A unified theory for Inertial Manifolds, Exponential Dichotomy and the Saddle Point Property

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8th IST-IME Meeting - In honor of Giorgio Fusco
In memory of Jorge Sotomayor
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This is a picture of Giorgio in the ICMC-Summer Meeting on Differential Equations - 2008 Chapter together with Jack Hale, Carlos Rocha and Xiaobiao Lin. Me and all of them were Hale's students. The first applications of the inertial manifold theory that I have studied are the works of Carlos Rocha on large diffusivity and the work of Giorgio Fusco "On a explicity construction of an ODE which has the same dynamics as a scalar parabolic PDE" (see next).

Thank you Giorgio for the enlightenment!

On the Explicit Construction of an ODE Which Has the Same Dynamics as a Scalar Parabolic PDE*

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1. Introduction

Suppose $a \in C^2([0,1], \mathbf{R})$, $a(x) > 0$, $0 \leqslant x \leqslant 1$, $f \in C^3([0,1] \times \mathbf{R}, \mathbf{R})$ and consider the parabolic equation

$$\begin{aligned} &u_t = (au_x)_x + f(x,u), && 0 < x < 1, \\ &-\rho u + (1-\rho)\, au_x = 0, && x = 0, \quad t \geqslant 0 \qquad (1) \\ &\sigma u + (1-\sigma)\, au_x = 0, && x = 1, \quad t \geqslant 0, \end{aligned}$$

The interval $[0,1]$ is split in several sub-intervals with the diffusion a being large in the interior of the subintervals and small in the end points. This forces the solutions to be approximately constant in each subinterval leading to a finite dimensional dynamics governed by a explicitly constructed ode.

Outline of the Presentation

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Joint work with P. Lappicy, E. Moreira, A. Oliveira-Sousa

Main goal

Inertial manifold theory, saddle point property and robustness of exponential dichotomy are topics that have been treated separately in the literature with distinct proofs.

As a common feature, they all have the purpose of 'splitting' the space in order to understand the dynamics.

Our goal is to give a unified treatment to these topics and to some further applications to autonomous and non-autonomous problems.

Basic Terminology

Let $(X, \|\cdot\|)$ be a Banach space and Consider the IVP

$$\begin{aligned} \dot{u} &= A(t)u + f(t,u), \;\; t > \tau \\ u(\tau) &= u_0 \in X, \end{aligned} \tag{1}$$

where $f : \mathbb{R} \times X \to X$ is continuous, $f(t,0) = 0$, for all $t \in \mathbb{R}$, and $\exists\, \ell > 0$ such that $\|f(t,u) - f(t,\tilde{u})\| \leqslant \ell \|u - \tilde{u}\|$, $\forall\, (t,u), (t,\tilde{u}) \in \mathbb{R} \times X$.

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The assumption that $f(t,\cdot)$ be globally Lipschitz and $f(t,0) = 0$ may seem too restrictive but this can be circumvented for the intended analysis and are chosen in to simplify the calculations.

Assume that the linear (unbounded) operators $\{A(t) : t \in \mathbb{R}\}$ are such that, for each $(\tau, u_0) \in \mathbb{R} \times X$ there exists a 'unique solution' (denoted by $u(t, \tau, u_0) =: L(t, \tau)u_0,\ t \geqslant \tau$) of

$$
\begin{aligned}
\dot{u} &= A(t)u, \quad t > \tau, \\
u(\tau) &= u_0 \in X,
\end{aligned}
\tag{2}
$$

and $\{L(t, \tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$ is an evolution process, that is,

- $L(t,t) = Id_X$,
- $L(t, s)L(s, \tau) = L(t, \tau),\ t \geqslant s \geqslant \tau$ and, for each $u_0 \in X$,
- $\{(t, \tau) \in \mathbb{R}^2 : t \geqslant \tau\} \ni (t, \tau) \mapsto L(t, \tau)u_0 \in X$ is continuous.

With this, the solutions of (1) are given by a nonlinear evolution process $\{T(t,\tau) : t \geqslant \tau\} \subset \mathcal{C}(X)$ which is implicitly defined by the variation of constants formula, that is, for $t \geqslant \tau$ and $u_0 \in X$,

$$T(t,\tau)u_0 = L(t,\tau)u_0 + \int_\tau^t L(t,s)f(s, T(s,\tau)u_0)\, ds, \qquad (3)$$

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$$\dot{u} = A(t)u + f(t,u), \ t > \tau$$
$$u(\tau) = u_0 \in X,$$

Exponential Splitting & Dichotomy

Exponential dichotomy is the nonautonomous notion of hyperbolicity which, for each time, splits the space into two invariant linear subspaces, one with exponential expansion and another with exponential attraction.

