Stability of the Lyapunov exponents of randomly perturbed quasi-periodic cocycles

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In Memoriam of

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Joint work with

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Linear Schrödinger 1-dimensional Equation

$$i\,\hbar \frac{\partial \psi}{\partial t} = (-\Delta + V)\,\psi =: \mathcal{H}_V\,\psi$$

$\mathcal{H}_V : \mathcal{U} \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is called the **Schrödinger Operator**

The dynamics of the LSE depends on the **spectral properties** of the Schrödinger operator.

Discrete 1-dimensional Schrödinger Operator

$V : \mathbb{Z} \to \mathbb{R}$ bounded **potential** $\quad V = (V_n)_{n\in\mathbb{Z}}$

$H_V : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$

$$(H_V\psi)_n := -(\psi_{n+1} + \psi_{n-1}) + V_n\,\psi_n$$

Question How does the behavior of the potential V influences the spectral properties of the Schrödinger operator H_V ?

Spectral types of the Schrödinger Operators

- **Absolutely continuous** spectrum
- **Singular continuous** spectrum
- **Pure point** spectrum

Spectral types of the Schrödinger Operators

- **Absolutely continuous** spectrum $\Rightarrow$ **Conductive medium**
- **Singular continuous** spectrum
- **Pure point** spectrum $\Rightarrow$ **Insulating medium**

Quasiperiodic Schrödinger Operators

$\alpha \in \mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$ **irrational frequency**, $\alpha \notin \mathbb{Q}/\mathbb{Z}$
$V : \mathbb{T}^1 \to \mathbb{R}$ **smooth** or **analytic**

The family of sequences $V_\theta(n) := V(\theta + n\,\alpha)$, determines the family of **quasiperiodic Schrödinger operators** $H_\theta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$,

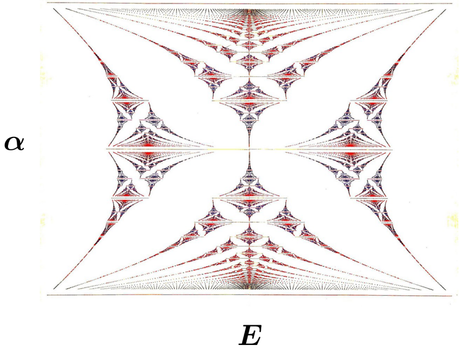
$$(H_\theta \psi)_n := -(\psi_{n+1} + \psi_{n-1}) + V(\theta + n\,\alpha)\,\psi_n$$

'Quasiperiodic Schrödinger-type operators naturally arise in solid state physics*, describing the influence of an external magnetic field on the electrons of a crystal'* **S. Jitomirskaya**

Hofstadter's Butterfly

The **Almost Mathieu Operator** is the quasi-periodic Schrödinger Operator determined by the family of potentials

$$V_\lambda : \mathbb{T} \to \mathbb{R}, \quad V_\lambda\,(x) := 2\,\lambda\,\cos(2\pi x).$$



Random Schrödinger Operators

$(\Omega, \mathcal{F}, \mathbb{P})$ probability space
$\{V_n(\omega)\}_{n\in\mathbb{Z}}$ i.i.d. random process, $\quad \omega \in \Omega$

This i.i.d. process determines the family of **Anderson-Bernoulli** also known as **Random Schrödinger operators**

$$(H_\omega \psi)_n := -(\psi_{n+1} + \psi_{n-1}) + V_n(\omega)\, \psi_n$$

These operators have **pure point spectrum** with an eigen-basis consisting of eigen-functions with exponential decay as $n \to \pm\infty$, a phenomenon known as **Anderson Localization** (AL).

Dynamically Defined Schrödinger Operators

$f : M \to M$ map preserving an **ergodic measure** $f_* \mu = \mu \in \mathrm{Prob}(M)$
$V : M \to \mathbb{R}$ bounded measurable function

The **Schrödinger operator dynamically defined** by (f, μ, V) is the family of operators $H_x : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$

$$\left[H_x \psi\right]_n := -(\psi_{n+1} + \psi_{n-1}) + V(f^n(x))\psi_n$$

for $\psi = \{\psi_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ and $x \in M$.

Theorem (Pastur–1980)

If (f, μ) *is* ergodic *then the Schrödinger operators* H_x *have the* same spectrum *for* μ*-almost every* $x \in M$.

