VI The Glimm Scheme in a nutshell

Goal: Solve the Cauchy problem (for $u(t,x) \in \mathbb{R}^N$)

$$\begin{cases} u_t + f(u)_x = 0 \quad , (x \in \mathbb{R},\ t > 0) & (1) \\ u(0,x) = u_0(x) . & (2) \end{cases}$$

Assumptions:
- (1) is strictly hyperbolic & each characteristic field is genuinely nonlinear (or linearly degenerate) in a neighborhood $\mathcal{U} \subset \mathbb{R}^N$ of a constant state $\bar{u} \in \mathbb{R}^N$.
- $$T.V.(u_0) + \|u_0 - \bar{u}\|_{L^\infty} \leq C_0 \qquad (3)$$

  for $C_0 > 0$ sufficiently small,

  where $T.V.(u_0)$ is the "total variation" of u_0.

$$\left[\begin{array}{l} T.V.\Big|_a^b (u_0) := \sup\left\{ \sum_{i=0}^{k} |u_0(x_{i+1}) - u_0(x_i)| \;\middle|\; \{x_0, \ldots, x_k\} \text{ partition of } [a,b] \right\} \\ T.V.(u_0) := \lim_{a \to \infty} T.V.\Big|_{-a}^{a} (u_0) \end{array}\right.$$

Thm 1 (Glimm, 1965)

Under above assumptions and for $C_0 > 0$ sufficiently small, there exist a weak solution $u(t,x)$ of (1)-(2), defined $\forall t \geq 0$ $\forall x \in \mathbb{R}$. Moreover, there exist a constant $C > 0$ such that

$$\begin{cases} T.V.\big(u(t,\cdot)\big) + \|u(t,\cdot) - \bar{u}\|_{L^\infty} \leq C \left(T.V.(u_0) + \|u_0 - \bar{u}\|_{L^\infty} \right) & (4) \\ \int_{-\infty}^{\infty} |u(t_2,x) - u(t_1,x)|\, dx \leq C\, T.V.(u_0) \cdot |t_2 - t_1| . & (5) \end{cases}$$

Remarks:

- The proof of Thm 1 is based on the following steps:
  1) Approximate solutions by piecewise solutions of Riemann problems at random "spatial locations" in each time step. ("Glimm scheme")
     ↳ Family of iteration schemes labeled by random parameter θ.
  2) Interaction estimates for solutions of neighboring Riemann problems.
  3) Introduce and control functional equivalent to $T.V.(\cdot) + \|\cdot\|_{L^\infty}$-norm
  4) Prove convergence of iterates (actually a subsequence) to a weak solution of (1)-(2).
     ↳ This is accomplished for almost every random parameter θ.
- Simplifying assumption: $\bar{u} = 0$ and no linear degenerate fields
- Recall (page 10): u is called a weak solution of (1)-(2) if

$$\int_0^\infty \int_{-\infty}^\infty \left(u \cdot \phi_t + f(u)\, \phi_x \right) dx\, dt + \int_{-\infty}^\infty u_0\, \phi\Big|_{\{t=0\}} dx = 0 \quad \forall \phi \in C_0^\infty(\mathbb{R}$$

- For uniqueness "results" see work of Alberto Bressan, (starting '94).

