Two time scale stochastic approximation for reinforcement learning with linear function approximation

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Joint work with Diogo Carvalho and Francisco Melo

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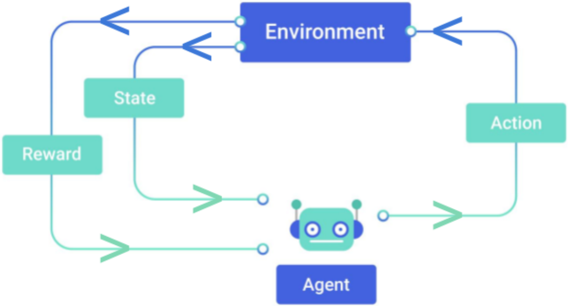
Reinforcement learning



$x_0, a_0, r_0, x_1, a_1, r_1, x_2, a_2, r_2, x_3 \; \ldots$

Contents

Markov Decision Processes

Definition (MDP)

A Markov decision process is a tuple $\{\mathcal{X}, \mathcal{A}, \mathcal{P}, r, \gamma\}$, where

- $\mathcal{X}$ denotes a finite set of n states;
- $\mathcal{A}$ a finite set of m actions;
- $\mathcal{P}$ is a set of $n \times n$ stochastic matrices P_a associated with each action $a \in \mathcal{A}$ with entries $[P_a]_{x,y} \in [0, 1]$ representing the probability that the state transitions from x to y given that the action a was performed;
- $R : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$ is a reward stochastic mapping;
- $\gamma \in [0, 1[$ is a discount factor.

The Markov Zoo

States	Actions	State Knowledge	
No	Yes		Multi-armed Bandit
Yes	No	Yes	Markov Process
Yes	No	No	Hidden MM
Yes	Yes	Yes	MDP
Yes	Yes	No	POMDP

Policy

Definition (Policy)

A policy $\pi : \mathcal{X} \to \Delta(\mathcal{A})$ is a stochastic map from states to actions.

The state value function of a given policy π:

Definition (State value function)

The state value function $V_\pi : \mathcal{X} \to \mathbb{R}$ is

$$V_\pi(x) := \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t R_t | x_0 = x],$$

where the expectation is taken with respect to the states $x_{t+1} \sim [P_{a_t}]_{x_t,\cdot}$ and the actions $a_t \sim \pi(x_t)$.

The Q function

Definition (State action value function)

The state action value function $Q_\pi : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$ is

$$Q_\pi(x, a) := \mathbb{E}[\sum_{t=0}^{\infty} \gamma^t R_t | x_0 = x, a_0 = a],$$

where the expectation is taken with respect to the states $x_{t+1} \sim [P_{a_t}]_{x_t,\cdot}$ and the actions $a_t \sim \pi(x_t)$.

The Q function

,	Action 1	Action 2
State 1	0	5
State 2	0	5
State 3	0	5
State 4	20	0

The Prediction Problem

Evaluate a policy π by approximating its state value function V_π

Proposition (Fixed point equation for the state value function)

The following relation holds:

$$V_\pi(x) = \mathbb{E}[R(x,a) + \gamma V_\pi(y)], \tag{1}$$

where the expectation is taken with respect to the next state $y \sim [P_a]_{x,\cdot}$ *and the action* $a \sim \pi(x)$.

The Control Problem

Find a policy π^* that maximizes the cumulative reward $V_{(\cdot)}$

A policy π is better than another policy π' if $V_\pi(x) \geq V_{\pi'}(x)$ for all $x \in \mathcal{X}$.

Definition

An optimal policy π^* is any such that, for any policy π,

$$V_{\pi^*}(x) \geq V_\pi(x),$$

for all $x \in \mathcal{X}$.

Optimal Policy

Theorem (Existence of solution)

There exists an optimal policy π^*.

Corollary (Optimal Value Equation)

The state value function of the optimal policy π^*, which we shall denote by V^*, verifies

$$V^*(x) = \max_{a \in \mathcal{A}} \mathbb{E}[R(x,a) + \gamma V^*(y)] \tag{2}$$

for all $x \in \mathcal{X}$, where the expectation is taken with respect to the next state $y \sim P_{a_{x},\cdot}$.