Linear Cocycle

A **linear cocycle** is a bundle map
$F : X \times \mathbb{R}^m \to X \times \mathbb{R}^m, \qquad F(\omega, v) = (f\omega, A(\omega)\, v),$

where

- (X, μ) is a **probability space**;
- $f : X \to X$ is a **measurable transformation** preserving μ;
- $A : X \to \mathrm{GL}_m(\mathbb{R})$ is a **bounded measurable** function.

The iterates of F are given by

$$F^n(\omega, v) = (f^n\omega, A^n(\omega)\, v)$$

$$A^n(\omega) := A(f^{n-1}\omega) \cdots A(f\omega)\, A(\omega)$$

Lyapunov exponents

Theorem (Furstenberg-Kesten – 1960)

*Under the previous assumptions, the following limit (*fiber growth rate*) exists and is constant for μ-almost every $\omega \in X$,*

$$L := \lim_{n \to +\infty} \frac{1}{n} \log \|A^n(\omega)\|.$$

This number is denoted by $L_1(A) = L_1(f, \mu, A)$ and referred to as

the **first Lyapunov exponent** of the linear cocycle (f, μ, A).

The Tangent Map

X smooth manifold
$f : X \to X$ smooth map preserving some ergodic measure μ

The **tangent map** $Tf : TX \to TX$ defined by

$$Tf(x, v) := (fx, Df_x\, v).$$

is the prototype model of a **linear cocycle**.

The Lyapunov exponent

$$L_1(f, \mu) := \lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n\| = \lim_{n\to\infty} \frac{1}{n} \log \|Df_{f^{n-1}x} \cdots Df_{fx}\, Df_x\|$$

measures the infinitesimal **divergence speed** of orbits of the map f.

Schrödinger Cocycles

$f : X \to X \quad f_* \mu = \mu$ **ergodic**
$V : X \to \mathbb{R}$ bounded and **measurable**.

Schrödinger cocycle
$F_E \colon X \times \mathbb{R}^2 \to X \times \mathbb{R}^2$, $F_E(\omega, v) := (f\omega, A_E(\omega) v)$, where

$$A_E(\omega) := \begin{bmatrix} V(\omega) - E & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

Orbits of F_E are formal solutions of the Eigenvalue equation

$$\begin{bmatrix} \psi_{n+1} \\ \psi_n \end{bmatrix} = A_E^n(\omega) \begin{bmatrix} \psi_0 \\ \psi_{-1} \end{bmatrix} \quad \Leftrightarrow \quad H_\omega \, \psi = E \, \psi$$

Operator spectrum *versus* Lyapunov spectrum

$$\begin{aligned} A_E \text{ is uniformly hyperbolic} \quad &\Rightarrow \quad H_\omega \text{ has no eigenvalues} \\ &\Rightarrow \quad L_1(A_E) > 0 \end{aligned}$$

Theorem (R. Johnson – 1986)

If $\mathrm{supp}(\mu) = X$ then for μ-almost every $\omega \in X$ and $E \in \mathbb{R}$,

$$A_E \text{ is uniformly hyperbolic} \quad \Leftrightarrow \quad E \notin \mathrm{spec}(H_\omega)$$

$$\mathrm{spec}(H_\omega) = \{E \in \mathbb{R} \colon L_1(A_E) = 0 \text{ or } A_E \text{ is non uniformly hyperbolic}\}$$

Thouless Formula and the Integrated Density of States

Theorem

$f : X \to X \quad f_*\mu = \mu \quad$ ergodic
$V : X \to \mathbb{R}$ *bounded and* measurable.

$$L_1(A_E) = \int_{\mathbb{R}} \log |E - x| \, dN(x)$$

where $N(E)$ *is the so called* integrated density of states *of* $\{H_\omega\}_{\omega \in X}$.

$N(E)$ measures the asymptotic relative number
of eigenvalues $\leq E$ of finite dimensional truncations of H_ω.

Useful Ergodic Properties for Anderson Localization

- $L_1(A_E) > 0$, for all $E \in \mathbb{R}$;
- A good modulus of continuity (e.g. Hölder) for $E \mapsto L_1(A_E)$;
- **Large Deviation Estimates**: $\exists C < \infty\ \forall \varepsilon > 0\ \exists c(\varepsilon) > 0,\ n_0(\varepsilon) \in \mathbb{N}$ such that $\forall n \geq n_0(\varepsilon) \quad \forall E \in \mathbb{R}$,

$$\mu \left\{ \omega \in X \colon \ \left| \frac{1}{n} \log \|A_E^n(\omega)\| - L_1(A_E) \right| > \varepsilon \right\} < C \exp\left(-c(\varepsilon)\, n\right)$$

