A local/nonlocal diffusion model

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Organization of the presentation and Main goals

- Introduction
- Propose a coupled local/nonlocal evolution problem in 1-d
- Study existence and uniqueness of solutions, mass conservation and asymptotic behaviour as t goes to infinity
- Study the asymptotic behavior of the solution as t goes to infinity
- Recover the local heat equation in the whole domain taking the limit in the rescaled nonlocal kernel
- Extend the results to higher dimensions

Introduction

A large amount of data are available to describe current epidemiological events. We need to offer reliable models to be used in each event.

Example: Risk studies for COVID-19 pandemic

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RESEARCH ARTICLE

Modeling future spread of infections via mobile geolocation data and population dynamics. An application to COVID-19 in Brazil

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Table 1. Descriptive statistics of the daily number of recordings in March 2019 and 2020 for each state on the weekends and weekdays.

State	Year	Day Week	Mean	SD	Min	1st Quart.	Median	3rd Quart.	Max
RJ	2019	Weekday	1,053,615	259,721	528,805	790,609	1,140,444	1,180,351	1,465,666
		Weekend	938,472	163,672	679,678	811,883	962,988	1,026,558	1,201,777
	2020	Weekday	870,920	445,958	214,521	509,431	870,189	1,196,752	1,682,386
		Weekend	624,417	425,258	179,309	238,466	442,211	1,023,705	1,332,701
SP	2019	Weekday	3,708,276	850,839	2,221,510	2,790,522	3,927,577	4,169,805	5,011,449
		Weekend	3,256,169	563,222	2,456,231	2,756,478	3,550,141	3,569,211	4,172,801
	2020	Weekday	4,353,782	1,652,625	1,661,284	2,816,090	4,465,741	5,545,852	7,384,012
		Weekend	3,561,949	1,495,902	1,681,135	2,118,311	3,964,115	4,999,465	5,527,734

SD = Standard Deviation.

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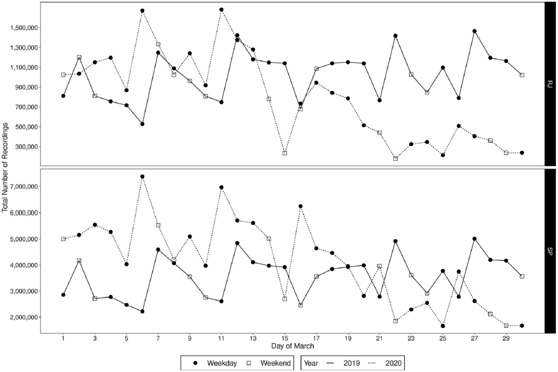

Fig 1. Total number of recordings for each day of March in São Paulo (SP) and Rio de Janeiro (RJ), in 2019 and 2020.

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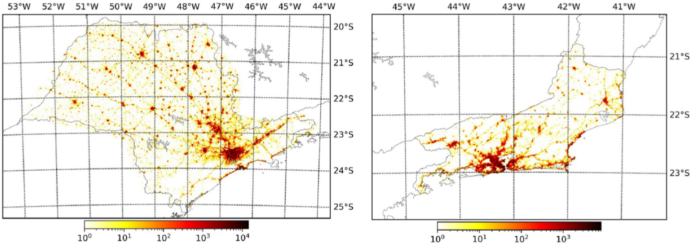

Fig 3. Typical distribution of the location of app usage in one day for the states of São Paulo (left) and Rio de Janeiro (right) considering a resolution of 0.01 degree on each geographical coordinate. This data refers to March 1st, 2020 and the color represents the number of recordings, first or subsequent, in each location.