Optimal state value function

Letting Q^* denote Q_{π^*}, we have that

Proposition (Optimal Q equation)

The following relation holds:

$$Q^*(x, a) = \mathbb{E}[R(x, a) + \gamma \max_{a' \in \mathcal{A}} Q^*(y, a')], \tag{3}$$

where the expectation is taken with respect to the next state $y \sim [P_a]_{x,\cdot}$ *and the action* $a' \sim \pi(x)$

Finding V_π

- As a system of $n \times n$ linear equations:

$$\begin{aligned} V_\pi(x) &= \mathbb{E}[R(x,a) + \gamma V_\pi(y)] \\ &= \sum_a \pi(a|x) \sum_{y,r} p(y,r|x,a)(r + \gamma V_\pi(y)) \end{aligned}$$

$$V_\pi = TV_\pi$$

- Iterative policy evaluation:

$$V_{k+1} = TV_k$$

Approximating π^*

Given V_π,

$$\pi(x) \leftarrow \underset{a}{\operatorname{argmax}} \sum_{y,r} p(y, r|x, a)(r + \gamma V_\pi(y))$$

Policy Iteration

$$\pi_0 \rightarrow V_{\pi_0} \rightarrow \pi_1 \rightarrow V_{\pi_1} \rightarrow \pi_2 \rightarrow \ldots \approx \pi^*$$

Problems with this simplistic approach

- We need to work with the whole state space

- We need to know the model of the world

Contents

Who needs a model?

Model-free Methods

- Monte Carlo Methods
- Temporal-Difference Learning

Q-learning

Q-learning algorithm for estimating π^*

Initialize $Q(x, a)$ for all $x \in \mathcal{X}$ and $a \in \mathcal{A}$ (e.g. $Q(x, a) = 0$);
repeat for each Episode
    Choose an initial state x;
    repeat for each step of Episode
        Choose action a using policy derived from Q (e.g. ϵ-greedy);
        Execute a, observe reward r and new state x';
        $Q(x, a) \leftarrow (1 - \alpha)Q(x, a) + \alpha\big(r + \gamma \max_{a'} Q(x', a')\big)$;
        $x \leftarrow x'$
    until x *is terminal*;
until *satisfied*;

Q-learning

$$Q(x, a) \leftarrow (1 - \alpha)Q(x, a) + \alpha\big(\ r\ +\ \gamma \max_{a'} Q(x', a')\ \big)$$

Q-learning

$$Q(x,a) \leftarrow (1-\alpha)Q(x,a) + \alpha\big(\, r \,+\; \gamma \max_{a'} Q(x',a')\;\big)$$

Q-learning

$$Q(x,a) \leftarrow (1-\alpha)Q(x,a) + \alpha\big(r + \gamma \max_{a'} Q(x',a') \big)$$

Q-learning

$$Q(x,a) \leftarrow Q(x,a) + \alpha\big(\, r \,+\, \gamma \max_{a'} Q(x',a') \,- Q(x,a)\big)$$

$$Q(x,a) \leftarrow Q(x,a) + \alpha\delta$$

with $\delta = r + \gamma \max_{a'} Q(x',a') - Q(x,a)$

Q-learning with function approximation

We wish to approximate Q^* using $\mathcal{Q} = \{Q_w : w \in \mathbb{R}^k\}$

Fixed point equation for the optimal state action value function

$$Q^*(x,a) = \mathbb{E}[R(x,a) + \gamma \max_{a' \in \mathcal{A}} Q^*(y,a')].$$

Loss function:

$$L(w) = \frac{1}{2}\mathbb{E}_\mu[(Q^*(x,a) - Q_w(x,a))^2]$$

$$w \leftarrow w + \alpha \mathbb{E}_\mu[(\ Q^*(x,a)\ - Q_w(x,a))\nabla_w Q_w(x,a)]$$

$$w \leftarrow w + \alpha \mathbb{E}_\mu[(\boxed{Q^*(x,a)} - Q_w(x,a))\nabla_w Q_w(x,a)]$$