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$$\mu \left\{ \omega \in X\colon \ \left| \frac{1}{n} \log \| A_E^n(\omega) \| - L_1(A_E) \right| > \varepsilon \right\} < C \exp\left(-c(\varepsilon)\, n\right)$$

Jiangong You's Problem

Consider a quasiperiodic potential to which one adds a small random noise

$$V_n(\theta,\omega) := \underbrace{V(\theta + n\,\alpha)}_{\text{qp potential}} \quad + \quad \underbrace{\varepsilon\,\omega_n}_{\text{i.i.d. rand. noise}}$$

Denote by $A_{\varepsilon,E}$ the corresponding family of Schrödinger cocycles.

Question

Is the Lyapunov exponent stable, i.e.,

$$L_1(A_E) = \lim_{\varepsilon\to 0} L_1(A_{\varepsilon,E})\ ?$$

Mixed Random-Quasiperiodic Operators

$V : \mathbb{T}^d \to \mathbb{R}$ class C^1-function;
(Σ, μ) Polish probability space with $\Sigma = \text{supp}(\mu)$ compact;
$\alpha : \Sigma \to \mathbb{T}^d$ **random frequency**;
$\nu : \Sigma \to \mathbb{R}$ **random noise** ;

The **base map** $f : \Sigma^{\mathbb{Z}} \times \mathbb{T}^d \to \Sigma^{\mathbb{Z}} \times \mathbb{T}^d$,

$$f(\omega, \theta) := (\sigma\omega, \theta + \alpha(\omega_0))$$

preserves $\mu^{\mathbb{Z}} \times m$, which we assume to be **ergodic**.

The **mixed random-quasiperiodic potential** is $\mathcal{V} : \Sigma^{\mathbb{Z}} \times \mathbb{T}^d \to \mathbb{R}$,

$$\mathcal{V}(\omega, \theta) := V(\theta) + \nu(\omega_0),$$

determines the so called **mixed Schrödinger operators**.

Positivity of the Lyapunov exponent

Theorem (Cai-D-Klein)

Under very mild conditions the Lyapunov exponent of a mixed random-quasiperiodic cocycle is positive at all energies $E \in \mathbb{R}$.

For instance,:

- *If the frequency $\alpha : \Sigma \to \mathbb{T}^d$ is constant* it is enough *that the noise $\nu : \Sigma \to \mathbb{R}$ is not constant;*
- *If the noise ν is constant and $V : \mathbb{T}^d \to \mathbb{R}$ is analytic* it is enough *that $\exists a, b \in \Sigma$ such that $\alpha(b) - \alpha(a)$ is ergodic.*

This result is one of many extensions of a classical Theorem of **Harry Furstenberg** which gives a criterion for the positivity of the LE for random linear cocycles.

Continuity of the Lyapunov exponents

The mixed Schrödinger cocycles depend on the data

$$(V, \mu, \alpha, \nu, E) \mapsto A_{V,\mu,\alpha,\nu,E}$$

Theorem (Cai-D-Klein)

Under the same mild conditions, if either

- α *is constant and ergodic (but the noise* ν *is non-constant); or else*
- *the measure* $\mu := \alpha_* \mu \in \mathrm{Prob}(\mathbb{T}^d)$ *satisfies a* mixing Diophantine condition, *i.e.,* $\exists C < \infty$ *and* $\tau > 0$ *such that*

$$|\hat{\mu}(k)| \leq 1 - \frac{C}{\|k\|^\tau} \quad \forall k \in \mathbb{Z}^d \setminus \{0\}.$$

then the Lyapunov exponent is a locally Hölder *continuous function of the above data.*

Uniform Large Deviations

The mixed Schrödinger cocycles depend on the data

$$(V, \mu, \alpha, \nu, E) \mapsto A_{V,\mu,\alpha,\nu,E}$$

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$$|\hat{\mu}(k)| \leq 1 - \frac{C}{\|k\|^\tau} \quad \forall k \in \mathbb{Z}^d \setminus \{0\}.$$

then uniform Large Deviation Estimates of exponential type hold, where the uniformity refers to the above parameters.