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$$\frac{dI_i}{dt}(t) = (1+r)I_i(t)\left(\frac{N_i - I_i(t)}{N_i}\right) + s\left[\sum_{j\neq i}\omega_{ji}(t)I_j(t) - \sum_{j\neq i}\omega_{ij}(t)I_i(t)\right]$$

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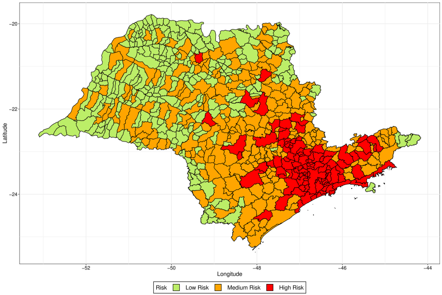
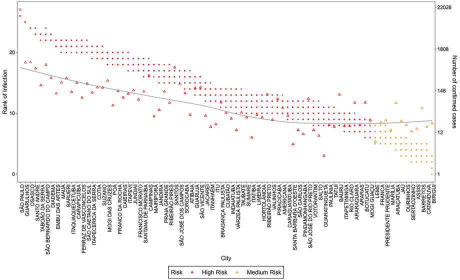


Fig 14. Rank of infection and number of confirmed COVID19 cases for São Paulo state.
The points refer to ranks estimated for different values of s, the triangles refer to the official number of confirmed COVID19 cases registered on the 1st of May 2020 for each city in the state of São Paulo. The line is a smooth approximation of the confirmed cases (triangles). The colors refer to the risk evaluated by k-means clustering of the ranks attributed by the simulated models.



Introduction

Example: Impact of lock down strategies during COVID-19 pandemic

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A snapshot of a pandemic: The interplay between social isolation and COVID-19 dynamics in Brazil

Cláudia P. Ferreira,[2] Diego Marcondes,[3] Mariana P. Melo,[1] Sérgio M. Oliva,[3] Cláudia M. Peixoto,[3] and Pedro S. Peixoto[3,4,*]
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Stage + 1 + 2 + 3

Figure 5. Dispersion between the mean relative isolation index of 1 week ago and the 7-day moving average of incidence for 32 cities during the upward phase.
Colors refer to the stage. The line is a LOESS . The value of *S* is the skewness coefficient of the normalized LOESS curve.

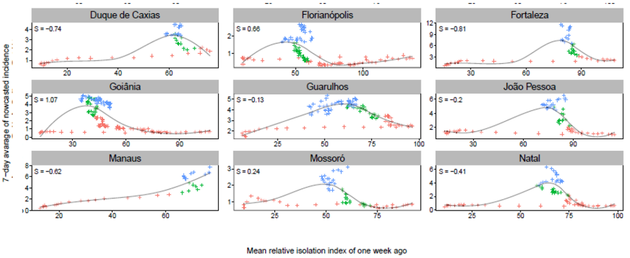

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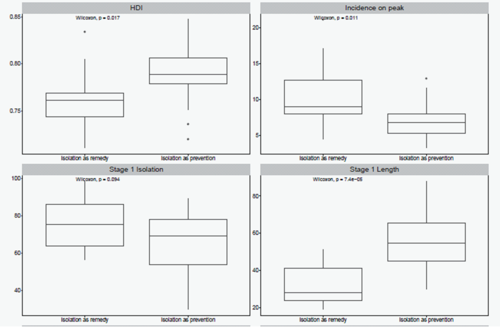
Figure 8. Box plots of HDI, incidence on the peak, median isolation on each stage, length of each stage, and length of the upward phase for the cities that employed isolation as a remedy (negative skewness coefficient of the smooth curve) and cities that employed it as prevention (positive skewness)

p values refer to the Wilcoxon test comparing the two groups of cities.



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Introduction

A classical example of a local linear diffusion equation is the heat equation,

$$u_t(x,t) = u_{xx}(x,t).$$

Properties

- Existence, uniqueness and continuous dependence on the initial data;
- Maximum and comparison principles;
- If u_0 is a nonnegative and nontrivial initial data, then $u(x,t) > 0$ for every $x \in \mathbb{R}^N$ and every $t > 0$.

Nonlocal diffusion equation

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy,$$

where $J : \mathbb{R}^N \to \mathbb{R}$ is assumed nonnegative, continuous, radially symmetric, with compact support and verifies

$$\int_{\mathbb{R}^N} J(r)dr = 1.$$

Properties (the same as for the heat equation, but there is no regularizing effect)

- Existence, uniqueness and continuous dependence on the initial data;
- Maximum and comparison principles;
- If u_0 is a nonnegative and nontrivial initial data, then $u(x,t) > 0$ for every $x \in \mathbb{R}^N$ and every $t > 0$.