Its study goes back to Perron, Massera and Schäffer [P-MZ-30, MS-Ann-58, MS-Ann-59, MS-AcPrss-66].

Many developments have been achieved ever since. We mention the books [DK-AMST-74, Coppel-LNM-78, H-LNM-81, SY-AMS-00, BV-LNM-08, CLR-AMS-13].

Definition (Exponential Splitting)

A linear evolution process $\{L(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$ has **exponential splitting**, with constant $M \geqslant 1$, exponents $\gamma, \rho \in \mathbb{R}$, with $\gamma > \rho$, and projections $\{Q(t) : t \in \mathbb{R}\} \subset \mathcal{L}(X)$, if

i) $Q(t)L(t,\tau) = L(t,\tau)Q(\tau)$, for all $t \geqslant \tau$,

ii) $L(t,\tau) : Im(Q(\tau)) \to Im(Q(t))$ is an isomorphism, with inverse denoted by $L(\tau,t)$, $t \geqslant \tau$,

iii) the following estimates hold

$$\begin{aligned} \|L(t,\tau)Q(\tau)\|_{\mathcal{L}(X)} &\leqslant Me^{-\rho(t-\tau)}, \quad t \leqslant \tau, \\ \|L(t,\tau)(I - Q(\tau))\|_{\mathcal{L}(X)} &\leqslant Me^{-\gamma(t-\tau)}, \quad t \geqslant \tau. \end{aligned} \tag{4}$$

If $\omega = -\rho = \gamma > 0$, $\{L(t,\tau) : t \geqslant \tau\}$ has **exponential dichotomy** with constant $M \geqslant 1$, exponent ω and projections $\{Q(t) : t \in \mathbb{R}\}$.

$\{L(t,\tau) : t \geqslant \tau\}$ has **exponential splitting**, with constant M, exponents $\gamma > \rho$ and projections $\{Q(t) : t \in \mathbb{R}\}$ if and only if $\{e^{(\gamma - \frac{\gamma-\rho}{2})(t-\tau)} L(t,\tau) : t \geqslant \tau\}$ has **exponential dichotomy**, with constant M, exponent $\omega = \frac{\gamma-\rho}{2}$ and projections $\{Q(t) : t \in \mathbb{R}\}$.

Its most important properties are the description of the local dynamics and its robustness under perturbations.

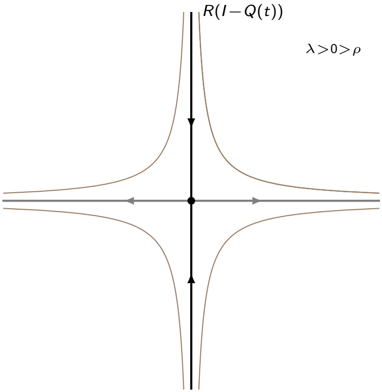


Figure: Exponential Splitting.

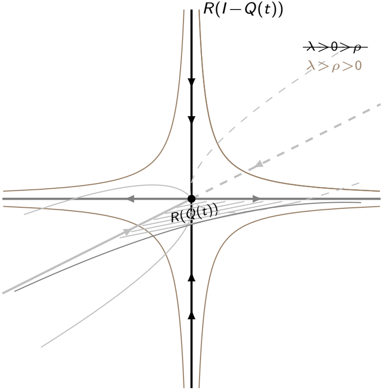


Figure: Exponential Splitting.

Inertial Manifolds & Saddle Point Property

The saddle-point property is of local nature. It arises as a nonlinear version of the linear exponential dichotomy.

The inertial manifold is of global nature. It arises as a nonlinear version of exponential splitting, with the purpose to reduce the relevant dynamics to that of an invertible dynamical system.

Our proof is inspired in the work of Henry [H-LNM-81] which in turn draws its inspiration from Hale [H-ODE-69]. The idea goes back to Lyapunov [Lyapunov-50] and Pliss [Pliss-Iz-64] for the reduction principle with the splitting at zero (center manifolds).

The terminology was introduced by Foias, Sell & Temam in [FST-JDE-88]. See [Z-PRSEA-14] and for a recent account and [KS-JDDE-02] for the non-autonomous case. We introduce the idea of a stable manifold of an inertial manifold, given as a graph.

Let us define an inertial manifold for a nonlinear evolution process.