Stability of the Lyapunov exponents

- $V : \mathbb{T}^d \to \mathbb{R}$ analytic potential;
- $\alpha \in \mathbb{T}^d$ satisfying a **Diophantine Condition**;
- $\rho \in \mathrm{Prob}([-1,1])$ such that $\exists C < \infty,\ \beta > 0$
$$\forall x \in [-1,1],\ \forall 0 < r < 1 \;\; \rho([x, x+r]) \leq C\, r^{\beta}.$$

The **base map** $f : [-1,1]^{\mathbb{Z}} \times \mathbb{T}^d \to [-1,1]^{\mathbb{Z}} \times \mathbb{T}^d$,

$$f(\omega, \theta) := (\sigma\omega, \theta + \alpha)$$

preserves the **ergodic** measure $\rho^{\mathbb{Z}} \times m$.

The previous data determines the **mixed Schrödinger operator** with potential $\mathcal{V}_{V,\varepsilon} : [-1,1]^{\mathbb{Z}} \times \mathbb{T}^d \to \mathbb{R}$,

$$\mathcal{V}_{V,\varepsilon}(\omega, \theta) := V(\theta) + \varepsilon\, \omega_0.$$

Stability of the Lyapunov exponents

Theorem (Cai-D-Klein)

Given the following data,

- $V : \mathbb{T}^d \to \mathbb{R}$ *is analytic;*
- $\alpha \in \mathbb{T}^d$ *satisfies a* Diophantine Condition;
- $\rho \in \mathrm{Prob}([-1,1])$ *is such that* $\exists C < \infty,\ \beta > 0$

$$\forall x \in [-1,1],\ \forall 0 < r < 1 \quad \rho([x, x+r]) \leq C\, r^{\beta}.$$

then

$$L_1(A_{V,\alpha,E}) = \lim_{\varepsilon \to 0} L_1(A_{V,\alpha,\rho,\varepsilon,E})$$

with locally uniform convergence in the data.

Stability of the Lyapunov exponents

Theorem (Cai-D-Klein)

Given the following data,

- $V : \mathbb{T}^d \to \mathbb{R}$ *is analytic;*
- $\alpha \in \mathbb{T}^d$ *satisfies a* Diophantine Condition*;*
- $L_1(A_{V,\alpha,E_0}) > 0$;
- $\rho \in \mathrm{Prob}([-1,1])$ *is such that* $\exists C < \infty,\ \beta > 0$
$$\forall x \in [-1,1],\ \forall 0 < r < 1 \quad \rho([x, x+r]) \leq C\, r^\beta .$$

then in a neighborhood of $(V, \alpha, \rho, 0, E_0)$ *the map*

$$(V, \alpha, \rho, \varepsilon, E) \mapsto L_1(A_{V,\alpha,\rho,\varepsilon,E})$$

is weak-Hölder continuous. Moreover, uniform large deviation estimates of sub-exponential type hold in this neighborhood.

Strategy of the proof

Abstract Continuity Theorem (D-Klein 2016)
Uniform large deviation estimates of exponential (sub-exponential) type

$$\Downarrow$$

$A \mapsto L_1(A)$ is locally Hölder (weak-Hölder) continuous.

By proximity for $\varepsilon \approx 0$,
Uniform LDE for $A_E \Rightarrow$ LDE for $A_{V,\varepsilon,E}$ up to time $n_0(\varepsilon) := \log^{1/2}(\varepsilon^{-1})$

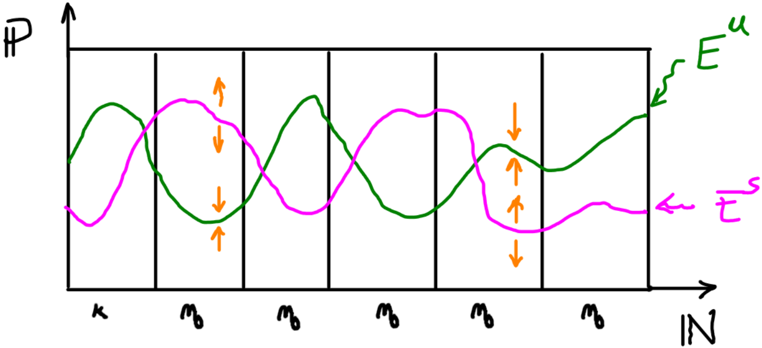
Strategy of the proof

- For $\varepsilon \approx 0$,
  Uniform LDE for $A_E \quad \Rightarrow$ LDE for $A_{V,\varepsilon,E}$ up to time $n_0(\varepsilon) := \log^{1/2}(\varepsilon^{-1})$.
- Because of the random noise $\quad \Rightarrow$ LDE for $A_{V,\varepsilon,E}$ **of exponential type** hold beyond the time scale $n_1(\varepsilon) := \log^2(\varepsilon^{-1}) = n_0(\varepsilon)^4$.
- A sort of interpolating argument, based on the **Avalanche Principle**, gives LDE for $A_{V,\varepsilon,E}$ **of sub-exponential type** at all intermediate time scales $n_0(\varepsilon) \leq n \leq n_1(\varepsilon)$.