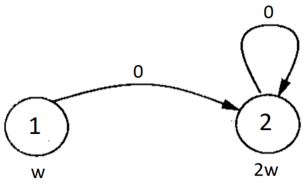
$$w \leftarrow w + \alpha \mathbb{E}_\mu[(\boxed{R(x,a) + \gamma \max_{a' \in \mathcal{A}} Q_w(y,a')} - Q_w(x,a))\nabla_w Q_w(x,a)]$$

Q-learning with function approximation

The $w \to 2w$ example (Tsitsiklis and Van Roy 1996)

Consider the state space $\mathcal{X} = \{x_1, x_2\}$, one action, all rewards 0, and the transition matrix

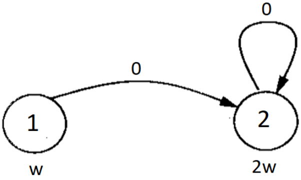
$$P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$



$$Q^* = V = 0$$

Q-learning with function approximation

The $w \to 2w$ example (Tsitsiklis and Van Roy 1996)



$$Q^* = V = 0$$

$$\mathcal{Q} = \{w\phi,\ w \in \mathbb{R}\}$$

with $\phi : \mathcal{X} \to \mathbb{R}$ such that $\phi(x_1) = 1, \phi(x_2) = 2$

The spiral example (Tsitsiklis and Van Roy 1997)

Markov chain

$\mathcal{X} = \{s_1, s_2, s_3\}$, $\mathcal{A} = \{a\}$ and

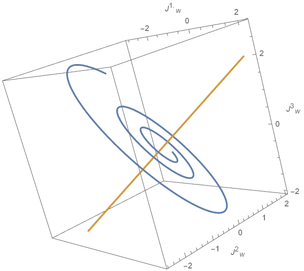
$$\mathrm{P} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

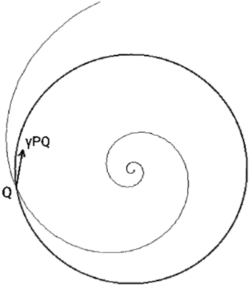
Approximation architecture

$$\frac{dQ_w}{dw} = (S + \epsilon \mathrm{I}) Q_w,$$

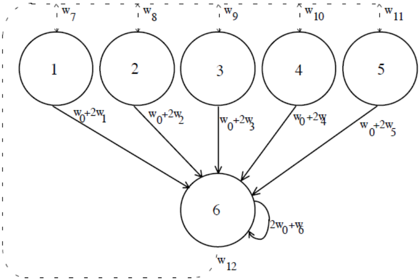
where ϵ is very small and

$$S = \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} & 1 \end{bmatrix}.$$





MDP Example (Baird 1995)