Definition

A family $\{\mathcal{M}(t) \subseteq X : t \in \mathbb{R}\}$ is called an **inertial manifold** for the evolution process $\{T(t,\tau) : t \geqslant \tau\}$, if

1. $\mathcal{M}(t)$ is a Lipschitz manifold, for each $t \in \mathbb{R}$.
2. $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ is invariant $(T(t,\tau)\mathcal{M}(\tau) = \mathcal{M}(t),\ t \geqslant \tau)$.
3. $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ is exponentially attracting with respect to the elapsed time,

We give a somewhat different proof of the existence of the inertial manifold and its exponential attraction.

Our proof includes the possibility that the inertial manifold be a graph of a possibly unbounded map.

We introduce the notion and prove the existence of a stable manifold of an inertial manifold, which again is given by a graph.

We also include the possibility that the manifold be repelling instead of attracting. This is suitable for the fine description of the behavior of solutions inside an unstable manifold.

With this, the saddle point property becomes a corollary of the invariant manifold theorem.

Also, if f is linear and $-\rho = \gamma > 0$, the inertial manifold is a family of linear spaces which, together with their stable manifolds give a decomposition of the space and an exponential dichotomy.

This will lead to the result on robustness of exponential dichotomy. The same result can be used to obtain robustness of an exponential splitting.

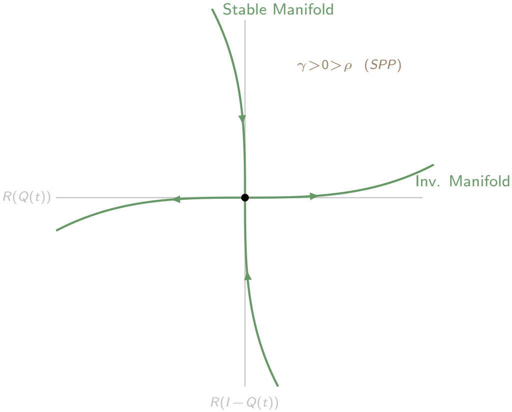
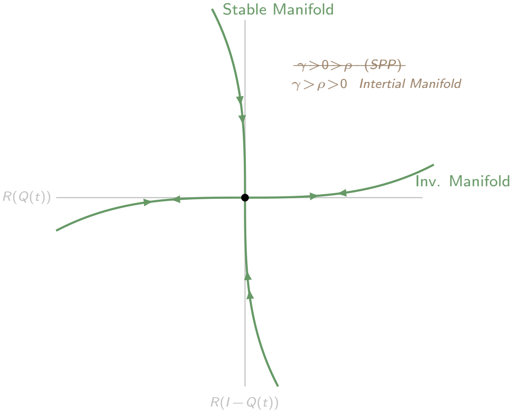


Figure: Invariant manifold and its stable manifold.



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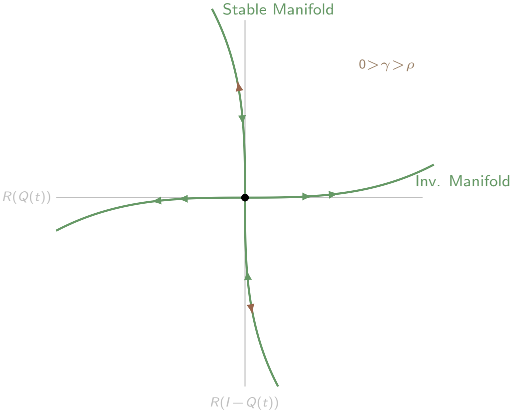

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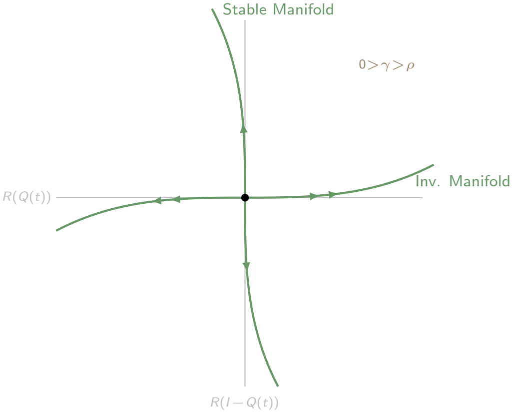

Figure: Invariant manifold and its stable manifold.

