Periodic solutions for systems of impulsive delay differential equations

Teresa Faria

CMAFCIO, Faculty of Sciences

University of Lisbon
teresa.faria@fc.ul.pt

joint work with Rubén Figueroa (Univ. Santiago de Compostela)

8th IST-IME Meeting
Lisbon, September 5-9, 2022

Supported by project UIDB/04561/2020 (FCT).

Contents:

In: TF, R. Figueroa, DCDS-B (2022), doi: 10.3934/dcdsb.2022070

1. Introduction

Our model: A class of periodic systems of DDEs with (finite) **delay** and **impulses**:

$$\begin{cases} x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}(t)x_j(t) + g_i(t, x_{it}) \text{ for } t \neq t_k, \\ \Delta(x_i(t_k)) := x_i(t_k^+) - x_i(t_k) = I_{ik}(x_i(t_k)), \ k \in \mathbb{Z}, \quad i = 1, \dots, n, \end{cases} \tag{1}$$

where: τ is the time-delay,

· $x_t = (x_{1t}, \dots, x_{nt}) = x_{|_{[t-\tau.t]}}$ is the *past history* of the state, defined by

$$x_t(s) = x(t+s) \text{ for } s \in [-\tau, 0]$$

· the solutions $x(t)$ are piecewise continuous, left continuous, with jump discontinuities at t_k ($k \in \mathbb{Z}$) given by $I_{ik}(x_i(t_k))$

· $d_i(t), a_{ij}(t), g_i(t, \varphi)$ continuous, nonnegative and ω-periodic in t ($\omega > 0$),

· the impulses at times t_k occur with periodicity ω

- GOAL: To find sufficient conditions for the existence of (at least) **one positive periodic solution** for the impulsive delay differential equation (IDDE) (1)

• GOAL: To find sufficient conditions for the existence of (at least) **one positive periodic solution** for the impulsive delay differential equation (IDDE) (1)

• Our criteria can be applied to broad families of concrete models:

- discrete/distributed, finite/infinite delay
- systems incorporating general impulses whose signs may vary
- and a very general nonlinearity g

(in general, the nonlinearities g are **non-monotone**)

• GOAL: To find sufficient conditions for the existence of (at least) **one positive periodic solution** for the impulsive delay differential equation (IDDE) (1)

• Our criteria can be applied to broad families of concrete models:

· discrete/distributed, finite/infinite delay

· systems incorporating general impulses whose signs may vary

· and a very general nonlinearity g

(in general, the nonlinearities g are **non-monotone**)

• As usual, our method uses a fixed point argument (Krasnoselskii)

• Our technique is based on the construction of an original operator, whose fixed points are the periodic solutions we are looking for

• **Scalar** periodic impulsive DDEs:

(See e.g. Du and Feng 04; A. Wan et al. 04; D. Jiang et al 04,08; Yan 05,07; X. Li et al. 05; Liu and Takeuchi 07; Chu and Nieto 08; Saker and Alzabut 09; Meng and Yan 15; Zhang and Feng 15, Dai and Bao 16, etc....)

... but

• There are only few results for periodic **systems** of DDEs, almost all for the situation **without impulses**.

Here:

Generalization of results obtained for **scalar** IDDE in Faria & Oliveira, JDDE (2019), Buedo-Fernandez & Faria, MMAS (2020)

Some references for systems:

• Periodic (nonimpulsive) **systems** of DDEs:

⋆ periodic n-dim LV models:
Li, JMAA (2000)
Tang & Zou, PAMS 2006
Benhadri, Caraballo & Zeghdoudi, Opuscula Math. 2020

⋆ periodic n-dim DDEs $x'(t) = f(t, x(t), x(t-\tau))$ $(f \geq 0)$
Amster & Bondorevsky, AMC (2021)

⋆ periodic n-dim Nicholson systems:
Ding & Fu, J Exp Theor Artificial Intel (2020)
TF, JDE (2017)
Huang, Wang & Huang, EJDE (2020)
Troib, FDE (2014)
Wang, Liu & Chen, MMAS (2019)

• **Periodic impulsive systems of DDEs:**

Liu & Gong, Abstr. Appl. Anal. (2013), on neural networks
Zhang, Huang & Wei, Adv. Diff. Equ.(2015), on a 2-dim impulsive Nicholson system

finite delay $\tau > 0$: $I = [-\tau, 0]$

- $PC := PC([-\tau, 0], \mathbb{R}^n)$, space of functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$ which are piecewise continuous (i.e., finite number of jump discontinuities) and left continuous,

- Phase space: Banach space of normalized (from the left) *regulated functions* $R(I; \mathbb{R}^n) := \overline{PC}$ in the space of bounded fcs $B(I; \mathbb{R}^n)$ with

$$\|\varphi\| = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$$

finite delay $\tau > 0$: $I = [-\tau, 0]$

• $PC := PC([-\tau, 0], \mathbb{R}^n)$, space of functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$ which are piecewise continuous (i.e., finite number of jump discontinuities) and left continuous,

• Phase space: Banach space of normalized (from the left) *regulated functions* $R(I; \mathbb{R}^n) := \overline{PC}$ in the space of bounded fcs $B(I; \mathbb{R}^n)$ with

$$\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$$

• **IC** at $t = 0$: $\quad x(s) = \phi(s), \; s \in [-\tau, 0],$
i.e.,

$$x_0 := x_{|_{[-\tau,0]}} = \phi \in PC$$

· $\omega > 0$ is the period
· $(t_k)_{k\in\mathbb{Z}}$ is an "ω-periodic sequence" of given points where the impulses occur, $0 \le t_1 < \cdots < t_p < \omega$, $t_{k+np} = t_k + n\omega, \forall n \in \mathbb{Z}, k = 1, \ldots, p$

· $\omega > 0$ is the period
· $(t_k)_{k\in\mathbb{Z}}$ is an "ω-periodic sequence" of given points where the impulses occur, $0 \le t_1 < \cdots < t_p < \omega$, $t_{k+np} = t_k + n\omega, \forall n \in \mathbb{Z}, k = 1, \ldots, p$

- Define the **Banach** space

$$
\begin{aligned}
X := X(\mathbb{R}^n) = \{x : \mathbb{R} \to \mathbb{R}^n \mid x \text{ is } \omega\text{-periodic, continuous for all } t \neq t_k, \\
\text{there exist } x(t_k^-), x(t_k^+) \text{ and } x(t_k^-) = x(t_k), \text{ for } k \in \mathbb{Z}\} \\
\text{and } \tilde{X} := \{x_t : x \in X, t \in \mathbb{R}\}
\end{aligned}
\tag{2}
$$

- our models are from math biology $X^+ := \{x \in X : x(t) \ge 0, t \in [0, \omega]\}$

· $\omega > 0$ is the period
· $(t_k)_{k\in\mathbb{Z}}$ is an "ω-periodic sequence" of given points where the impulses occur, $0 \le t_1 < \cdots < t_p < \omega$, $t_{k+np} = t_k + n\omega, \forall n \in \mathbb{Z}, k = 1, \ldots, p$

• Define the **Banach** space

$$\begin{aligned} X := X(\mathbb{R}^n) = \{ & x : \mathbb{R} \to \mathbb{R}^n \mid x \text{ is } \omega\text{−periodic, continuous for all } t \ne t_k, \\ & \text{there exist } x(t_k^-), x(t_k^+) \text{ and } x(t_k^-) = x(t_k), \text{ for } k \in \mathbb{Z}\} \\ \text{and } \tilde{X} := \{ & x_t : x \in X, t \in \mathbb{R}\} \end{aligned} \tag{2}$$

• our models are from math biology $X^+ := \{x \in X : x(t) \ge 0, t \in [0,\omega]\}$

• X is endowed with the norm $\|\cdot\|_\infty$, simply denoted by $\|\cdot\|$, and with the partial order $\le$ induced by the **cone** X^+: $y_1 \le y_2$ if $y_2 - y_1 \in X^+$

- $x \in X \implies x_t \in PC$, i.e., $\tilde{X} := \{x_t : x \in X, t \in \mathbb{R}\} \subset PC$

- an isometry:

$$X \ni x \mapsto x_0 = x_{|_{[-\tau,0]}} \in PC \subset R([-\tau, 0]; \mathbb{R}^n)$$

(w.l.g. for $\tau \geq \omega$), so X can be interpreted as a complete subspace of the phase space $R([-\tau, 0]; \mathbb{R}^n)$.