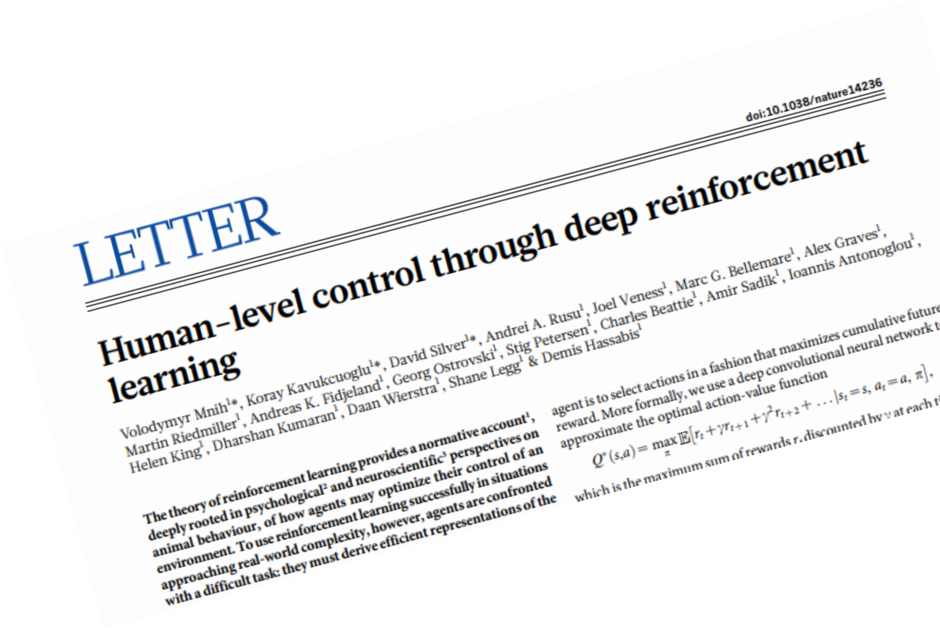


$Q_0(\text{Solid}) > Q_0(\text{Dashed})$
$Q_0(6 \to 6)$ largest

The Deadly Triad

- Function Approximation
- Bootstraping
- Off-policy training

Deepmind's breakthrough



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LETTER

Human-level control through deep reinforcement learning

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The theory of reinforcement learning provides a normative account[1], deeply rooted in psychological[2] and neuroscientific[3] perspectives on animal behaviour, of how agents may optimize their control of an environment. To use reinforcement learning successfully in situations approaching real-world complexity, however, agents are confronted with a difficult task: they must derive efficient representations of the

agent is to select actions in a fashion that maximizes cumulative future reward. More formally, we use a deep convolutional neural network to approximate the optimal action-value function

$$Q^*(s,a) = \max_\pi \mathbb{E}\left[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \ldots \mid s_t = s,\, a_t = a,\, \pi\right],$$

which is the maximum sum of rewards r_t discounted by γ at each ti

Deepmind's breakthrough

Two special techniques used:

- Replay Buffer
- Target Network

$$w \leftarrow w + \alpha \mathbb{E}_\mu[(\boxed{R(x,a) + \gamma \max_{a' \in \mathcal{A}} Q_w(y,a')} - Q_w(x,a))\nabla_w Q_w(x,a)]$$

$$w \leftarrow w + \alpha \mathbb{E}_\mu[(\boxed{R(x,a) + \gamma \max_{a' \in \mathcal{A}} Q_u(y,a')} - Q_w(x,a))\nabla_w Q_w(x,a)]$$

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Stochastic Approximation

$$w = h(w) = \mathbb{E}_\mu[H(w,X)]$$

$$w \leftarrow w + \alpha(\ \mathbb{E}_\mu[H(w,X)] \ - w)$$

$$w \leftarrow w + \alpha(\ \mathbb{E}_\mu[H(w,X)] \ - w)$$

$$\begin{aligned}
w_{t+1} &= w_t + \alpha\big(H(w_t, x_t) - w_t\big) \\
&= w_t + \alpha\big(H(w_t, x_t) - w_t + h(w) - h(w)\big) \\
&= w_t + \alpha\big(h(w) - w + \ H(w_t, x_t) - h(w) \ \big) \\
&= w_t + \alpha\big(h(w) - w + \ H(w_t, x_t) - h(w) \ \big) \\
&= w_t + \alpha\big(h(w) - w + M_t\big)
\end{aligned}$$

Stochastic Approximation

The O.D.E. approach

The equation

$$w_{t+1} = w_t + \alpha_t\big(h(w_t) - w + M_t\big)$$

can be thought as a noisy discretization for the o.d.e.

$$\dot{w} = h(w) - w$$

$$\sum_t \alpha_t = \infty, \qquad \sum_t \alpha_t^2 < \infty$$

M_t is a Martingale difference sequence

Two time scale stochastic approximation

$$\begin{cases} v_{t+1} = v_t + \alpha_t(f(v_t, u_t) + M_{t+1}) \\ u_{t+1} = u_t + \beta_t(g(v_t, u_t) + N_{t+1}) \end{cases}, t \in \mathbb{N}$$

Assumptions:

- $\sum_t \beta_t = \infty, \sum_t \alpha^2 < \infty,\ \beta_t/\alpha_t \to 0$
- $f : \mathbb{R}^{k+d} \to \mathbb{R}^k$, $g : \mathbb{R}^{k+d} \to \mathbb{R}^d$ are locally Lipschitz
- $\sup_t(||v_t|| + ||u_t||) < \infty \quad w.p.1.$
- $M_t \in \mathbb{R}^k$ and $N_t \in \mathbb{R}^d$ are Martingale difference sequences and

$$\mathbb{E}[||M_t||^2] \leq c_M(1 + ||v_t||^2 + ||u_t||^2)$$
$$\mathbb{E}[||N_t||^2] \leq c_N(1 + ||v_t||^2 + ||u_t||^2)$$