Invariant Manifolds &Their Stable Manifolds

Assume that $\{L(t,\tau):t\geqslant\tau\}$ has exponential splitting with constant $M\geqslant 1$, exponents $\gamma>\rho$ and projections $\{Q(t):t\in\mathbb{R}\}$. Assume also that $f:\mathbb{R}\times X\to X$ is continuous, $f(t,0)=0$, $f(t,\cdot):X\to X$ is Lipschitz with constant $\ell>0$, for all $t\in\mathbb{R}$ and that κ satisfies

$$0<\frac{M^2(1+\kappa)}{\frac{\gamma-\rho}{\ell}-2M(1+\kappa)}\leqslant\kappa<1, \tag{5}$$

which can be achieved if

$$\frac{\gamma-\rho}{\ell}>\max\{M^2+2M+\sqrt{8M^3},3M^2+2M\}, \tag{6}$$

Define $\mathcal{L}_\Sigma(\kappa)$ as the set of functions $\Sigma : \mathbb{R} \times X \to X$ such that

$$\begin{aligned} &\Sigma(t,0) = 0, \ t\in\mathbb{R}, \\ &\Sigma(t,u)=\Sigma(t,Q(t)u)\in Im(I-Q(t)), \ (t,u) \in \mathbb{R}\times X, \\ &\|\Sigma(t,u)-\Sigma(t,\tilde{u})\| \leqslant \kappa\|u-\tilde{u}\|, \ (t,u),(t,\tilde{u}) \in \mathbb{R}\times X, \end{aligned} \tag{7}$$

and $\mathcal{L}_\Theta(\kappa)$ as the set of functions $\Theta : \mathbb{R} \times X \to X$ such that

$$\begin{aligned} &\Theta(t,0) = 0, \ t\in\mathbb{R}, \\ &\Theta(t,u)=\Theta(t,(I-Q(t))u)\in Im(Q(t)), \ (t,u) \in \mathbb{R}\times X, \\ &\|\Theta(t,u)-\Theta(t,\tilde{u})\| \leqslant \kappa\|u-\tilde{u}\|, \ (t,u),(t,\tilde{u}) \in \mathbb{R}\times X, \end{aligned} \tag{8}$$

The Invariant Manifold Theorem

Theorem (IMT)

Under these assumptions, there is a $\Sigma^* \in \mathcal{L}_\Sigma(\kappa)$ *such that, if*

$$\mathcal{M}(t) := \{u \in X : u = q + \Sigma^*(t,q), q \in Im(Q(t))\}, \ t \in \mathbb{R}, \quad (9)$$

$\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *yields an invariant manifold for* $\{T(t,\tau) : t \geqslant \tau\}$.

Furthermore, if $P_{\Sigma^*}(t)u := Q(t)u + \Sigma^*(t, Q(t)u)$, $(t,u) \in \mathbb{R} \times X$,

(i) $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *has controlled growth: for any* $u \in X$, $t \leqslant \tau$,

$$\|T(t,\tau)P_{\Sigma^*}(\tau)u\| \leqslant M(1+\kappa)e^{-(\rho+\ell M(1+\kappa))(t-\tau)}\|P_{\Sigma^*}(\tau)u\|. \quad (10)$$

(ii) *If* $\delta := \gamma - M\ell - \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma-\rho-\ell M(1+\kappa)}$, $\{\mathcal{M}(t) : t \in \mathbb{R}\}$ *satisfies:*

$$\|(I-P_{\Sigma^*}(t))T(t,\tau)u\| \leqslant M\|(I-P_{\Sigma^*}(\tau))u\|e^{-\delta(t-\tau)}, \quad (11)$$

for $u \in X$, $t \geqslant \tau$. *If* $\delta > 0$ *the invariant manifold is exponentially attracting and it is called an inertial manifold.*

Skip to [??]

Stable Manifold of an Invariant Manifold

Theorem (SMIM)

There is a $\Theta^* \in \mathcal{L}_\Theta(\kappa)$ *such that, if*

$$\mathcal{N}(t) := \{u \in X : u = \Theta^*(t,p) + p,\, p \in Im(I - Q(t))\},\ \ t \in \mathbb{R}, \quad (12)$$

$\{\mathcal{N}(t) : t \in \mathbb{R}\}$ *yields a positively invariant family for* $\{T(t,\tau) : t \geqslant \tau\}$.

Moreover, if $P_{\Theta^*}(t)u := \Theta^*(t, (I - Q(t))u) + (I - Q(t))u,\ (t,u) \in \mathbb{R} \times X$,

$$\|T(t,\tau)P_{\Theta^*}(\tau)u\| \leqslant M(1+\kappa)e^{-(\gamma - \ell M(1+\kappa))(t-\tau)}\|P_{\Theta^*}(\tau)u\|, \quad (13)$$

$t \geqslant \tau,\ u \in X$, *and*

$$\|u - P_{\Theta^*}(\tau)u\| \leqslant Me^{\hat{\delta}(t-\tau)}\|(I - P_{\Theta^*}(t))T(t,\tau)u\|, \quad (14)$$

$t \geqslant \tau,\ u \in X$, *where* $\hat{\delta} = \rho + M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma - \rho - M\ell(1+\kappa)}$.

