_m \in S^m H^0(X, \Omega^1_X)$

- They come from maps to complex tori and hence $\implies \pi_1(X) = \infty$.

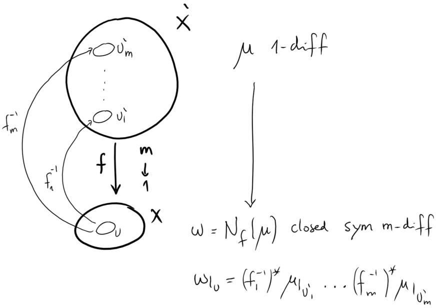
1)

Theorem (Bogomolov,- 11)

X projective manifold and w a sym m-diff of rank 1. Then:

1. *w is closed.*
2. *w comes from a map to a $\mathbb{Z}_d$ -quotient $(d|m)$ of a complex torus.*
3. *$\pi_1(X \setminus E) = \infty$, where E is a nonpositive divisor or emply.*

Examples



2) Such w can exist on X without the existence of 1-diff on X.

4) Next we look at certain types of closed sym diff. and derive the corresponding geometry.

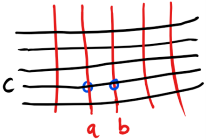
Closed of the 1st Kind and Abelian Rank

w is of the **1st kind** if w is everywhere locally the product of closed 1-diff.

The **abelian rank** of w is the abelian rank of the associated web.

It is the dimension of the space of decompositions of a constant function as the sum of non-constant functions constant along the foliations.

-A sym. m-diff. of 1st kind and rank 2 has trivial abelian rank.



- locally a general sym m-diff of 1st kind has trivial abelian rank.

Geometry and Local Systems

Theorem (Bogomolov,- 13)

X projective manifold with w sym. 2-diff. of rank 2 and of 1st kind. Then:

1. *$w = \phi_1\phi_2$ with ϕ_i 1-diff twisted by a local system $\mathbb{C}_{\rho_i}$.*
2. *The local systems $\mathbb{C}_{\rho_i}$ $i = 1, 2$ are dual and* **torsion**.
3. *w comes from a map to a quotient of an complex torus by a cyclic or dihedral group.*
4. *$\pi_1(X)$ is infinite.*

Recurent theme: closed sym diff on a projective variety X are associated to maps from X to a quotient of a complex torus by a finite group.

Products of Two Closed Meromorphic 1-forms

- Apriori, such examples do not necessarily force $\pi_1(X)$ large since logarithmic differentials can exist on simply connected manifolds.
- Provide examples of global closed sym 2-diff that are not of 1st kind.

Theorem (Bogomolov,- 13)

X projective manifold with w sym. 2-diff. of rank 2 of the form:

$$w = \phi_1\phi_2, \qquad \mu_i \in H^0(X, \Omega^1_{X,cl}(*))$$

$\Omega^1_{X,cl}(*)$ is the sheaf of closed meromorphic 1-diff. Then $\pi_1(X)$ is infinite, moreover the Albanese dimension ≥ 2 and one of the follwing holds:

1. *w is of 1st kind, i.e. ϕ_i are holomorphic.*
2. *$w = (f^*\varphi + u)\mu$, where $f : X \to C$, C smooth curve of genus $g \geq 1$, $(f^*\varphi + u)$ is not holomorphic with $\varphi \in H^0(C, \Omega^1_C(*))$ and $u \in H^0(X, \Omega^1_X)$, and $\mu \in H^0(C, \Omega^1_C)$.*

Extrinsic Geometry

-We study symmetric differentials whose coefficients are meromorphic (instead of holomorphic) functions with a bound on the type of poles.

- We consider poles defined extrinsically in terms hyperplane sections of a given embedding of X on $\mathbb{P}^N$.

Theorem (Schneider 92)

If $X^{(n)} \subset \mathbb{P}^N$ smooth and $n > N/2$, then no symmetric m-differentials even if poles of order "smaller" than m hyperplane sections are allowed, i.e.

$$Q(X, \alpha) = \bigoplus_{m\alpha \in \mathbb{Z}} H^0(X, S^m[\Omega^1_X(\alpha)]) = \mathbb{C}, \quad \alpha \in \mathbb{Q}, \alpha < 1$$

The boundary case $Q(X, 1) = \bigoplus_{m \geq 0} H^0(X, S^m[\Omega^1_X(1)])$ becomes of interest!