Two time scale stochastic approximation

$$\begin{cases} v_{t+1} = v_t + \alpha_t(f(v_t, u_t) + M_{t+1}) \\ u_{t+1} = u_t + \beta_t(g(v_t, u_t) + N_{t+1}), \end{cases}, t \in \mathbb{N}$$

Assumptions (cont):

- The o.d.e.

$$\dot{v}_t = f(v_t, u)$$

  has a unique globally asymptotically stable equilibrium $\lambda(u)$, where $\lambda : \mathbb{R}^d \to \mathbb{R}^k$ is a Lipschitz-continuous function.
- The o.d.e.

$$\dot{u}_t = g(\lambda(u_t), u_t)$$

  has a unique globally asymptotically stable equilibrium u^*.

Two time scale stochastic approximation

Theorem (Borkar 2008)

If the assumptions are true, then

$$(v_t, u_t) \rightarrow^{a.s.} (\lambda(u^*), u^*).$$

Contents

Coupled Q-learning

Update rule

$$v_{t+1} = v_t + \alpha_t\big(r_t + \max_{a'\in\mathcal{A}} Q_{u_t}(y_t, a') - Q_{v_t}(x_t, a_t)\big)\nabla Q_{v_t}(x_t, a_t)$$
$$u_{t+1} = u_t + \beta_t(\nabla Q_{v_t}(x_t, a_t)Q_{v_t}(x_t, a_t) - u_t)$$

Each function $Q_w : \mathcal{X} \times \mathcal{A} \to \mathbb{R}$ is such that

$$Q_w(x, a) = \phi^T(x, a)w$$

we call $\phi : \mathcal{X} \times \mathcal{A} \to \mathbb{R}^k$ the feature vector

Then $\nabla_w Q_w(x, a) = \phi(x, a)$

Convergence result

Assumptions

1. State-action pairs are independent and identically distributed to μ;
2. $\Sigma_\mu = \mathbb{E}_\mu[\phi(x_t, a_t)\phi^T(x_t, a_t)]$ is non-singular; $||\phi(x, a)||_2 \leq 1$;
3. $\sum_{t=0}^{\infty} \beta_t = \infty$, $\sum_{t=0}^{\infty} \alpha_t^2 < \infty$ and $\beta_t = o(\alpha_t)$.

Theorem (Carvalho, Melo and Santos, 2020)

Under assumptions 1 through 3, the sequence $\{u^{(t)}, v^{(t)}\}$ generated by coupled Q-learning converges a.s. to a single limit solution (u^*, v^*).

Error bounds

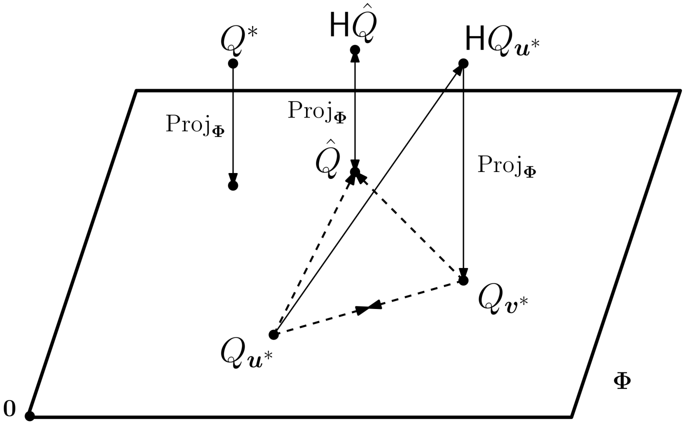
Assume $\{\phi_k\}$ are ortogonal and $\mathbb{E}_\mu[\phi_k] = \sigma$, i.e. $\Sigma_\mu = \sigma I$

Theorem

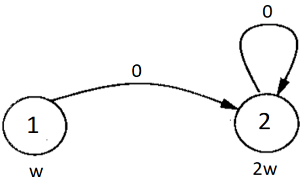
The limit solution Q_{v^*} of coupled Q-learning verifies