Furthermore, if $\gamma > 0$, $\{Im(P_{\Theta^*}(t)) : t \in \mathbb{R}\}$ *is the stable manifold of the inertial manifold* $\{Im(P_{\Sigma^*}(t)) : t \in \mathbb{R}\}$.

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Asymptotic Phase

If the projections have finite rank we also have.

Corollary (Asymptotic Phase)

Under the hypothesis of Theorem (IMT), if $\dim Im(Q(\tau)) < \infty$ *and*

$$\bar{\delta} := \ell \left[\frac{\gamma - \rho}{\ell} - M(2+\kappa) - \frac{M^2(1+\kappa)(1+M)}{\frac{\gamma-\rho}{\ell} - M(1+\kappa)} \right] > 0,$$

there exists $c > 0$ *such that, for any* $u_0 \in X$, *there exists a solution* $[\tau, \infty) \ni t \mapsto (\bar{q}(t), \Sigma^*(t, \bar{q}(t))) \in \mathcal{M}(t)$ *such that*

$$\| T(t,\tau)u_0 - (\bar{q}(t) + \Sigma^*(t, \bar{q}(t)))\|_X \leqslant ce^{-\delta(t-\tau)} \|(I - Q(\tau))u_0 - \Sigma^*(\tau, q(\tau)))\|_X$$

where δ *is given in Theorem 3 .* *Skip to [??]*

The Saddle Point Property

We now obtain the saddle point property as corollary of Theorems 3 and 4. First we define the **unstable** and **stable sets** of 0.

$$W^u(0) := \left\{ \begin{array}{l} (\tau, u_0) \in \mathbb{R} \times X : \text{there is a solution } u : (-\infty, \tau] \to X \\ \text{such that } u(\tau) = u_0 \text{ and } \lim_{t \to -\infty} u(t) = 0, \end{array} \right\} \tag{15a}$$

$$W^s(0) := \left\{ \begin{array}{l} (\tau, u_0) \in \mathbb{R} \times X : \text{there is a solution } u : [\tau, \infty) \to X \\ \text{such that } u(\tau) = u_0 \text{ and } \lim_{t \to +\infty} u(t) = 0. \end{array} \right\} \tag{15b}$$

For a characterization as graphs of the unstable and stable manifolds of global hyperbolic solutions see [CL-JDE-07]. Here they follow from Theorems (IMT) and (SMIM).

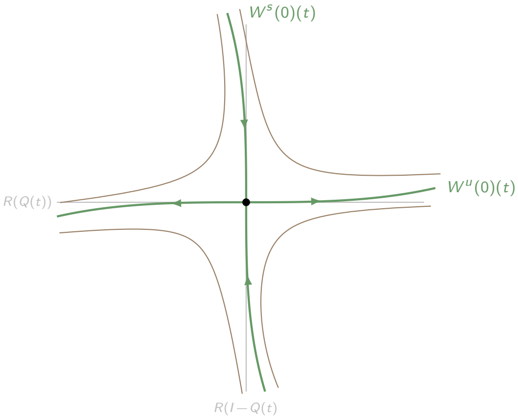


Figure: Unstable and stable manifolds.

Corollary

Suppose that the linear evolution process $\{L(t,\tau) : t \geqslant \tau\}$ *has exponential dichotomy, with constant* $M \geqslant 1$*, exponent* $\omega > 0$ $(\omega = \gamma = -\rho)$ *and a family of projections* $\{Q(t) : \ t \in \mathbb{R}\}$*. Then, for suitably small* $\ell > 0$*, there are continuous functions* $\Sigma^u \in \mathcal{L}_\Sigma(\kappa)$ *and* $\Theta^s \in \mathcal{L}_\Theta(\kappa)$ *such that the unstable and stable manifolds of* $u_* = 0$ *are given by*

$$W^u(0) = \{(\tau, u) \in \mathbb{R} \times X : u = Q(\tau)u + \Sigma^u(\tau, Q(\tau)u)\}, \tag{16a}$$

$$W^s(0) = \{(\tau, u) \in \mathbb{R} \times X : u = \Theta^s(\tau, (I - Q(\tau))u)) + (I - Q(\tau))u\}. \tag{16b}$$