• $x \in X \implies x_t \in PC$, i.e., $\tilde{X} := \{x_t : x \in X, t \in \mathbb{R}\} \subset PC$

• an isometry:

$$X \ni x \mapsto x_0 = x_{|_{[-\tau,0]}} \in PC \subset R([-\tau,0];\mathbb{R}^n)$$

(w.l.g. for $\tau \geq \omega$), so X can be interpreted as a complete subspace of the phase space $R([-\tau,0];\mathbb{R}^n)$.

THUS:

• From now on, for the purpose of finding periodic solutions of an IDDE, we "forget" the phase space $R([-\tau,0];\mathbb{R}^n)$ and work on

$$X \quad \text{with} \quad \|\cdot\| = \|\cdot\|_\infty$$

A remark on infinite memory

HERE: To simplify the exposition, we only consider systems with **finite delay**.

The consideration of models with infinite delay goes back to *Volterra's population models (1920's, 1930's)* (where typically the "memory functions" appear as integral kernels) e.g. in predator-prey models:

$$\dot{x}(t) = x(t)[a - bx(t) - cy(t) - \int_0^\infty k_1(s)x(t-s)ds - \int_0^\infty k_2(s)y(t-s)ds]$$
$$\dot{y}(t) = y(t)[-d + px(t) - qy(t) + \int_0^\infty k_3(s)x(t-s)ds - \int_0^\infty k_4(s)y(t-s)ds]$$

$a, b, c, d, p, q > 0$, $k_i(s) \geq 0$ continuous, $k_i \in L^1[0, \infty)$

(the delay effects diminish gradually when going back in time)

A remark on infinite memory

HERE: To simplify the exposition, we only consider systems with **finite delay**.

The consideration of models with infinite delay goes back to *Volterra's population models (1920's, 1930's)* (where typically the "memory functions" appear as integral kernels) e.g. in predator-prey models:

$$\dot{x}(t) = x(t)[a - bx(t) - cy(t) - \int_0^\infty k_1(s)x(t-s)ds - \int_0^\infty k_2(s)y(t-s)ds]$$
$$\dot{y}(t) = y(t)[-d + px(t) - qy(t) + \int_0^\infty k_3(s)x(t-s)ds - \int_0^\infty k_4(s)y(t-s)ds]$$

$a, b, c, d, p, q > 0$, $k_i(s) \geq 0$ continuous, $k_i \in L^1[0, \infty)$

(the delay effects diminish gradually when going back in time)

IC at $t = 0$: $(x(s), y(s)) = \phi(s), s \leq 0,$ i.e., $(x, y)|_{(-\infty,0]} = \phi \in \mathcal{B} \subset C((-\infty, 0]; \mathbb{R}^2)$

The treatment of infinite delay requires a careful choice of an *admissible* phase space, which must satisfy some axiomatic (Hale & Kato, Funkcial. Ekvac.'78)

impulses, infinite delay: $I=(-\infty,0]$:

• $PC(I;\mathbb{R}^n)$: the space of functions $\varphi:(-\infty,0]\to\mathbb{R}^n$ whose restrictions $\varphi_{|_{[-\tau,0]}}$ to any interval $[-\tau,0]$ are piecewise continuous and left continuous

However: $PC(I;\mathbb{R}^n)$ is not a good space; moreover, $PC(I;\mathbb{R}^n)\not\subset\; B(I;\mathbb{R}^n)$, for $B(I;\mathbb{R}^n)$ the space of bounded fcs.

• With the identification $x\equiv x_0=x_{|_{(-\infty,0]}}$, the space X of ω-periodic functions with jump discontinuities at (t_k) is also seen as a (closed) subspace of an appropriate phase space $\mathcal{B}\subset PC$,

$$X\subset\mathcal{B},\quad \|x\|_\infty\sim\|x_t\|_{\mathcal{B}}\quad \forall x\in X$$

impulses, infinite delay: $I = (-\infty, 0]$:

• $PC(I; \mathbb{R}^n)$: the space of functions $\varphi : (-\infty, 0] \to \mathbb{R}^n$ whose restrictions $\varphi_{|_{[-\tau,0]}}$ to any interval $[-\tau, 0]$ are piecewise continuous and left continuous

However: $PC(I; \mathbb{R}^n)$ is not a good space; moreover, $PC(I; \mathbb{R}^n) \not\subset B(I; \mathbb{R}^n)$, for $B(I; \mathbb{R}^n)$ the space of bounded fcs.

• With the identification $x \equiv x_0 = x_{|_{(-\infty,0]}}$, the space X of ω-periodic functions with jump discontinuities at (t_k) is also seen as a (closed) subspace of an appropriate phase space $\mathcal{B} \subset PC$,

$$X \subset \mathcal{B}, \quad \|x\|_\infty \sim \|x_t\|_{\mathcal{B}} \quad \forall x \in X$$

—

HERE: To simplify the exposition, we only consider systems with **finite delay**.

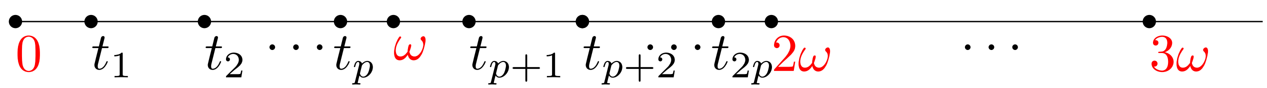
(See e.g. Buedo-Fernandez & Faria, MMAS 2020, for scalar IDDE with ∞ delay...)

2. Preliminary results

$$\begin{cases} x_i'(t) = -d_i(t)x_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}(t)x_j(t) + g_i(t, x_{it}) \text{ for } t \neq t_k, \\ \Delta(x_i(t_k)) := x_i(t_k^+) - x_i(t_k) = I_{ik}(x_i(t_k)), \ k \in \mathbb{Z}, \quad i = 1, \ldots, n \end{cases} \tag{1}$$

(H1) $I_{ik} : \mathbb{R}^+ \to \mathbb{R}$ are continuous and $\exists\, p \in \mathbb{N}$ such that $0 \leq t_1 < \cdots < t_p < \omega$ (for some $\omega > 0$) and

$$t_{k+p} = t_k + \omega, \quad I_{i,k+p} = I_{ik}, \quad k \in \mathbb{Z}, i = 1, \ldots, n$$



(H2) There exist $\alpha_{ik} > -1$ and η_{ik} such that

$$\alpha_{ik} u \leq I_{ik}(u) \leq \eta_{ik} u, \quad u \geq 0, \ k \in \{1, \ldots, p\}$$

and, if $\boxed{n > 1}$ there exist $\lim_{u \to 0^+} \dfrac{u}{u + I_{ik}(u)}$, $i = 1, \ldots, n, k = 1, \ldots, p$;

(H3) $\prod_{k=1}^{p}(1 + \eta_{ik}) < e^{\int_0^\omega d_i(t)\, dt}$, $i = 1, \ldots, n$

(H4) (i) the functions $d_i, a_{ij}\ g_i : \mathbb{R} \times \tilde{X}(\mathbb{R}) \to \mathbb{R}^+$ are **continuous,** ω-**periodic** in $t \in \mathbb{R}$ ($\forall i, j$), with $\int_0^\omega d_i(s)\, ds > 0$,

$$g(t, x_t) := (g_1(t, x_{1t}), \dots, g_n(t, x_{nt}))$$

is bounded on bounded sets of $\mathbb{R} \times \tilde{X}$;

(ii) if $\boxed{n > 1}$ either $\int_0^\omega a_{ij}(s)\, ds > 0$ for all $i \neq j$ or $\int_0^\omega g_i(s, 0)\, ds > 0$, for each $i = 1, \dots, n$.