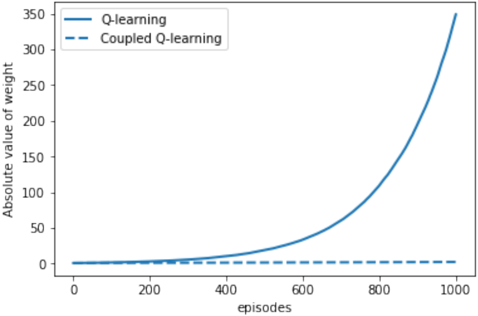
$$\|Q^* - Q_{v^*}\|_\infty \leq \frac{1}{1-\gamma}\|Q^* - Proj_\Phi\, Q^*\|_\infty + \xi_\sigma, \;\; where\ \xi_\sigma = \frac{1-\sigma}{\sigma}\frac{\gamma\rho}{(1-\gamma)^2}$$

Error bounds

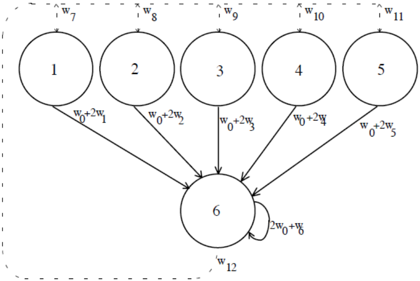


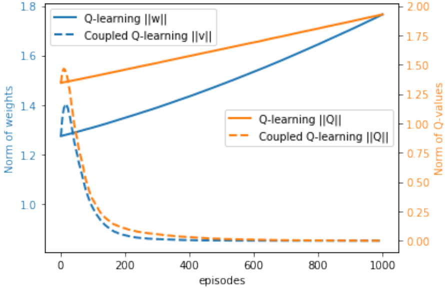
The $\theta \to 2\theta$ example



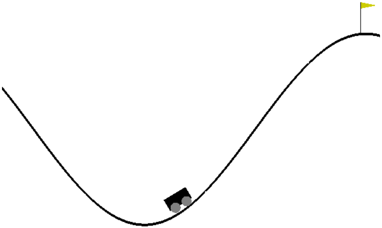


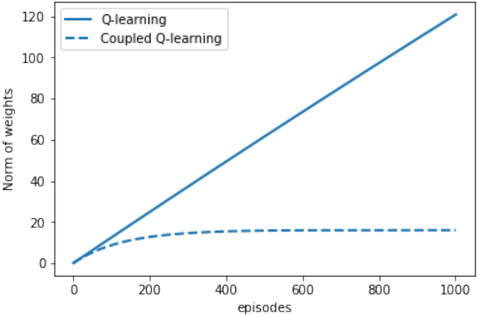
The modified star problem





The mountain car





Other applications of two-time scale stochastic approximation

Double-tap: approximate policy iteration (ongoing work)

$$w_{t+1} = w_t + \alpha_t\big(r_t + \gamma Q_{w_t}(x_{t+1}, a_{t+1}) - Q_{w_t}(x_t, a_t)\big)\nabla Q_{w_t}(x_t, a_t)$$
$$\pi_{t+1} = \pi_t + \beta_t(\Gamma Q_{w_t} - \pi_t),$$

where Γ projects a state action value function to an ϵ-soft policy and is Lipschitz-continuous.

Other applications of two-time scale stochastic approximation

Regular gradient actor critic

$$\begin{aligned} w_{t+1} &= w_t + \alpha_t \big(r_t - j_t + \gamma V_{w_t}(x_{t+1}) - V_{w_t}(x_t) \big) \nabla V_{w_t}(x_t) \\ \theta_{t+1} &= \theta_t + \beta_t \big(r_t - j_t + \gamma V_{w_t}(x_{t+1}) - V_{w_t}(x_t) \big) \nabla \pi_{\theta_t}(x_t, a_t), \end{aligned}$$

where j_t is the average reward at time t.

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Conclusions

Coupled Q-learning

- Is variant of Q-learning
- The algorithm converges with linear function approximation
- This method also allows to transform outer-inner-cycle algorithms into one cycle - two-time step algorithms