V.1 Glimm's random choice method

Introduce mesh on $[0,\infty]\times\mathbb{R}$ with mesh-length

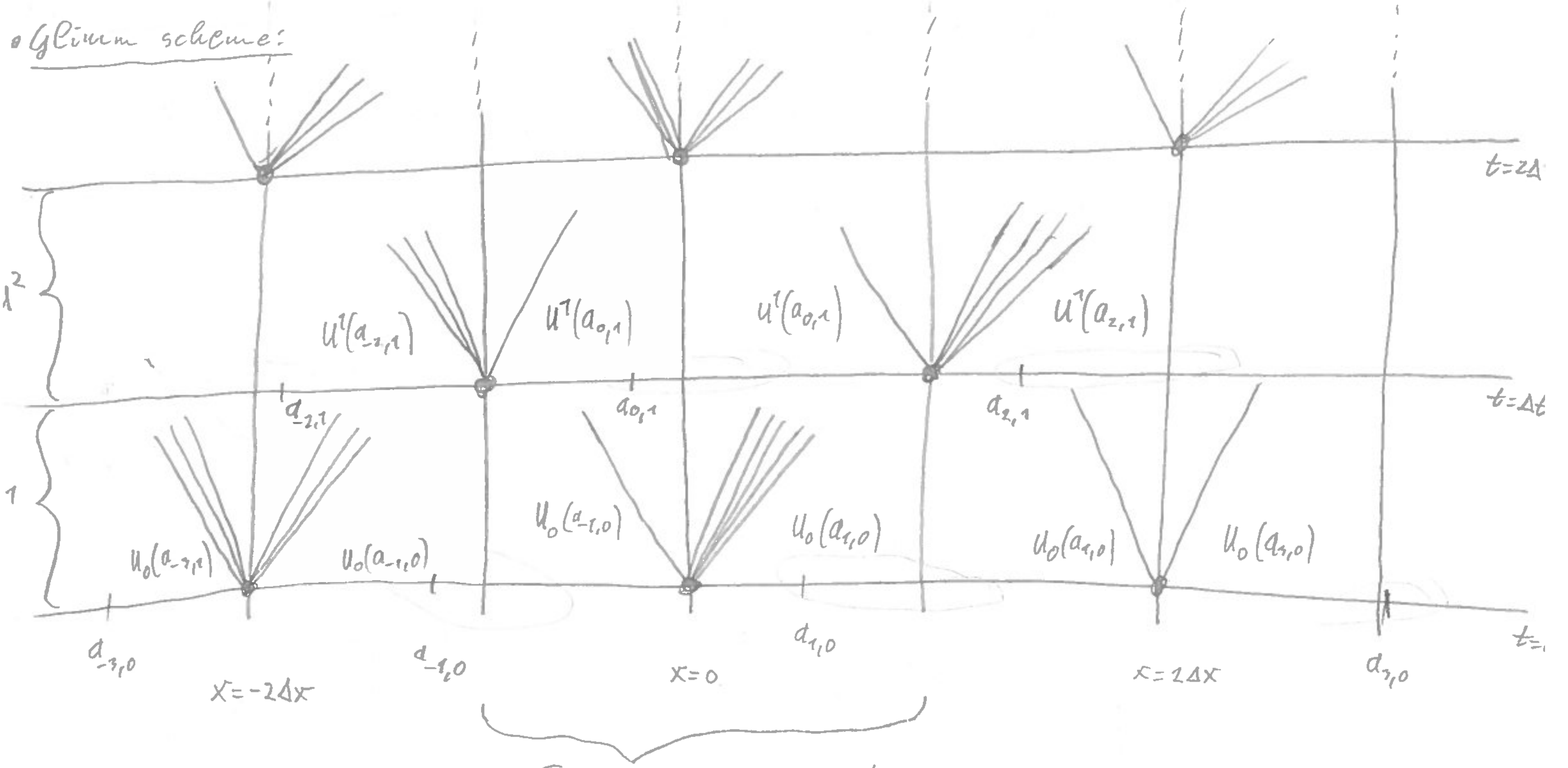
$$\Delta x = c\,\Delta t \quad \text{for} \quad c > \sup_{u\in\mathcal{U}} \{|\lambda_1(u)|, \dots, |\lambda_n(u)|\}$$

characteristic speeds

- For each $n\in\mathbb{N}_0$ choose $\theta_n \in [-1,1]$ at random.

  Define "mesh points" $a_{m,n} := ((m+\theta_n)\Delta x, n\Delta t)$ $\forall m\in\mathbb{Z}$ with $\underline{m+n\in 2\mathbb{Z}}$.

- <u>Glimm scheme:</u>



Solve Riemann problem:

$$v_t + f(v)_x = 0, \quad v(0,x) = \begin{cases} u_0(a_{-1,0}), & x\in[-\Delta x,0] \\ u_0(a_{1,0}), & x\in[0,\Delta x] \end{cases}$$

- Assume u^n is defined on line $\{t = n\Delta t\}$. (Begin iteration at $u^0 = u_0$.)

  For any $t\in[n\Delta t,(n+1)\Delta t]$ and $x\in[(m-1)\Delta x,(m+1)\Delta x]$,

  define $u^{n+1}_m(t,x) = v(t,x)$, where v solves the Riemann problem

$$v_t + f(v)_x = 0, \quad v(n\Delta t, x) = \begin{cases} u^n(a_{m-1,n}), & x\in[(m-1)\Delta x, m\Delta x] \\ u^n(a_{m+1,n}), & x\in[m\Delta x,(m+1)\Delta x] \end{cases}$$

  Define u^{n+1} on $[n\Delta t,(n+1)\Delta t]\times\mathbb{R}$ by setting $u^{n+1}(t,x) = u^{n+1}_m(t,x)$ when $x\in[(m-1)\Delta x,(m+1)\Delta x]$.