(H5) The function

$$G(t, x) := g(t, x_t) \quad \text{for} \quad t \in \mathbb{R}, x \in X^+$$

is **uniformly equicontinuous** for $t \in [0, \omega]$ on bounded sets of X^+, i.e.,

$\forall A \subset X^+$ bounded and $\forall \varepsilon > 0$, $\exists \delta > 0$:

$\max_{t \in [0,\omega]} \|G(t, x) - G(t, y)\| < \varepsilon$ for all $x, y \in A$ with $\|x - y\| < \delta$.

Remarks about the hypotheses

⋆ Conditions (H1) and (H4)(i) give the ω-periodicity of system (1).

⋆ The situation without impulses is included in our setting: $\boxed{I_{ik} \equiv 0}$ for all k

⋆ Here, the impulses are allowed to be **negative or to change signs**!

⋆ (H2) guarantees that, at the impulsive points t_k, solutions of (1) with $x(t_k^-) = x(t_k) > 0$ must satisfy

$$x_i(t_k^+) = x_i(t_k) + I_{ik}(x_i(t_k)) \geq (1 + \alpha_{ik})x_i(t_k) > 0, k \in \mathbb{N}.$$

$\star$ Conditions (H1) and (H4)(i) give the ω-periodicity of system (1).

$\star$ The situation without impulses is included in our setting: $\boxed{I_{ik} \equiv 0}$ for all k

$\star$ Here, the impulses are allowed to be **negative or to change signs**!

$\star$ (H2) guarantees that, at the impulsive points t_k, solutions of (1) with $x(t_k^-) = x(t_k) > 0$ must satisfy

$$x_i(t_k^+) = x_i(t_k) + I_{ik}(x_i(t_k)) \geq (1 + \alpha_{ik})x_i(t_k) > 0, k \in \mathbb{N}.$$

$\star$ Assumptions (H2) and (H3) are **significantly weaker** than the ones usually considered in the literature for the **scalar** case ($n = 1$), where:

· the impulses are nonnegative & additional restrictions on $\frac{I_k(u)}{u}$ close to 0 and ∞ or
· the impulses are **linear** $I_k(u) = \alpha_k u$, with $\prod_{k=1}^{p}(1 + \alpha_k) = 1$

[*Simple example:* If $\alpha_k \equiv \alpha\, \forall k$, the latter condition is satisfied only if $\alpha = 0$ (no impulses!), whereas our setting only requires $-1 < \alpha < e^{\frac{1}{p}\int_0^\omega a(t)\,dt} - 1$.]

Remarks about the hypotheses (cont.)

⋆ For $\boxed{n > 1}$: for $a_{ij}, g_i(\cdot, 0) \in C^+_\omega(\mathbb{R})$, we have imposed

(H4)(ii): $a_{ij} \not\equiv 0 \ \forall j \neq i$, or $g_i(\cdot, 0) \not\equiv 0$

(note that $a_{ij} \not\equiv 0$ iff $\int_0^\omega a_{ij}(s)\, ds > 0 \ (j \neq i)$ and $g_i(\cdot, 0) \not\equiv 0$ iff $\int_0^\omega g_i(s, 0)\, ds > 0$)

The role of (H4)(ii) is to preclude the existence of periodic solutions with **one component positive but with others that may vanish**.

⋆ For $\boxed{n > 1}$: for $a_{ij}, g_i(\cdot, 0) \in C^+_\omega(\mathbb{R})$, we have imposed

(H4)(ii): $a_{ij} \not\equiv 0 \ \forall j \neq i$, or $g_i(\cdot, 0) \not\equiv 0$

(note that $a_{ij} \not\equiv 0$ iff $\int_0^\omega a_{ij}(s)\, ds > 0$ $(j \neq i)$ and $g_i(\cdot, 0) \not\equiv 0$ iff $\int_0^\omega g_i(s, 0)\, ds > 0$)

The role of (H4)(ii) is to preclude the existence of periodic solutions with **one component positive but with others that may vanish**.

⋆ The role of (H5) is to guarantee that the operator Φ defined below is **compact**.

- X and X^+, with $\|\cdot\| = \|\cdot\|_\infty$

Define ($i = 1, \ldots, n, k \in \mathbb{Z}$):

$$D_i(t) = \int_0^t d_i(s)\, ds, \quad J_{ik}(u) = \begin{cases} \dfrac{u}{u + I_{ik}(u)}, & u > 0, \\ \lim\limits_{u \to 0^+} \dfrac{u}{u + I_{ik}(u)}, & u = 0, \end{cases}$$

$$B_i(t; x_i) = \prod_{k: t_k \in [0,t)} J_{ik}(x_i(t_k)) \quad \text{and}$$

$$\tilde{B}_i(s, t; x_i) = \frac{B_i(s; x_i)}{B_i(t; x_i)} = \prod_{k: t_k \in [t,s)} J_{ik}(x_i(t_k)) \text{ for } 0 \le t \le s \le t + \omega, x \in X^+;$$

$$\text{for } t = \omega: \qquad D_i(\omega) = \int_0^\omega d_i(s)\, ds,$$

$$\Gamma_i(x_i) = \left(B_i(\omega; x_i) e^{D_i(\omega)} - 1\right)^{-1} \quad \text{for} \quad i = 1, \ldots, n, x \in X^+.$$

Recall: (H3) $\prod_{k=1}^{p} (1 + \eta_{ik}) < e^{\int_0^\omega d_i(t)\, dt} \Rightarrow B_i(\omega; x_i) e^{D_i(\omega)} > 1, \forall x \in X^+$, so Γ_i are well-defined.

Properties of these auxiliary functions:

Assume (H1)–(H4). For $i = 1, \dots, n, k \in \mathbb{Z}, x = (x_1, \dots, x_n) \in X^+$:

- J_{ik} are continuous and <u>bounded</u>: $J_{ik}(u) = \frac{1}{1+\frac{I_{ik}(u)}{u}}$ with $\frac{I_k(u)}{u} \in [\alpha_{ik}, \eta_{ik}] \implies$

$$\boxed{(1+\eta_{ik})^{-1} \le J_{ik}(u) \le (1+\alpha_{ik})^{-1}, \quad u \ge 0}$$

- $\Gamma_i : X^+(\mathbb{R}) \to (0, \infty)$ are continuous and <u>bounded</u>: $0 < \underline{\Gamma_i} \le \Gamma_i(x_i) \le \overline{\Gamma_i}$
where $\underline{\Gamma_i} := \left(\prod_{k=1}^p (1+\alpha_{ik})^{-1} e^{D_i(\omega)} - 1\right)^{-1}, \overline{\Gamma_i} := \left(\prod_{k=1}^p (1+\eta_{ik})^{-1} e^{D_i(\omega)} - 1\right)^{-1}$

- $\tilde{B}_i(s,t;x_i)$ are <u>bounded</u>:[1] $0 < \underline{B_i} \le \tilde{B}_i(s,t;x_i) \le \overline{B_i}$

- $\tilde{B}_i(s+\omega, t+\omega; x_i) = \tilde{B}_i(s,t;x_i)$ for $t \le s \le t+\omega$ and $\varphi \in X^+(\mathbb{R})$

[1] Recall that there is a finite number of impulses on each interval of length $\le \omega$.
NOTE THAT, with **linear** impulses $I_{ik}(u) = \eta_{ik} u$,
$J_{ik} \equiv (1+\eta_{ik})^{-1}$ (constants) and $B_i(t;x) \equiv B_i(t) = \prod_{k:t_k \in [0,t)} (1+\eta_{ik})^{-1}$.