In this model, $u(x,t)$ stands for the density of individuals in x at time t and $J(x-y)$ is the probability distribution of jumping from y to x. Then

$$(J * u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$$

is the rate at which the individuals are arriving to x from other places.

$$-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$$

is the rate at which they are leaving from x to other places.

Associated to the classical heat equation

$$u_t(x,t) = \triangle u(x,t) \tag{1}$$

we have the following energy

$$E(u) = \int \frac{|\bigtriangledown u|^2}{2},$$

in the sense that (1) is the gradient flow associated to $E(u)$.
Similarly, associated to the nonlocal diffusion problem

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy \tag{2}$$

we have the following energy

$$E(u) = \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))^2 dydx,$$

in the sense that (2) is the gradient flow associated to $E(u)$.

We split the domain $\Omega = (-1,1)$ in two subdomains $\Omega_l = (-1,0)$ and $\Omega_{nl} = (0,1)$. We propose and evolution problem consisting of two parts: a local part, composed of a heat equation with Neumann/Robin type boundary conditions, for $x \in (-1,0)$, $t > 0$,

$$\begin{cases} u_t(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t), \\ u_x(-1,t) = 0, \\ u_x(0,t) = C_{J,2} \displaystyle\int_{-1}^{0} \int_{0}^{1} J(x-y)(v(y,t) - u(0,t))dydx, \\ u(x,0) = u_0(x). \end{cases} \tag{3}$$

For the nonlocal domain we have, for $x \in (0,1)$, $t > 0$,

$$\begin{cases} v_t(x,t) = C_{J,1} \displaystyle\int_{0}^{1} J(x-y)\left(v(y,t) - v(x,t)\right)dy, \\ \qquad\qquad\qquad -C_{J,2} \displaystyle\int_{-1}^{0} J(x-y)(v(x,t) - u(0,t))dy, \\ v(x,0) = v_0(x). \end{cases} \tag{4}$$

The complete problem can be summarized as we look for w defined by

$$w(x,t) = \begin{cases} u(x,t), & \text{if} \quad x \in (-1,0) \\ v(x,t), & \text{if} \quad x \in (0,1) \end{cases} \tag{5}$$

where (u, v) is a solution to (3)–(4).

- The local/nonlocal problem is well posed in the sense that there is existence and uniqueness of solutions.
- There is an energy functional such that the evolution problem can be viewed as the associated gradient flow,

$$(u, v)'(t) = -\partial E\left[(u, v)(t)\right]$$

for

$$E(u, v) = \frac{1}{2}\int_{-1}^{0} |u_x|^2 + \frac{C_{J,1}}{4}\int_{0}^{1}\int_{0}^{1} J(x-y)|v(y)-v(x)|^2 + \frac{C_{J,2}}{2}\int_{-1}^{0}\int_{0}^{1} J(x-y)|v(y)-u(0)|^2.$$

- Solutions preserve the total mass of the initial condition.
- Solutions converge exponentially fast to the mean value of the initial condition.

Theorem (S.- Oliva - Rossi. Applicable Analysis 2021)

Given $w_0 \in L^2(-1,1)$, there exists a unique mild solution

$$w(t,\cdot) \in \mathcal{B} := \left\{ w \in L^2(-1,1) : u|_{\Omega_l} \in H^1(-1,0), v \in L^2(0,1) \right\}$$

to the local/nonlocal problem, with (u,v) satisfying (3)–(4).

A comparison principle holds: if $w_0 \geq z_0$ then the corresponding solutions verify $w \geq z$ in $(-1,1) \times \mathbb{R}_+$.

Moreover, the total mass of the solution is preserved along the evolution, that is,

$$\int_{-1}^{1} w(\cdot,t) = \int_{-1}^{1} w_0.$$

Fixed point argument

Theorem

Given $w_0 \in L^2([-1,1])$ (or given $w_0 \in C([-1,1])$), there exists a unique solution to problem (3)–(4), *which has w_0 as initial condition.*

Idea of the proof:

- Given $u \in (-1,0)$ we obtain the solution for v;
- Given $v \in (0,1)$ we obtain the solution for u;
- Find a $T > 0$, such that we have a contraction;
- Use a fixed point argument.