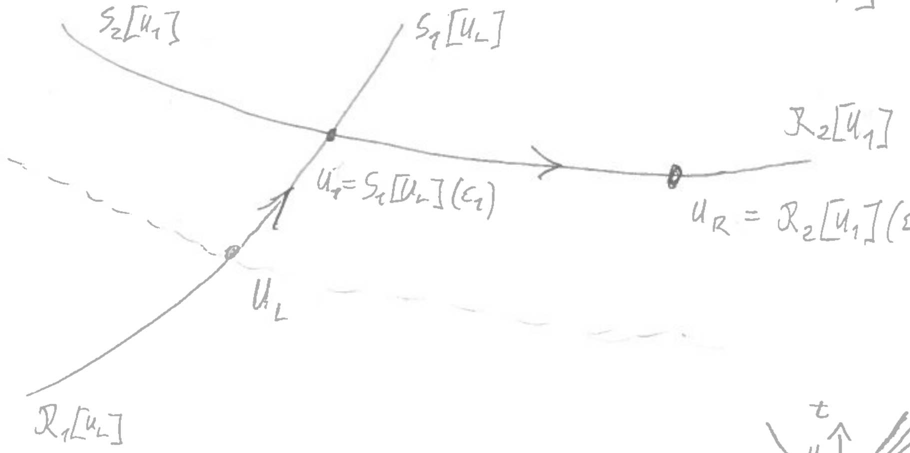
  Then u^{n+1} solves (1) on $[n\Delta t,(n+1)\Delta t]\times\mathbb{R}$, since u^n_m's match continuously.

V.2 Interaction estimates

Goal: Figure out how waves of neighboring Riemann problem at $t=n\Delta t$ interact and effect the wavestrength of subsequent solution of Riemann problem at $t=(n+1)\Delta t$.

Recall: Solution $u=(u_L,u_R)$ of Rieman problem $u_t+f(u)_x=0$, $u(0,x)=\begin{cases}u_L, & x<0\\ u_R, & x>0\end{cases}$ consists of $n-1$ intermediate states $u_1,\dots,u_{n-1}$, separated by either shock or rarefaction waves.
That is, in state space $\mathbb{R}^N$, u_L can be connected to u_R by moving a parameter length ε_k along the k-shock/rarefaction curve, emanating from u_{k-1}, to reach u_k, for any $k=1,\dots,n$.

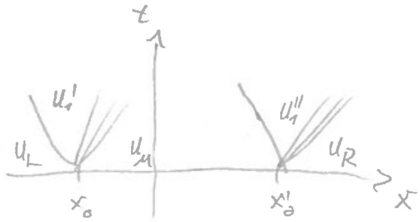
Write: $(u_L,u_R) := \left[(u_L,u_1,\dots,u_{n-1},u_R)/(\varepsilon_1,\dots,\varepsilon_n)\right]$



Consider now two adjacent solution of Riemann problems:

$$(u_L,u_M)=\left[(u_L,u_1',\dots,u_{n-1}',u_M)/(\gamma_1,\dots,\gamma_n)\right]$$

$$(u_M,u_R)=\left[(u_M,u_1'',\dots,u_{n-1}'',u_R)/(\delta_1,\dots,\delta_n)\right]$$



Def: We say the j-wave u_j'/γ_j (connecting u_{j-1}' to u_j') approaches the k-wave u_k''/δ_k if one of the following holds:

(i) $j\neq k$ & wave on left is faster than wave on the right ($j>k$)

(ii) $j=k$ & at least one wave is a shock.

Thm 2

If u_L, u_M, u_R are sufficiently close (for soln's (u_L,u_M), (u_M,u_R) & (u_L,u_R) to exi

then

$$\varepsilon_i = \gamma_i + \delta_i + D(\gamma,\delta)\,\overbrace{O(1)}^{\text{constant}, \in \mathbb{R}}$$

for $D(\gamma,\delta) := \sum\limits_{\substack{i\text{-wave} \\ \&\ \delta\text{-wave} \\ \text{approach}}} |\gamma_i|\,|\delta_j|$.