- a new cone $K = K(\sigma) := \{x \in X^+ : x_i(t) \geq \sigma_i \|x_i\|, t \in [0, \omega], i = 1, \dots, n\}$, with $\sigma \in (0,1)^n$
- an operator $\Phi = (\Phi_1, \dots, \Phi_n) : X^+ \to X^+$,

$$(\Phi_i x)(t) = \Gamma_i(x_i) \int_t^{t+\omega} \tilde{B}_i(s,t;x_i) e^{\int_t^s d_i(r)\,dr} \left(\sum_{j \neq i} a_{ij}(s) x_j(s) + g_i(s, x_{is}) \right) ds$$

for $x = (x_1, \dots, x_n) \in X^+, t \geq 0$

- a new cone
$K = K(\sigma) := \{x \in X^+ : x_i(t) \geq \sigma_i \|x_i\|, t \in [0, \omega], i = 1, \ldots, n\}$, with $\sigma \in (0,1)^n$

- an operator $\Phi = (\Phi_1, \ldots, \Phi_n) : X^+ \to X^+$,

$$(\Phi_i x)(t) = \Gamma_i(x_i) \int_t^{t+\omega} \tilde{B}_i(s,t;x_i) e^{\int_t^s d_i(r)\, dr} \left(\sum_{j \neq i} a_{ij}(s) x_j(s) + g_i(s, x_{is}) \right) ds$$

for $x = (x_1, \ldots, x_n) \in X^+, t \geq 0$

LEMMA 1. Assume (H1)–(H4), take $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $0 < \sigma_i \leq \underline{B_i}\,\overline{B_i}^{-1} e^{-D_i(\omega)}$ for $i = 1, \ldots, n$, and $K = K(\sigma)$. THEN:

(i) $\Phi(K) \subset K$.
(ii) If $x \in K \setminus \{0\}$, x is a **fixed point** of Φ iff x is a **positive** ω-periodic solution of (1).
(iii) If in addition (H5) holds, Φ is completely continuous on $K \setminus \{0\}$

Remark:

• If $x(t) = (x_1(t), \ldots, x_n(t))$ is a solution of (1), the function $y(t) = (y_1(t), \ldots, y_n(t))$, where $y_i(t) = B_i(t; x_i)x_i(t), \quad i = 1, \ldots, n,$ is **continuous**, because

$$J_{ik}(x_i(t_k)) = \frac{x_i(t_k)}{x_i(t_k^+)}$$

• Rather than using **sums** of the impulses, the key idea is to account for the impulses in a **multiplicative** mode by means of the products of the functions $J_{ik}(u)$:

in this way, $B_i(t; x_i) = \prod_{k: t_k \in [0,t)} J_{ik}(x_i(t_k))$ are used to "glue" the pieces of the solution's graph at impulse times t_k, so that it becomes continuous.

• If $x(t) = (x_1(t), \ldots, x_n(t))$ is a solution of (1), the function $y(t) = (y_1(t), \ldots, y_n(t))$, where $y_i(t) = B_i(t; x_i)x_i(t), \quad i = 1, \ldots, n,$ is **continuous**, because

$$J_{ik}(x_i(t_k)) = \frac{x_i(t_k)}{x_i(t_k^+)}$$

• Rather than using **sums** of the impulses, the key idea is to account for the impulses in a **multiplicative** mode by means of the products of the functions $J_{ik}(u)$:

in this way, $B_i(t; x_i) = \prod_{k:t_k \in [0,t)} J_{ik}(x_i(t_k))$ are used to "glue" the pieces of the solution's graph at impulse times t_k, so that it becomes continuous.

• Instead of Φ, the following operator has been considered for **scalar** IDDEs:

$$(\Psi y)(t) = (e^{D(\omega)} - 1)^{-1} \left[\int_t^{t+\omega} g(s, y_s) e^{\int_t^s d(u)\, du} \, ds + \sum_{k:t_k \in [t, t+\omega)} I_k(y(t_k)) e^{\int_t^{t_k} d(u)\, du} \right]$$

(the impulses multiplied by the Green function $G(t,s) = \frac{e^{\int_t^s d(u)\, du}}{e^{D(\omega)} - 1}$ are *summed up* to time t).

Krasnoselskii Theorem:

Let X be a Banach space, K a cone in $X r, R \in \mathbb{R}^+$ with $r \neq R$ and $A_{r_0,R_0} := \{x \in K : r_0 \leq \|x\| \leq R_0\}$, where $r_0 = \min\{r, R\}, R_0 = \max\{r, R\}$. Let $T : A_{r_0,R_0} \longrightarrow K$ be a completely continuous operator such that

(i) $Tx \neq \lambda x$ for all $x \in K$ with $\|x\| = R$ and $\lambda > 1$;

(ii) There exists $\psi \in K \setminus \{0\}$ such that $x \neq Tx + \lambda\psi$ for all $x \in K$ with $\|x\| = r$ and all $\lambda > 0$.

Then T has a fixed point $x \in A_{r_0,R_0}$ which moreover satisfies $r_0 < \|x\| < R_0$.

Let X be a Banach space, K a cone in $X r, R \in \mathbb{R}^+$ with $r \neq R$ and
$A_{r_0,R_0} := \{x \in K : r_0 \leq \|x\| \leq R_0\}$, where $r_0 = \min\{r, R\}, R_0 = \max\{r, R\}$.
Let $T : A_{r_0,R_0} \longrightarrow K$ be a completely continuous operator such that

(i) $Tx \neq \lambda x$ for all $x \in K$ with $\|x\| = R$ and $\lambda > 1$;

(ii) There exists $\psi \in K \setminus \{0\}$ such that $x \neq Tx + \lambda\psi$ for all $x \in K$ with $\|x\| = r$ and all $\lambda > 0$.

Then T has a fixed point $x \in A_{r_0,R_0}$ which moreover satisfies $r_0 < \|x\| < R_0$.

Next: Under some additional conditions,

one shows that $\exists\ 0 < r < R$ such that Φ satisfies (i),(ii) [2]

$\implies$ there is an ω-periodic solution $x^* > 0$ in a conical sector A_{r_0,R_0} of K

[2] Combination of both the compressive ($r > R$) and expansive ($r < R$) forms can lead to the existence of more than one positive period solution to (1).