Asymptotic behavior

Let us take β_1 as the first nonnegative eigenvalue

$$0 < \beta_1 = \inf_{w: \int_{-1}^{1} w = 0} \frac{E(u,v)}{\displaystyle\int_{-1}^{1} (w(x))^2 dx}. \tag{6}$$

Remark: Due to the lack of compactness of the nonlocal part, it is not clear that the infimum defining β_1 is attained.

Theorem

Given $w_0 \in L^2(-1,1)$, the solution of the problem, with initial condition w_0 converges to its mean value as $t \to \infty$ with an exponential rate,

$$\left\| w(\cdot,t) - ⨍ w_0 \right\|_{L^2(-1,1)} \le C\left(\|w_0\|_{L^2(-1,1)}\right) e^{-\beta_1 t}, \qquad t > 0, \tag{7}$$

where β_1 is given by (6) and $C = (w_0) > 0$.

Semigroup theory

$$B_J u = \begin{cases} -u_{xx} \quad \text{for} \quad x \in (-1,0), \\ \text{with } u_x(-1) = 0 \text{ and } u_x(0) = -C_{J,2}\displaystyle\int_{-1}^{0} J(x-y)(u(y)-u(0))dy, \\ -\dfrac{C_{J,1}}{2}\displaystyle\int_0^1 J(x-y)(u(y)-u(x))dy + C_{J,2}\displaystyle\int_{-1}^{0} J(x-y)(u(x)-u(0))dy \\ \quad \text{for} \quad x \in (0,1). \end{cases}$$

Let

$$D(B_J) := \Big\{(u,v) : u \in H^2(-1,0), v \in L^2(0,1) \text{ with } \frac{\partial u}{\partial x}(-1) = 0 \text{ and}$$

$$\frac{\partial u}{\partial x}(0) = -C_{J,2}\int_{-1}^{0} J(x-y)(v(y)-u(0))dy\Big\}$$

denote the domain of the operator, that is, we have that

$$B_J : D(B_J) \to L^2(-1,1).$$

According to Andreu et all (2010), we can define a mild solution in $L^2(-1,1)$, of the abstract Cauchy problem by:

$$\begin{cases} u'(t) = B_J(u(t)), & t > 0 \\ u(0) = u_0. \end{cases} \tag{8}$$

Note that this operator is linear in a Banach space so the general theory of existence also can be given by the Hille-Yosida Theorem.

Existence of a mild solution

Theorem

Given and initial condition $w_0 \in L^2(-1,1)$, there exists an unique mild solution w of the problem.

Idea of the proof:

- B_J is completely accretive in $L^2(-1,1)$;
- B_J satisfies the range condition, $L^2(-1,1) \subset R(I + B_J)$;
- By the Crandall-Ligget's Theorem and the linear semigroup theory will give existence and uniqueness of a mild solution of the evolution problem.

Rescaling the nonlocal kernel

Le J be the rescaled kernel

$$J^{\epsilon}(x) := \frac{C_{J,1}}{\epsilon^3} J\left(\frac{x}{\epsilon}\right),$$

where $C_{J,1} = \frac{2}{M(J)}$, $C_{J,2} = 1$, and

$$M(J) := \int_{\mathbb{R}} J(z)|z|^2 dz < \infty.$$

The associated energy functional to the rescaled problem is given by

$$E^{\varepsilon}(w^{\varepsilon}) := \frac{1}{2}\int_{-1}^{0}(u_x^{\varepsilon})^2 + \frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^{\varepsilon}(x-y)\left(v^{\varepsilon}(y)-v^{\varepsilon}(x)\right)^2 dydx \quad (9)$$
$$+ \frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^{\varepsilon}(x-y)\left(u^{\varepsilon}(0)-v^{\varepsilon}(x)\right)^2 dxdy,$$

if $w^{\varepsilon} \in D(E^{\varepsilon}) := H^1(-1,0)\times L^2(0,1)$, and $E^{\varepsilon}(w^{\varepsilon}) := \infty$ if not. Analogously, we define the limit energy functional as

$$E(w) := \frac{1}{2}\int_{-1}^{1}|w_x|^2 dx, \quad (10)$$

if $w \in D(E) := H^1(-1,1)$, and $E(w) := \infty$ if not.