Moreover, solutions within the unstable (resp. stable) manifold exponentially decay to zero backwards (resp. forwards) in time, according to (10) *and* (13)*.*

Proof: For $\ell > 0$ sufficiently small, (6) is satisfied and $\delta > 0$, and we obtain the graph of Σ^* from Theorem 3. We now prove that the unstable set $W^u(0)$ defined in (15a) coincides with the graph of $\Sigma^u := \Sigma^*$. Clearly, from (10), the graph of Σ^u is contained in the unstable set. Now, from (11), for any solution $z(t) = T(t,\tau)z(\tau)$, $t_0 \geqslant t \geqslant \tau$ in the unstable set,

$$\begin{aligned} \|(I-Q(t))T(t,\tau)z(\tau)-\Sigma^u(t,Q(t)T(t,\tau)z(\tau))\|_X \\ \leqslant e^{-\delta(t-\tau)}\|(I-P_{\Sigma^u}(\tau))z(\tau)\|_X,\ t \geqslant \tau. \end{aligned} \tag{17}$$

Since $\delta > 0$, we obtain that $(I - Q(t))z(t) = \Sigma^u(t, Q(t)z(t))$ for all $t \in \mathbb{R}$ as $\tau \to -\infty$, and thus such backward solution in the unstable set lies in the graph of Σ^u. The case of stable manifold is analogous applying Theorem 4.□

Fine Description within Invariant Manifolds

No, we describe a finer growth and decay structure within invariant manifolds, in case of an additional spectral splitting.

This allows for the comparison between two different growth (decay) rates within the unstable (stable) manifold, which dictates the directions solutions approach towards the past (future).

We may apply this result to asymptotically autonomous PDEs extending known results in the autonomous case, see [BF-NA-86, Lemma 2.2] and [A-JDE-86, Lemma 6].

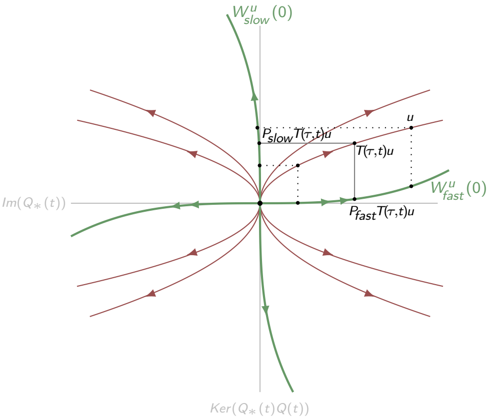


Figure: The local dynamics inside the unstable manifold of $u_* \equiv 0$. Note that solutions in $W^u(0)\backslash W^u_{fast}(0)$ are tangent space of $W^u_{slow}(0)$ as $t \to -\infty$.

Corollary

Suppose that the linear evolution process $\{L(t,\tau) : t \geqslant \tau\}$ *has exponential splitting, with constant* M*, exponents* $\gamma > \rho$*, and projections* $\{Q(t) : t \in \mathbb{R}\}$*. If* $\{L(t,\tau) : t \geqslant \tau\}$ *has another exponential splitting with constant* $M_* \geqslant 1$*, exponents* $\gamma_* > \rho_*$*,* $\rho \geqslant \gamma_*$ *and projections* $\{Q_*(t) : \ t \in \mathbb{R}\}$*, then, for* $\ell > 0$ *is suitably small, then there are graphs corresponding to the fast and slow submanifolds represented by* $W^u_{fast}(0), W^u_{slow}(0)$ *of the invariant manifold represented by* $W^u(0)$ *such that*

$$\lim_{\tau \to -\infty} \frac{\|(I - P_{slow}(\tau))T(\tau,t)u\|}{\|(I - P_{fast}(\tau))T(\tau,t)u\|} = 0, \tag{18}$$

for any $(\tau, u) \in W^u(0) \backslash W^u_{fast}(0)$*, where* $P_{fast}(\cdot)$ *and* $P_{slow}(\cdot)$ *denote the nonlinear projections onto* $W^u_{fast}(0)$ *and* $W^u_{slow}(0)$*.*

Proof: For $(\tau, u_0) \in W^u(0) \setminus W^u_{fast}(0)$, we obtain from (11) that

$$\|(I - P_{fast}(t))T(t,\tau)u_0\| \leqslant Me^{-\delta_*(t-\tau)}\|(I - P_{fast}(\tau))u_0\|, \quad t \geqslant \tau. \quad (19)$$

where $\delta_* = \gamma_* - M\ell - \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma_* - \rho_* - \ell M(1+\kappa)}$. Similarly, from (14) we have

$$\|(I - P_{slow}(\tau))u_0\| \leqslant Me^{\hat{\delta}_*(t-\tau)}\|(I - P_{slow}(t))T(t,\tau)u_0\|, \quad t \geqslant \tau. \quad (20)$$

where $\hat{\delta}_* = \rho_* - M\ell + \frac{M^2\ell^2(1+\kappa)(1+M)}{\gamma_* - \rho_* - M\ell(1+\kappa)}$.