3. Main results

(H6) There are constants r_0, R_0 with $0 < r_0 < R_0$ and functions $b_{1i}, b_{2i} \in C^+_\omega(\mathbb{R})$ with $\int_0^\omega b_{qi}(t)\,dt > 0\,(q = 1, 2)$, such that for $i = 1, \dots, n$, $x \in K$ and $t \in [0, \omega]$ it holds:

$$g_i(t, x_{it}) \geq b_{1i}(t)u \quad \text{if } 0 < u \leq x_i \leq r_0,$$
$$g_i(t, x_{it}) \leq b_{2i}(t)u \quad \text{if } R_0 \leq x_i \leq u.$$

3. Main results

(H6) There are constants r_0, R_0 with $0 < r_0 < R_0$ and functions $b_{1i}, b_{2i} \in C_\omega^+(\mathbb{R})$ with $\int_0^\omega b_{qi}(t)\,dt > 0\,(q = 1, 2)$, such that for $i = 1, \ldots, n$, $x \in K$ and $t \in [0, \omega]$ it holds:

$$g_i(t, x_{it}) \geq b_{1i}(t)u \quad \text{if } 0 < u \leq x_i \leq r_0,$$
$$g_i(t, x_{it}) \leq b_{2i}(t)u \quad \text{if } R_0 \leq x_i \leq u.$$

Theorem 1. Assume (H1)–(H6) and that, for b_{1i}, b_{2i} as in (H6),

$$c_i^0 := \underline{\Gamma_i}\,\underline{B_i} \min_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big(\sum_{j\neq i} a_{ij}(s) + b_{1i}(s)\Big)\,ds \geq 1,$$
$$C_i^\infty := \overline{\Gamma_i}\,\overline{B_i} \max_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big(\sum_{j\neq i} a_{ij}(s) + b_{2i}(s)\Big)\,ds \leq 1, i = 1, \ldots, n.$$

THEN there exists (at least) one positive ω-periodic solution $x^*(t)$ of (1)
(Moreover, $x^*(t) \in K$, for $\sigma_i = B_i \overline{B_i}^{-1} e^{-D_i(\omega)}$ $(1 \leq i \leq n)$ as in LEMMA 1.)

(i) $\Phi x \neq \lambda x$ for all $x \in K$ with $\|x\| = R$ and $\lambda > 1$:
⋆ Fix r_0, R_0 as in (H6), let $R \geq R_0(\min_{1\leq i\leq n} \sigma_i)^{-1}$ and $x \in K$ with $\|x\| = R$. Choose i such that $\|x\| = \|x_i\| = R$.
⋆ $x_i(t) \leq R$ and $x_i(t) \geq \sigma_i \|x_i\| = \sigma_i R \geq R_0$ for $t \in [0, \omega]$, thus, from the 2nd inequality in (H6),

$$g_i(t, x_{it}) \leq b_{2i}(t)R.$$

Using the properties in Lemma 1 and $C_i^\infty \leq 1$, we have

$$\|\Phi_i x\| \leq R\,\overline{\Gamma_i}\,\overline{B_i} \max_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big[\sum_{j\neq i} a_{ij}(s) + b_{2i}(s))\Big]\,ds = RC_i^\infty \leq R.$$

In particular, we conclude that $\Phi x \neq \lambda x$ for all $\lambda > 1$ and $x \in K$ with $\|x\| = R$.

(i) $\Phi x \neq \lambda x$ for all $x \in K$ with $\|x\| = R$ and $\lambda > 1$:

⋆ Fix r_0, R_0 as in (H6), let $R \geq R_0(\min_{1\leq i\leq n} \sigma_i)^{-1}$ and $x \in K$ with $\|x\| = R$. Choose i such that $\|x\| = \|x_i\| = R$.

⋆ $x_i(t) \leq R$ and $x_i(t) \geq \sigma_i \|x_i\| = \sigma_i R \geq R_0$ for $t \in [0, \omega]$, thus, from the 2nd inequality in (H6),

$$g_i(t, x_{it}) \leq b_{2i}(t)R.$$

Using the properties in Lemma 1 and $C_i^\infty \leq 1$, we have

$$\|\Phi_i x\| \leq R\,\overline{\Gamma_i}\,\overline{B_i} \max_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big[\sum_{j\neq i} a_{ij}(s) + b_{2i}(s)) \Big] \, ds = RC_i^\infty \leq R.$$

In particular, we conclude that $\Phi x \neq \lambda x$ for all $\lambda > 1$ and $x \in K$ with $\|x\| = R$.

(ii) $\exists \psi \in K \setminus \{0\}$ such that $x \neq \Phi x + \lambda \psi$ for all $x \in K$ with $\|x\| = r$ and all $\lambda > 0$:

⋆ Take $r \leq \min_{1\leq i\leq n} \sigma_i r_0$, $\psi \equiv \mathbf{1} := (1, \ldots, 1)$ and consider any $\lambda > 0$. For $x \in K$ with $\|x\| = r$, we claim that $x \neq \Phi x + \lambda \psi$.

⋆ Suppose otherwise that there are $\lambda > 0, x \in K$ with $\|x\| = r$ and $x = \Phi x + \lambda \mathbf{1}$.

⋆ Let $\mu := \min_{t\in[0,\omega]} \min_{1\le i\le n} x_i(t)$. We have $0 < \lambda \le \mu \le x_i(t) \le r \le r_0$, thus the 1st inequality in (H6) implies

$$g_i(t, x_{it}) \ge b_{1i}(t)\mu,$$

which, together with the constraint $c_i^0 \ge 1$, yields

$$(\Phi_i x)(t) \ge \mu\, \underline{\Gamma_i}\, \underline{B_i} \min_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big[\sum_{j\ne i} a_{ij}(s) + b_{1i}(s)\Big]\, ds = \mu c_i^0 \ge \mu.$$

⋆ Next, choose $t^* \in [0,\omega]$ and $i^* \in \{1,\dots,n\}$ such that $x_{i^*}(t^*) < \mu + \lambda$.
From $\boxed{x = \Phi x + \lambda \mathbf{1}}$,

$$\mu > x_{i^*}(t^*) - \lambda = (\Phi_{i^*} x)(t^*) \ge \mu,$$

a contradiction.

⋆ Let $\mu := \min_{t\in[0,\omega]} \min_{1\le i\le n} x_i(t)$. We have $0 < \lambda \le \mu \le x_i(t) \le r \le r_0$, thus the 1st inequality in (H6) implies

$$g_i(t, x_{it}) \ge b_{1i}(t)\mu,$$

which, together with the constraint $c_i^0 \ge 1$, yields

$$(\Phi_i x)(t) \ge \mu\, \underline{\Gamma_i}\, \underline{B_i} \min_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big[\sum_{j\ne i} a_{ij}(s) + b_{1i}(s)\Big]\, ds = \mu c_i^0 \ge \mu.$$

⋆ Next, choose $t^* \in [0,\omega]$ and $i^* \in \{1,\dots,n\}$ such that $x_{i^*}(t^*) < \mu + \lambda$. From $\boxed{x = \Phi x + \lambda \mathbf{1}}$,

$$\mu > x_{i^*}(t^*) - \lambda = (\Phi_{i^*} x)(t^*) \ge \mu,$$

a contradiction.

(i),(ii) are proven, thus Krasnoselskii Theorem gives the existence of a fixed point x^* for Φ in $K_{r,R} = \{x \in K : r \le \|x\| \le R\}$, i.e., a positive ω-periodic solution of (1).

Theorem 1$^+$. Assume (H1)–(H6) and that there is $v = (v_1, \dots, v_n) > 0$ such that, for b_{1i}, b_{2i} as in (H6),

$$c_i^0(v) := \underline{\Gamma}_i \, \underline{B}_i \min_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big(\sum_{j\neq i} v_i^{-1} v_j a_{ij}(s) + b_{1i}(s) \Big) \, ds \geq 1,$$

$$C_i^\infty(v) := \overline{\Gamma}_i \, \overline{B}_i \max_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big(\sum_{j\neq i} v_i^{-1} v_j a_{ij}(s) + b_{2i}(s) \Big) \, ds \leq 1,$$

$$i = 1, \dots, n.$$

THEN there exists (at least) one positive ω-periodic solution $x^*(t)$ of (1).