Understanding the maps of sections in the structural exact sequences enveloping Ω^1_X have a clear meaning once twisted with $\mathcal{O}(1)$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N^*_{X/\mathbb{P}^N}(1) & \xrightarrow{id} & N^*_{X/\mathbb{P}^N}(1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^1_{\mathbb{P}^N}|_X(1) & \longrightarrow & \bigoplus_{i=1}^{N+1} \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow q & & \downarrow id & & \\
0 & \longrightarrow & \Omega^1_X(1) & \longrightarrow & \widetilde{\Omega^1}_X(1) & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
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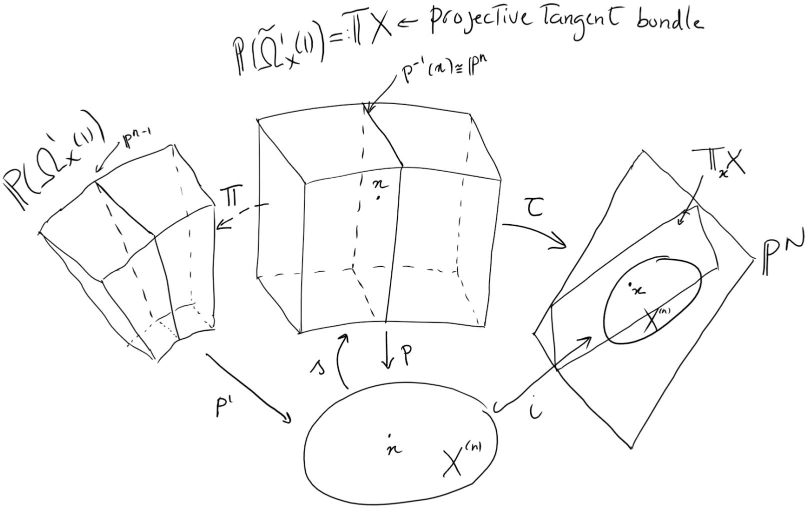
\end{array}
$$



$$\pi^* \mathcal{O}_{\mathbb{P}(\Omega^1_X(1))}(m) \simeq \mathcal{O}_{\mathbb{T}X}(m) \simeq \tau^* \mathcal{O}_{\mathbb{P}^N}(m)$$

- sections of $\mathcal{O}_{\mathbb{P}(\Omega^1_X(1))}(m)$ are the twisted sym. m-differentials.
- sections of $\mathcal{O}_{\mathbb{P}^N}(m)$ are the homogeneous polynomials of degree m on $\mathbb{C}^{N+1}$.

Question:

Which polynomials correspond to twisted symmetric differentials on X?

Tangentially Homogeneous Polynomials

Let $H \in \mathbb{C}[Z_0, ..., Z_N]^{(m)}$, h a dehomogenezation of H and $h_x := h|_{T_x X}$.

$$h_x = h_x^{(0)} + ... + h_x^{(m)} \quad \text{(Taylor exp. at x)}$$

H is **tangentially homogenous** relative to $X \subset \mathbb{P}^N$ if:

$$h_x = h_x^{(m)}, \quad \forall x \in X$$

$$\mathbb{C}[X_0, ..., X_N]^{(m)}_{TX} = \{H \text{ tang. hom. relative to } X \text{ of degree } m\}$$

The algebra generated by tang. hom. polynomials:

$$\mathbb{C}[X_0, ..., X_N]^{h}_{TX} = \bigoplus_{m \in \mathbb{N}_0} \mathbb{C}[X_0, ..., X_N]^{(m)}_{TX}$$

Theorem (Langdon, -, 17)

If the tangent map $\tau : \mathbb{T}X \to \mathbb{P}^N$ is surjective and with connected fibers, then

$$Q(X,1) = \mathbb{C}[Z_0, ..., Z_N]^h_{TX}$$

Theorem (Bogomolov, -, 08)

The tangent map $\tau : \mathbb{T}X \to \mathbb{P}^N$ is surjective and with connected fibers if $\dim X > 2/3(N-1)$.

$n > 2/3(N-1)$ remind us of the Hartshorne conjecture dimensional range:

If $X^{(n)} \subset \mathbb{P}^N$ is smooth with $n > 2/3N$, then X is a complete intersection.

Tangentially Homogeneous Polynomials

Question: which polynomials are tangentially homogeneous relative to X?

- Trivially, polynomials H whose zero locus $V(H)$ contains the tangent variety of X, $\mathrm{Tan}(X) := \tau(\mathbb{T}X)$.
- Quadratic polynomials Q such that $X \subset V(Q)$.

$q_x = q_x^{(2)}$, since $q_x^{(0)} = 0$ ($X \subset V(Q)$) and $q_x^{(1)} = 0$ ($T_x X \subset T_x V(Q)$).