About proof:

- Follows by induction, carefully studying single waves of solutions of Riemann problems.
- See [Smoller], Theorem 19.2 for details.

$\blacksquare$

V.3 The Glimm functional

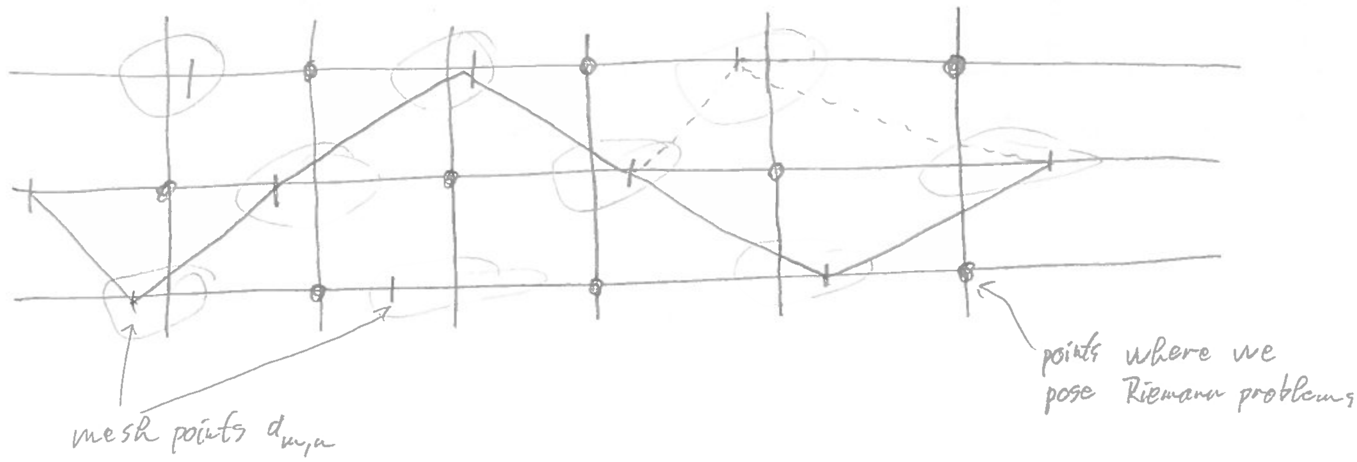
Let $\theta = (\theta_1, \theta_2, \dots)$ be a choice of random parameters.
Denote with $u_{\theta,\Delta x}$ approximate solutions generated by Glimm scheme for fixed θ and Δx.

Goal: Derive bounds on $T.V.(\cdot)$, $\|\cdot\|_{L^\infty}$ and L^1-Lipschitz norms (of $u_{\theta,\Delta x}$) that are independent on θ and Δx.

Def: A "mesh curve" is a continuous (unbounded) curve J that connects mesh points $a_{m,n}$ such that: if J is defined $\forall x \le m\Delta x$ and $a_{m,n} \in J$, then either

$a_{m+1,n+1} \in J$ or $a_{m+1,n-1} \in J$.

($\hookrightarrow$ Mesh curves move from West to either North or South.)



- Given mesh curves J & I, we say $\boxed{J > I}$ if every point on I lies either on J or lies in the region bounded by J and $\{t=0\}$.
  This defines a partial ordering!

Def: Given an approximate solution $u_{\theta,\Delta x}$, we define on a mesh curve J the functionals

$$L(J) := \sum_{\alpha \text{ crosses } J} |\alpha| \qquad \text{and} \qquad Q(J) := \sum_{\substack{\alpha,\beta \text{ cross } J \\ \text{and approach}}} |\alpha|\cdot|\beta|$$

where α and β are waves in $u_{\theta,\Delta x}$, (and $|\alpha|$ parameter length of α-wave)

[Note, we identify waves connecting states with the parameter required to connect these states along the corresponding wave curves.

Thm 3

Let J, I be mesh curves with $J > I$ and that I lies in domain of definition of $u_{\theta,\Delta x}$.
If $L(I)$ is sufficiently small, then J lies in domain of definition of $u_{\theta,\Delta}$ and $Q(J) \le Q(I)$. Moreover, $\exists\, k > 0$ independent of J such that

$$L(J) + k\,Q(J) \le \underbrace{L(I) + k\,Q(I)}_{\text{"Glimm functional"}}.$$

about proof:

- Is based on applying interaction estimates (Thm 2) to an immediate successor J of I.
- See Theorem 19.5 in [Smoller] for details.