"superlinear case":

Theorem 2. Assume (H1)–(H5) and

(H6*) There are constants r_0, R_0 with $0 < r_0 < R_0$ and functions $b_{1i}, b_{2i} \in C^+_\omega(\mathbb{R})$ with $\int_0^\omega b_{qi}(t)\,dt > 0\,(q = 1, 2)$, such that for $i = 1, \dots, n$, $x \in K$ and $t \in [0, \omega]$ it holds:

$$g_i(t, x_{it}) \le b_{1i}(t)u \quad \text{if } 0 < x_i \le u \le r_0,$$
$$g_i(t, x_{it}) \ge b_{2i}(t)u \quad \text{if } x_i \ge u \ge R_0.$$

If there is a vector $v = (v_1, \dots, v_n) > 0$ such that for $i = 1, \dots, n$

$$C_i^0(v) := \overline{\Gamma_i}\,\overline{B_i} \max_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big(\sum_{j\neq i} v_i^{-1} v_j a_{ij}(s) + b_{1i}(s) \Big)\, ds \le 1,$$

$$c_i^\infty(v) := \underline{\Gamma_i}\,\underline{B_i} \min_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} \Big(\sum_{j\neq i} v_i^{-1} v_j a_{ij}(s) + b_{2i}(s) \Big)\, ds \ge 1.$$

THEN there exists (at least) one positive ω-periodic solution $x^*(t)$ of (1).

- For (1), define the $n \times n$ matrices of functions in $C_\omega^+(\mathbb{R})$ given by

$$D(t) = \operatorname{diag}\left(d_1(t), \ldots, d_n(t)\right), \quad A(t) = \left[a_{ij}(t)\right], \tag{3}$$

(with $a_{ii}(t) := 0 \, \forall i$)

- Assume (H1)–(H6).

- For $b_{1i}(t), b_{2i}(t)$ as in (H6), define

$$B_1(t) = \operatorname{diag}\left(b_{11}(t), \ldots, b_{1n}(t)\right), \quad B_2(t) = \operatorname{diag}\left(b_{21}(t), \ldots, b_{2n}(t)\right)$$

Corollary 1. Existence of a positive ω-periodic solution of (1) IF

- $\exists\, v > 0$:

$$M_2\Big[B_2(t) + A(t)\Big]v \le D(t)v \le M_1\Big[B_1(t) + A(t)\Big]v,\ \ t \in [0,\omega] \qquad \text{(H7)}$$

i.e., $$m_{2i}\Big(\sum_{j\neq i} v_j a_{ij}(t) + v_i b_{2i}(t)\Big) \le v_i d_i(t) \le m_{1i}\Big(\sum_{j\neq i} v_j a_{ij}(t) + v_i b_{1i}(t)\Big),\ \forall i, t \in [0,\omega]$$

$$\text{where}\quad M_1 = \mathrm{diag}(m_{11},\dots,m_{1n}),\ M_2 = \mathrm{diag}(m_{21},\dots,m_{2n}),$$
$$m_{1i} := \underline{\Gamma_i}\,\underline{B_i}(e^{D_i(\omega)} - 1),\ m_{2i} := \overline{\Gamma_i}\,\overline{B_i}(e^{D_i(\omega)} - 1),\ \ i = 1,\dots,n; \qquad (4)$$

Proof. From (H7),

$$\sum_{j\neq i} v_j a_{ij}(s) + v_i b_{1i}(s) \ge m_{1i}^{-1} v_i d_i(s), \quad \sum_{j\neq i} v_j a_{ij}(s) + v_i b_{2i}(s) \le m_{2i}^{-1} v_i d_i(s) \implies$$

$$c_i^0(v) \ge m_{1i}^{-1}\underline{\Gamma_i}\,\underline{B_i} \min_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} d_i(s)\,ds = m_{1i}^{-1}\underline{\Gamma_i}\,\underline{B_i}(e^{D_i(\omega)} - 1) = 1,$$

$$C_i^\infty(v) \le m_{2i}^{-1}\overline{\Gamma_i}\,\overline{B_i} \max_{t\in[0,\omega]} \int_t^{t+\omega} e^{\int_t^s d_i(r)\,dr} d_i(s)\,ds = m_{2i}^{-1}\overline{\Gamma_i}\,\overline{B_i}(e^{D_i(\omega)} - 1) = 1.$$

Corollary 2. Existence of a positive ω-periodic solution of (1) IF

- $\exists\, v > 0$:

$$\int_0^\omega N_2\Big[B_2(t) + A(t)\Big] v\, dt \le v \le \int_0^\omega N_1\Big[B_1(t) + A(t)\Big] v\, dt, \qquad \text{(H8)}$$

i.e., $n_{2i} \int_0^\omega \Big(\sum_{j\neq i} v_j a_{ij}(s) + v_i b_{1i}(s)\Big)\, ds \le v_i \le n_{1i} \int_0^\omega \Big(\sum_{j\neq i} v_j a_{ij}(s) + v_i b_{2i}(s)\Big)\, ds \ \forall i$

for

$$N_1 = \operatorname{diag}(n_{11}, \ldots, n_{1n}),\ N_2 = (n_{21}, \ldots, n_{2n}), \qquad (5)$$
$$n_{1i} := \underline{\Gamma_i}\, \underline{B_i},\ n_{2i} := \overline{\Gamma_i}\, \overline{B_i} e^{D_i(\omega)},\ \ i = 1, \ldots, n.$$

(*Proof.* Trivial)

Recall:

$\underline{\Gamma_i} := \Big(\prod_{k=1}^p (1+\alpha_{ik})^{-1} e^{D_i(\omega)} - 1\Big)^{-1}, \overline{\Gamma_i} := \Big(\prod_{k=1}^p (1+\eta_{ik})^{-1} e^{D_i(\omega)} - 1\Big)^{-1}$

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j \neq i} a_{ij}(t)x_j(t) + g_i(t, x_{it}), \quad i = 1, \ldots, n, \qquad (6)$$

⋆ $\Gamma_i(u) \equiv \left(e^{D_i(\omega)} - 1\right)^{-1}$ and $B_i(t, u) \equiv 1$

⋆ $M_1 = M_2 = I$ and

$N_1 = diag\left(e^{D_1(\omega)} - 1, \ldots, e^{D_n(\omega)} - 1\right)^{-1}, N_2 = diag\left(1 - e^{-D_1(\omega)}, \ldots, 1 - e^{-D_n(\omega)}\right)^{-1}$

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j\neq i} a_{ij}(t)x_j(t) + g_i(t, x_{it}), \quad i = 1, \ldots, n, \qquad (6)$$

⋆ $\Gamma_i(u) \equiv \left(e^{D_i(\omega)} - 1\right)^{-1}$ and $B_i(t, u) \equiv 1$
⋆ $M_1 = M_2 = I$ and
$N_1 = diag\left(e^{D_1(\omega)} - 1, \ldots, e^{D_n(\omega)} - 1\right)^{-1}, N_2 = diag\left(1 - e^{-D_1(\omega)}, \ldots, 1 - e^{-D_n(\omega)}\right)^{-1}$

Corollaries 1'&2'. Assume (H4)–(H6). For the matrices in $D(t), A(t), B_1(t), B_2(t)$ as above, suppose that for some $v > 0$:

(a) either $B_2(t)v \le \left[D(t) - A(t)\right]v \le B_1(t)v$ for $t \in [0, \omega]$;

(b) or
$$\begin{cases} \int_0^\omega \left[B_2(t) + A(t)\right] v\, dt \le \mathrm{diag}\left(1 - e^{-D_1(\omega)}, \ldots, 1 - e^{-D_n(\omega)}\right) v \\ \int_0^\omega \left[B_1(t) + A(t)\right] v\, dt \ge \mathrm{diag}\left(e^{D_1(\omega)} - 1, \ldots, e^{D_n(\omega)} - 1\right) v. \end{cases}$$

Then, there exists a positive ω-periodic solution of (6).