Theorem (Langdon,-, 17)

Let $X^{(n)} \subset \mathbb{P}^N$ *be a nondegenerate smooth complete intersection with* $n > 2/3(N-1)$ *(and* $n > 1$*). Then:*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq \mathbb{C}[Q_0, ..., Q_r]$$

where $\{Q_0, ..., Q_r\}$ *is any basis of* $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$*.*

$I(X) = (F_1, ..., F_c)$ with $\deg F_i = d_i$, $d_1 \geq d_2 \geq ... \geq d_c$.

$\{F_k, ..., F_c\}$ form a basis for $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$.

H tang. hom., $H = \sum_{(i_1,...,i_c)\in I} G_{i_1...i_c} F_1^{i_1}...F_c^{i_c}$, with $G_{i_1...i_c} \notin I(X)$.

$x \in X$ general, what can we say about h_x?

h_x is homogeneous and deg h_x =deg H

The lowest possible term of h_x involves powers of the quadratic terms $f_{i,x}^{(2)}$ and 0-order terms $g_{i_1 \dots i_c, x}^{(0)}$.

If the lowest possible term is non-vanishing, then:

$G_{i_1 \dots i_c} \in \mathbb{C}$ and only quadratic F_i are involved for H.

The lowest possible term is non-vanishing if the $f_{i,x}^{(2)}$ are algebraically independent.

What is the geometrical meaning of the $f_{i,x}^{(2)}$?

They are the quadratic forms coming from the 2nd fundamental form of X (**curvature**).

Using the work of Terracini and then later of Griffiths and Harris on projective differential geometry, we get:

If X is a complete intersection, then the $f_{i,x}^{(2)}$ are algebraically independent iff $\text{Tan}(X) = \mathbb{P}^N$.

In motivational terms: if the $f_{i,x}^{(2)}$ are not algebraically independent, then the bending is not maximal making the tangent variety smaller than the ambient $\mathbb{P}^N$.

More Geometry

What to do if X is not known to be a complete intersection?

Remarks:

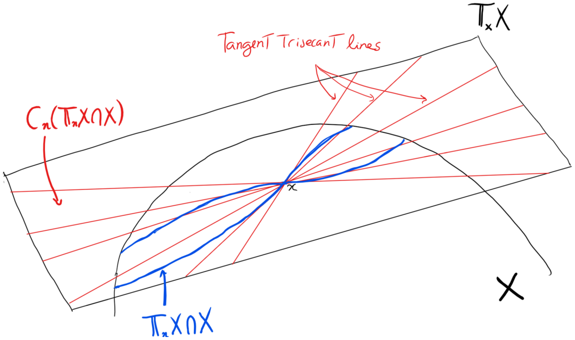
- If $X^{(n)} \subset \mathbb{P}^N$ has codimension 1, then it is a complete intersection, but already if codimension is 2 this is not known ($n > 2/3N$).

- The twisted cubic $C \subset \mathbb{P}^3$ is of degree 3 and not contained in a plane. If it was a complete intersection $C = V(F_1, F_2)$ its degree would be $d_1.d_2$ which is not prime, contradiction.

Goal: understand the locus where tang. homo. polynomials vanish?

Trisecant variety

If H tangentially homogeneous rel. to X, then H vanishes on the cones $C_x(\mathbb{T}_x X \cap X) \subset \mathbb{T}_x X$ for all x in X.



This implies that H vanishes on the (tangent-secant)-trisecant variety of X, ts-Trisec(X).

$Q(E)$ the intersection of all quadrics containing X

Theorem (Langdon,-, 17)

If $X^{(n)} \subset \mathbb{P}^N$, nondegenerate, codimension 2 and $n \geq 3$, then

$$\textit{ts-Trisec}(X) = QE(X)$$

By Severi, Ran and Kwak the **quadratic envelope** QE(X) is one of the following:
(i) $\mathbb{P}^N$, $h^0(\mathbb{P}^N, \mathcal{I}_X(2)) = 0$.
(ii) Q, the single quadric $X \subset Q$, $h^0(\mathbb{P}^N, \mathcal{I}_X(2)) = 1$.
(iii) X, where X must be a complete intersection or $X = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, $h^0(\mathbb{P}^N, \mathcal{I}_X(2)) > 1$.

Theorem (Langdon,-, 17)

If $X^{(n)} \subset \mathbb{P}^N$, nondegenerate, codimension 2 and $n \geq 3$, then

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq \mathbb{C}[Q_0, ..., Q_r]$$