Therefore, the bounds (19) and (20) applied to $u_0 = T(\tau, t)u$ yield

$$\frac{\|(I - P_{slow}(\tau))T(\tau,t)u\|}{\|(I - P_{fast}(\tau))T(\tau,t)u\|} \leqslant M^2 e^{-\hat{\hat{\delta}}_*(t-\tau)} \frac{\|(I - P_{slow}(t))u_0\|}{\|(I - P_{fast}(t))u_0\|}, \quad t \geqslant \tau.$$

where $\hat{\hat{\delta}}_* = \gamma_* - \rho_* - 2M\ell - \frac{2M^2\ell^2(1+\kappa)(1+M)}{\gamma_* - \rho_* - \ell M(1+\kappa)}$. The limit $\tau \to -\infty$ yields the desired claim, since $\hat{\hat{\delta}}_* > 0$, for suitably small $\ell > 0$. □

Robustness of Exponential Dichotomy

Now we obtain the robustness of exponential dichotomy applying Theorems 3 and 4 in a linear setting. Consider the linear equation,

$$\dot{u} = A(t)u + B(t)u, \ \ t \geqslant \tau, \ \ u(\tau) = u_0. \tag{21}$$

with $\mathbb{R} \ni t \mapsto B(t) \in \mathcal{L}(X)$ strongly continuous and $\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \leqslant \ell$.

The evolution process $\{T(t,\tau) : t \geqslant \tau\} \subset \mathcal{L}(X)$ associated to (21) is

$$T(t,\tau) = L(t,\tau) + \int_\tau^t L(t,s)B(s)T(s,\tau)\, ds, \quad t \geqslant \tau. \tag{22}$$

Thus, we can obtain a linear invariant manifold and its stable manifold for (22).

Corollary (Linear)

Suppose that $\{L(t,\tau) : t \geqslant \tau\}$ *has exponential splitting with constant* M*, exponents* $\gamma > \rho$ *and projections* $\{Q(t) : t \in \mathbb{R}\}$.

If (6) *is satisfied, there are* $\Sigma^* \in \mathcal{L}_\Sigma(\kappa)$ *and* $\Theta^* \in \mathcal{L}_\Theta(\kappa)$, $\Sigma^*(t,\cdot), \Theta^*(t,\cdot) \in \mathcal{L}(X)$ *such that*

- $\{P_{\Sigma^*}(t)(X) : t \in \mathbb{R}\}$ *is invariant and* $\{P_{\Theta^*}(t)(X) : t \in \mathbb{R}\}$ *is positively invariant;*
- $\{T(t,\tau) : t \geqslant \tau\}$ *given by* (22) *satisfies and* (11), (14) *and*

$$\begin{aligned} \|T(t,\tau)P_{\Sigma^*}(\tau)\|_{\mathcal{L}(X)} &\leqslant M(1+\kappa)e^{-(\rho+M\ell(1+\kappa))(t-\tau)}, \ t\leqslant\tau, \\ \|T(t,\tau)P_{\Theta^*}(\tau)\|_{\mathcal{L}(X)} &\leqslant M(1+\kappa)e^{-(\gamma-M\ell(1+\kappa))(t-\tau)}, \ t\geqslant\tau, \end{aligned} \tag{23}$$

Proof: The proof follows from Theorem 3 with $f(t,\cdot)$ being a bounded (uniformly with respect to t) linear operator . Since $f(t,\cdot)$ is linear, taking $\Sigma(t,\cdot)$ linear, $G(\Sigma)$ will also be linear and so will be the fixed point. Similarly, Θ^* will also be linear. □

Next, we show the robustness of the exponential dichotomy.