4. Applications

Example 1. A *periodic* Nicholson system with distributed delays:

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j\neq i} a_{ij}(t)x_j(t)$$
$$+ \underbrace{\sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) x_i(s) e^{-c_{il}(s)x_i(s)}\, ds}_{g_i(t,x_{it})}, \quad i = 1, \dots, n, \qquad \text{(N)}$$

Example 1. A *periodic* Nicholson system with distributed delays:

$$x_i'(t) = -d_i(t)x_i(t) + \sum_{j\neq i} a_{ij}(t)x_j(t)$$
$$+ \underbrace{\sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) x_i(s) e^{-c_{il}(s)x_i(s)}\, ds}_{g_i(t,x_{it})}, \quad i = 1, \dots, n, \qquad \text{(N)}$$

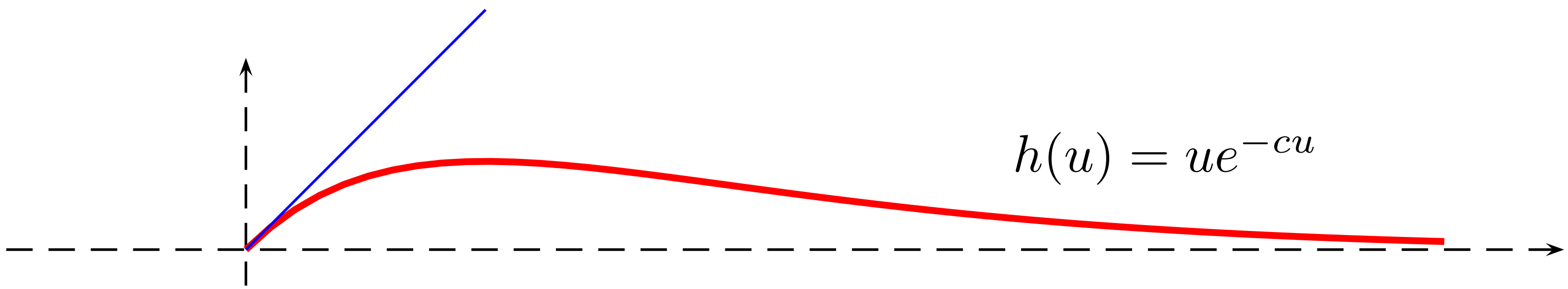
Biological interpretation:

One or multiple species, n classes or patches, with migration of the populations among classes

$x_i(t)$ - density of the species on class i
$a_{ij}(t)$ ($j \neq i$) - migration coefficient from class j to class i (w.l.g. $a_{ii} \equiv 0$)
$d_i(t)$ - coefficient of instantaneous loss for class i: death rate on class i plus the emigration rates of the population that leaves class i:
$d_i(t) = m_i(t) + \sum_{j\neq i} a_{ji}(t)$, $(m_i > 0)$
birth function on class i (Nicholson-type): $\sum_{k=1}^{m} \beta_{ik}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) h_{ik}(x_i(s))\, ds$

$h_{ik}(u) = xe^{-c_{ik}(t)u}$

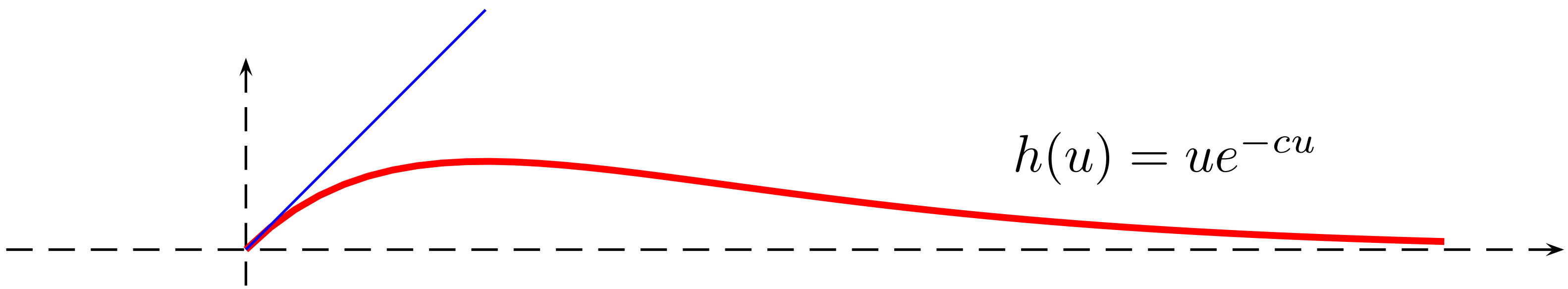
Ricker nonlinearities $h(u) = ue^{-cu}$ $(c > 0)$:



$h(0) = 0, h'(0) = 1$, $h(u)/u \to 0$ as $t \to \infty$

$h_{ik}(u) = xe^{-c_{ik}(t)u}$

Ricker nonlinearities $h(u) = ue^{-cu}$ $(c > 0)$:



$h(0) = 0, h'(0) = 1$, $h(u)/u \to 0$ as $t \to \infty$

- The nonlinearities are **bounded** $\implies$ $B_2(t)$ can be taken arbitrarily small
- With

$$b_i(t) := \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s)\, ds, \quad t \geq 0, \quad i = 1, \ldots, n$$

and $B(t) = \operatorname{diag}(b_1(t), \ldots, b_n(t))$, **(H6) holds** with

$$B_1(t) = (1 - \varepsilon)B(t), \; B_2(t) = \varepsilon I \quad (\forall \varepsilon > 0)$$

Proposition 1. [3]

IF (with $\mathbf{v} = (\mathbf{1}, \ldots, \mathbf{1})$):

(i) either $\sum_{j \neq i} a_{ij}(t) \leq_{\not\equiv} d_i(t) \leq_{\not\equiv} \sum_{j \neq i} a_{ij}(t) + b_i(t), \ \forall t \in [0, \omega]$

(ii) or $e^{D_i(\omega)} \int_0^\omega \sum_{j \neq i} a_{ij}(t)\, dt \leq (e^{D_i(\omega)} - 1) \leq \int_0^\omega (\sum_{j \neq i} a_{ij}(t) + b_i(t))\, dt$

THEN system (N) has a positive ω-periodic solution.

[3]with no impulses: $m_{1i} = m_{2i} = 1$, $n_{1i} = (e^{D_i(\omega)} - 1)^{-1}, n_{2i} = e^{D_i(\omega)}(e^{D_i(\omega)} - 1)^{-1}$

Impulsive version of (N):

$$\begin{cases} x_i'(t) = -d_i(t)x_i(t) + \sum_{j\neq i} a_{ij}(t)x_j(t) \\ \qquad\qquad + \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) x_i(s) e^{-c_{il}(s)x_i(s)}\, ds, \ \ t \neq t_k, \\ x_i(t_k^+) - x_i(t_k) = I_{ik}(x_i(t_k)), \ \ k \in \mathbb{Z}, \quad i = 1, \dots, n. \end{cases} \tag{IN}$$

where $0 \leq t_1 < t_2 < \cdots < t_p < \omega$, $t_{k+p} = t_k + \omega, \forall k$ (p impulses on $[0, \omega]$).

Take e.g. $I_{ik}(u) = I(u) := \sin u, \ u \geq 0, \forall k$.

$$\begin{cases} x_i'(t) = -d_i(t)x_i(t) + \sum_{j\neq i} a_{ij}(t)x_j(t) \\ \qquad\qquad + \sum_{l=1}^{m} \beta_{il}(t) \int_{t-\tau_{il}(t)}^{t} \gamma_{il}(s) x_i(s) e^{-c_{il}(s)x_i(s)}\, ds, \ t \neq t_k, \\ x_i(t_k^+) - x_i(t_k) = I_{ik}(x_i(t_k)), \ k \in \mathbb{Z}, \quad i = 1, \dots, n. \end{cases} \tag{IN}$$

where $0 \le t_1 < t_2 < \cdots < t_p < \omega$, $t_{k+p} = t_k + \omega, \forall k$ (p impulses on $[0, \omega]$).