Sketch of the proof

A sufficient condition for LDE of exponential type for all $n \geq n_1$ is that $\forall(\theta, \hat{p}) \in \mathbb{T}^d \times \mathbb{P}(\mathbb{R}^2)$, with large probability one has

$$\frac{1}{n_1} \log \| A^{n_1}_{\varepsilon,E}(\omega, \theta)\, p \| \geq \frac{1}{2} L_1(A_E)$$



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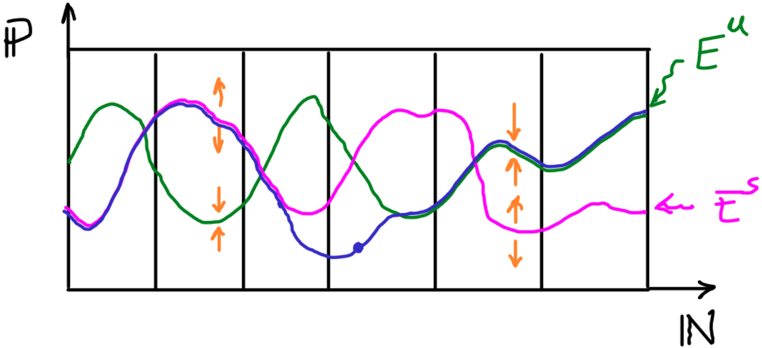

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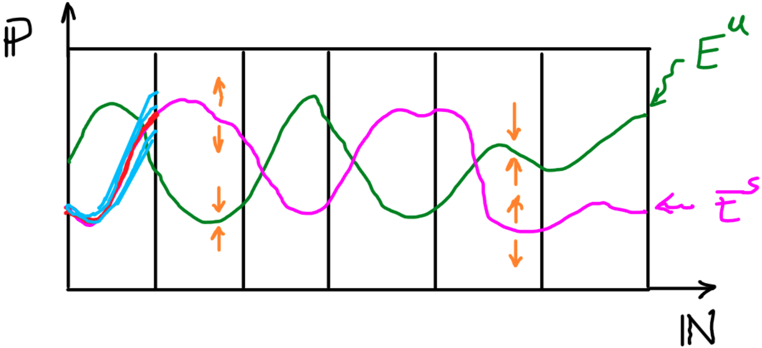

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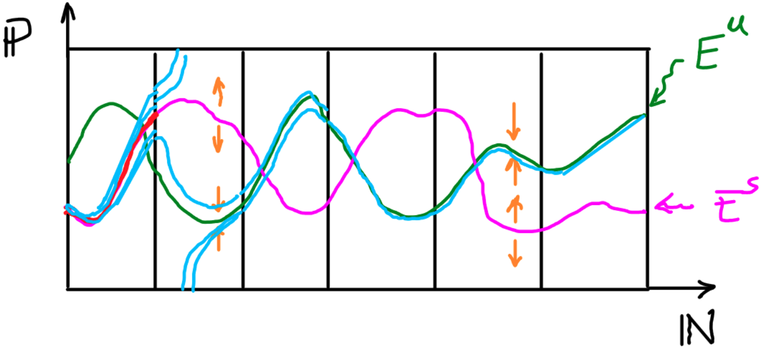

Problems

1. The assumption on the dimension of ρ plays a technical role in the proof. Can it be removed?
2. Can we obtain uniform large deviation estimates and a modulus of continuity when $L_1(A_{V,\alpha,E_0}) = 0$?
3. Can we obtain uniform large deviation estimates and a modulus of continuity when the frequency $\alpha_\varepsilon = \alpha + O(\varepsilon)$ is random?
4. Can these results be extended to cocycles over partially hyperbolic maps?

The End

Thank You

- Ao Cai, Pedro Duarte, Silvius Klein, **Furstenberg Theory of Mixed Random-Quasiperiodic Cocycles**, https://arxiv.org/abs/2201.04745
- Ao Cai, Pedro Duarte, Silvius Klein, **Mixed Random-Quasiperiodic Cocycles**, https://arxiv.org/abs/2109.09544