$\blacksquare$

Cor 1

If $T.V.(u_0)$ is small, then the approximate solution $u_{\theta,\Delta x}$ is defined $\forall t \geq 0$

Cor 2:

If $T.V.(u_0)$ is sufficiently small, then there exist $C>0$ independent of $n, \theta, \Delta x$ and Δt, such that

(Recall: $\bar{u}=0$, WLOG)

$$T.V.\left(u_{\theta,\Delta x}(\cdot, n\Delta t)\right) + \|u_{\theta,\Delta x}(\cdot, n\Delta t)\|_{L^\infty} \leq C\left(T.V.(u_0) + \|u_0\|_{L^\infty}\right) \tag{6}$$

$$\& \int_{-\infty}^{\infty} |u_{\theta,\Delta x}(x,t_2) - u_{\theta,\Delta x}(x,t_1)|\, dx \leq C\left(|t_2 - t_1| + \Delta t\right) \tag{7}$$

Idea of proof: $L(\cdot)$ & $T.V.(\cdot)$ are equivalent norms... $\blacksquare$

V.4 Convergence of $u_{\theta,\Delta x}$ as $\Delta x \to 0$

- By Helly's Theorem, the uniform bound (6) implies the existence of a subsequence $(u_{\theta,\Delta x_j})_{j\in\mathbb{N}}$ which converges to some function u in L^1_{loc} as $\Delta x_j \to 0$, and u satisfies (6):

$$T.V.(u(t,\cdot)) + \|u(t,\cdot)\|_{L^\infty} \leq C\left(T.V.(u_0) + \|u_0\|_{L^\infty}\right) \quad \forall t \geq 0.$$

- The hard part is to prove that u solves the Cauchy problem (1)-(2) for almost every θ.

That is, to show that

$$\boxed{\mathcal{L}_\phi(u) = 0} \quad \forall \phi \in C_0^\infty(\mathbb{R}^2),$$

where $\mathcal{L}_\phi(u) := \int_0^\infty \int_{-\infty}^\infty \left(u \cdot \phi_t + f(u)\phi_x\right) dx\, dt + \int_{-\infty}^\infty u_0\, \phi\big|_{\{t=0\}}\, dx$

- Now, $u_{\theta,\Delta x} = \bigcup_{n\in\mathbb{N}} u^n$ is a weak solution of (1) in each time strip $[(n-1)\Delta t, n\Delta t]\times\mathbb{R}$ and one can show that

$$u_{\theta,\Delta x_j}\big|_{\{t=0\}} \xrightarrow{\Delta x_j \to 0} u_0 \quad \text{in } L^1_{loc}.$$

$$\Rightarrow \mathcal{L}_\phi(u_{\theta,\Delta x_j}) \overset{\substack{\text{(Gauss Thm, for each } n)\\ \downarrow}}{=} \underbrace{\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \phi(x, n\Delta t)\Big(u_{\theta,\Delta x_j}(n\Delta t + 0, x) - u_{\theta,\Delta x_j}(n\Delta t - 0, x)\Big)\,dx}_{=:\, J(\theta,\Delta x_j,\phi)}$$

Thus: $u_\theta := \lim_{j\to\infty} u_{\theta,\Delta x_j}$ is a weak solution of (1)

if and only if $\underbrace{J(\theta,\Delta x_j,\phi) \xrightarrow{\Delta x_j \to 0} 0}$

One can prove this by integrating over θ, resulting in convergence for almost every θ. (Non-trivial! See Thm 19.14 in [Smoller].)

- We finally obtain:

Thm 4:

If T.V.(u_0) is sufficiently small, then $\exists\, \mathcal{N} \subset [-1,1]\times\mathbb{N}$ a set of measure zero and a subsequence $(u_{\theta,\Delta x_j})_{j\in\mathbb{N}}$, such that $u_\theta := \lim_{j\to\infty} u_{\theta,\Delta x_j}$ is a weak solution of the Cauchy problem (1)–(2) for any $\theta \in ([-1,1]\times\mathbb{N})\setminus\mathcal{N}$.

proof:

See Thm 19.5 in [Smoller] ☒

- Thm 4 together with estimates (6) & (7) complete the proof of Thm 1.

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