Take e.g. $I_{ik}(u) = I(u) := \sin u, \ u \geq 0, \forall k$.

Then:

$\star$ $-\frac{1}{\pi} \le \frac{I_{ik}(u)}{u} \le 1, \ u > 0$, thus (H2) holds with $\alpha_{ik} = -\frac{1}{\pi}, \eta_{ik} = 1$

$\star$ (H3) is satisfied if $2^p < e^{\int_0^\omega d_i(t)\,dt} \ \forall i$

With the above notations: $J_{ik}(u) = (1 + \frac{\sin u}{u})^{-1}$, $\exists J_{ik}(0) = \lim_{u\to 0^+} J(u) = \dfrac{1}{2}, \overline{B_i} = 1, \underline{B_i} = 2^{-p}, \overline{\Gamma_i} = ((e^{D_i(\omega)}2^{-p} - 1)^{-1}, \underline{\Gamma_i} = (e^{D_i(\omega)} - 1)^{-1}$

Plugging these constants to evaluate $\boxed{M_q(t), N_q(t)}$ $(q = 1, 2)$ in (4), (5), from **Cor 1 & 2** (with $v = \vec{1}$):

Proposition 2. IF

(a) either

$$\frac{e^{D_i(\omega)}-1}{2^{-p}e^{D_i(\omega)}-1}\sum_{j\neq i}a_{ij}(t)\leq_{\not\equiv} d_i(t)\leq_{\not\equiv} 2^{-p}\Big(\sum_{j\neq i}a_{ij}(t)+b_i(t)\Big),\ \forall t\in[0,\omega],\forall i$$

(b) or

$$\frac{(e^{D_i(\omega)}-1)^2}{2^{-p}e^{D_i(\omega)}-1}\int_0^\omega\sum_{j\neq i}a_{ij}(t)dt\leq(e^{D_i(\omega)}-1)\leq\int_0^\omega\Big(\sum_{j\neq i}a_{ij}(t)+b_i(t)\Big)dt,\forall i$$

THEN (IN) admits at least one positive ω-periodic solution.

Example 2. A simple autonomous planar Nicholson system

$$\begin{cases} x_1'(t) = -d_1 x_1(t) + a_1 x_2(t) + \beta_1 x_1(t-\tau_1) e^{-c_1 x_1(t-\tau_1)} \\ x_2'(t) = -d_2 x_2(t) + a_2 x_1(t) + \beta_2 x_2(t-\tau_2) e^{-c_2 x_2(t-\tau_2)} \end{cases} \qquad (d_i, a_i, \beta_i, c_i, \tau_i > 0)$$

$$\begin{cases} x_1'(t) = -d_1x_1(t) + a_1x_2(t) + \beta_1x_1(t-\tau_1)e^{-c_1x_1(t-\tau_1)} \\ x_2'(t) = -d_2x_2(t) + a_2x_1(t) + \beta_2x_2(t-\tau_2)e^{-c_2x_2(t-\tau_2)} \end{cases} \qquad (d_i, a_i, \beta_i, c_i, \tau_i > 0)$$

• **community matrix**: $M = \begin{bmatrix} \beta_1 - d_1 & a_1 \\ a_2 & \beta_2 - d_2 \end{bmatrix}$.

• $s(M) \le 0 \iff$ 0 is **GAS** (globally asymptotically stable) (in the set of all non-negative solutions), where $s(M) = \max\{Re\,\lambda : \lambda \in \sigma(M)\}$.

(e.g. with $d_i = 2, a_i = \beta_i = 1$, $i = 1, 2$, $\sigma(M) = \{0, -2\} \implies$ 0 is GAS)

$$\begin{cases} x_1'(t) = -d_1 x_1(t) + a_1 x_2(t) + \beta_1 x_1(t-\tau_1) e^{-c_1 x_1(t-\tau_1)} \\ x_2'(t) = -d_2 x_2(t) + a_2 x_1(t) + \beta_2 x_2(t-\tau_2) e^{-c_2 x_2(t-\tau_2)} \end{cases} \qquad (d_i, a_i, \beta_i, c_i, \tau_i > 0)$$

• **community matrix**: $M = \begin{bmatrix} \beta_1 - d_1 & a_1 \\ a_2 & \beta_2 - d_2 \end{bmatrix}$.

• $s(M) \leq 0 \iff$ 0 is **GAS** (globally asymptotically stable) (in the set of all non-negative solutions), where $s(M) = \max\{Re\,\lambda : \lambda \in \sigma(M)\}$.

(e.g. with $d_i = 2, a_i = \beta_i = 1$, $i = 1, 2$, $\sigma(M) = \{0, -2\} \implies$ 0 is GAS)

Remark. This contradicts the assertion in Zhang, Huang & Wei, Adv. Diff. Equ.(2015), of existence of a positive (ω-periodic) solution for the above system with ω-periodic coefficients (rather than autonomous).

• Fix any $\omega > 0$ and add e.g. a **single linear, constant, positive impulse** on each component and on each interval of length ω:

$$\Delta x_i(t_k) = \eta_i x_i(t_k), \quad i = 1, 2, k \in \mathbb{Z} \tag{7}$$

• With $v = (1, 1)$ in **Corol. 1**: this destroys the GAS of the trivial solution if

$$0 < \eta_i < \frac{e^{2\omega} - 1}{e^{2\omega} + 1}, \quad i = 1, 2 \implies \exists \text{ a positive } \omega\text{−periodic solution!}$$

Final Comments:

• There is only a couple of previous works proving the existence of positive periodic solutions for **systems** of differential equations with delays and **impulses**

(Moreover, as a particular case our work shows that same claims in Zang et al., Adv Dif Eqs 2015, are not correct!)

• Our approach also applies to impulsive systems (1) with **infinite delay**

• For the scalar case: very few papers have "average" criteria, relating the averages of the coefficients over $[0, \omega]$

Future:

• For $\boxed{n > 1}$, how to eliminate/weaken hypothesis (H4)(ii), and still derive the existence of an ω-periodic solution with all components **positive**?

Future:

• For $\boxed{n > 1}$, how to eliminate/weaken hypothesis (H4)(ii), and still derive the existence of an ω-periodic solution with all components **positive**?

• Once the **existence** of a positive periodic solution $x^*(t)$ is established: provide sufficient conditions for its global attractivity!

(This depends strongly on the particular $g_i(t, x_{it})$ and on the impulses!)

• For $\boxed{n > 1}$, how to eliminate/weaken hypothesis (H4)(ii), and still derive the existence of an ω-periodic solution with all components **positive**?

• Once the **existence** of a positive periodic solution $x^*(t)$ is established: provide sufficient conditions for its global attractivity!

(This depends strongly on the particular $g_i(t, x_{it})$ and on the impulses!)

• Apply the present technique to treat **other families** of impulsive systems of DDEs, such as Lotka-Volterra models, Nicholson systems with "nonlinear mortality terms", etc.

• For $\boxed{n > 1}$, how to eliminate/weaken hypothesis (H4)(ii), and still derive the existence of an ω-periodic solution with all components **positive**?

• Once the **existence** of a positive periodic solution $x^*(t)$ is established: provide sufficient conditions for its global attractivity!

(This depends strongly on the particular $g_i(t, x_{it})$ and on the impulses!)

• Apply the present technique to treat **other families** of impulsive systems of DDEs, such as Lotka-Volterra models, Nicholson systems with "nonlinear mortality terms", etc.

• Treat the case of **almost periodic** systems of DDEs with impulses.

(For non-impulsive equations and systems: the usual operators whose fixed points we are looking for are not **compact**, therefore other techniques have been used, by imposing conditions that allow the use of Lyapunov functionals, Banach contraction principle, monotone operators...)

THANK YOU!