Corollary

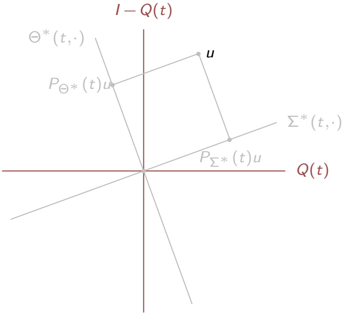
If $\{L(t,\tau) : t \geqslant \tau\}$ has exponential dichotomy with constant $M \geqslant 1$, exponent $\gamma > 0$ and projections $\{Q(t) : t \in \mathbb{R}\}$. and $\sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \leqslant \ell$, where $\ell > 0$ satisfies

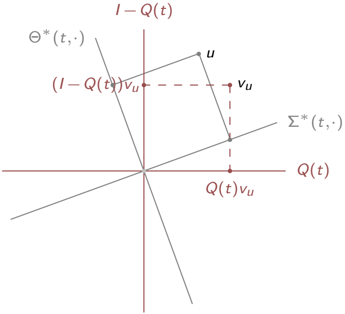
$$\ell < \frac{2\gamma}{3M(M+1)}. \tag{24}$$

Then $\{T(t,\tau): t \geqslant \tau\}$ has exponential dichotomy with constant $M_\ell := M(1+\kappa_\ell)$, exponent $\gamma_\ell := \gamma - \ell M(1+\kappa_\ell)$ and projections $\{Q_\ell(t): t\in\mathbb{R}\}$ with κ_ℓ given in Corollary (Linear). Moreover,

$$\sup_{t\in\mathbb{R}} \|Q(t) - Q_\ell(t)\|_{\mathcal{L}(X)} \leqslant \frac{2\kappa_\ell}{1-2\kappa_\ell}. \tag{25}$$

Proof: We prove that $X = Im(P_{\Sigma^*}(t)) \oplus Im(P_{\Theta^*}(t))$ showing that, for each $u \in X$ there exists a unique $v_u \in X$ such that,





$$\begin{aligned} u &= \overbrace{Q(t)v_u + \Sigma^*(t, v_u)}^{P_{\Sigma^*}(t)v_u} + \overbrace{(I - Q(t))v_u + \Theta^*(t, v_u)}^{P_{\Theta^*}(t)v_u} \\ &= v_u + \Sigma^*(t, v_u) + \Theta^*(t, v_u) = (I + \Sigma^*(t,\cdot) + \Theta^*(t,\cdot))v_u \end{aligned} \tag{26}$$

Clearly, if $\kappa < \frac{1}{2}$, $v_u = (I + \Sigma^*(t,\cdot) + \Theta^*(t,\cdot))^{-1}u$ and

$$\|v_u\|_X \leqslant \|(I + \Sigma^*(t,\cdot) + \Theta^*(t,\cdot))^{-1}u\| \leqslant \frac{\|u\|_X}{1 - 2\kappa}. \tag{27}$$

Define $Q_\ell(t)$ as the projection onto $Im(P_{\Sigma^*}(t))$ along $Im(P_{\Theta^*}(t))$, that is $Q_\ell(t)u := P_{\Sigma^*}(t)v_u$ and $(I - Q_\ell(t))u = P_{\Theta^*}(t)v_u$, $\forall\,(t,u) \in \mathbb{R} \times X$.

Since $\{Im(Q_\ell(t)) : t \in \mathbb{R}\}$ is ivariant $T(t,\tau)Q_\ell(\tau) = Q_\ell(t)T(t,\tau)$, $t \geqslant \tau$.

Equations (23) and (27) imply the desired exponential bounds.

Hence, $\{T(t,\tau) : t \geqslant \tau\}$ has exponential dichotomy with constant $M_\ell := M(1+\kappa)$ and exponent $\gamma_\ell := \gamma - \ell M(1+\kappa) > 0$.

Since $u = v_u + \Sigma^*(t, v_u) + \Theta^*(t, v_u)$, $Q(t)u = Q(t)v_u + \Theta^*(t, v_u)$, from

- $Q(t)\Sigma^*(t, v_u) = 0$ and
- $Q(t)\Theta^*(t, v_u) = \Theta^*(t, v_u)$.

Also, by definition $Q_\ell(t)u = P_{\Sigma^*}(t)v_u = Q(t)v_u + \Sigma^*(t, v_u)$.

Therefore,

$$Q(t)u - Q_\ell(t)u = (\Theta^*(t,\cdot) - \Sigma^*(t,\cdot))((I + \Sigma^*(t,\cdot) + \Theta^*(t,\cdot))^{-1})u.$$

Since Σ^*, Θ^* are bounded with norms in $(0, \frac{1}{2}) \ni \kappa_\ell$, (25) follows. □

Thank you very much for your attention!!

Muito obrigado pela atenção

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