Given $w_0 \in L^2(-1,1)$, for each $\varepsilon > 0$, let w^ε be the solution to the evolution problem associated with the energy E^ε, and w be the solution associated to the functional E, considering the same initial condition.

Theorem

Under the above assumptions, the solutions to the rescaled problems, w^ε, converge in the following sense, to the solution of the heat equation, w. For any finite $T > 0$ we have

$$\lim_{\varepsilon \to 0} \left(\max_{t \in [0,T]} \| w^\varepsilon(\cdot, t) - w(\cdot, t) \|_{L^2(-1,1)} \right) = 0. \tag{11}$$

Idea of the proof:

- $B_{J^\varepsilon} \in L^2(-1,1)$;
- $A \in H^2(-1,1)$;
- We show the convergence of the resolvents, that is, $\lim_{\varepsilon\to 0} (I + B_{J^\varepsilon})^{-1} \phi = (I + A)^{-1} \phi$, for every $\phi \in L^2(-1,1)$;
- By the Brezis-Pazy Theorem we get the convergence of the solutions, that is $w^\varepsilon \to w$ in the L^2-norm.

Higher dimension extension

Take Ω, as a bounded smooth domain in $\mathbb{R}^N$ and split it into two subdomains Ω_l and Ω_{nl},

$$\Omega = \Omega_l \cup \Omega_{nl}.$$

Let us call Σ, the interface between Ω_l and Ω_{nl} inside Ω, that is,

$$\Sigma = \overline{\Omega_l} \cap \overline{\Omega_{nl}} \cap \Omega.$$

We will assume that Ω_l has a Lipschitz boundary. For any

$$w = (u, v) \in \mathcal{B} := \left\{ w \in L^2(\Omega) : u|_{\Omega_l} \in H^1(\Omega_l), v \in L^2(\Omega_{nl}) \right\}$$

we define the energy

$$\begin{aligned} E(u,v) := \frac{1}{2} \int_{\Omega_l} |\nabla u|^2 dx + \frac{C_{J,1}}{4} \int_{\Omega_{nl}} \int_{\Omega_{nl}} J(x-y) \left(v(y) - v(x) \right)^2 dy dx \\ + \frac{C_{J,2}}{2} \int_{\Omega_{nl}} \int_{\Sigma} G(x,z) \left(v(x) - u(z) \right)^2 d\sigma(z) dx. \end{aligned} \tag{12}$$

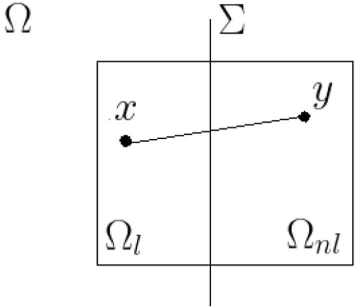

Figure 1: The local part (Ω_l) and the nonlocal part (Ω_{nl}).

With this energy the associated evolution problems reads as,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t), \\ \dfrac{\partial u}{\partial \eta}(z,t) = 0, \qquad z \in \partial\Omega_l \cap \partial\Omega, \\ \dfrac{\partial u}{\partial \eta}(z,t) = C_{J,2} \displaystyle\int_{\Omega_{nl}} G(x,z)(v(y,t) - u(z,t))dx, \qquad z \in \Sigma, \\ u(x,0) = u_0(x). \end{cases} \tag{13}$$

for $x \in \Omega_l$, $t > 0$, and

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = C_{J,1} \displaystyle\int_{\Omega_{nl}} J(x-y)\,(v(y,t) - v(x,t))\,dy - C_{J,2} \int_{\Sigma} G(x,z)(v(x,t) - u(z,t))d\sigma(z), \\ v(x,0) = v_0(x), \end{cases} \tag{14}$$

for $x \in \Omega_{nl}$, $t > 0$.

High dimension: Main results

i. The problem (13)–(14) posses existence and uniqueness of the solution;
ii. The total mass is preserved;
iii. Solutions converge to the mean value of the initial condition, as $t \to \infty$ with an exponential rate.

Remark: The approximation of the heat equation with Neumann boundary conditions under rescales of the kernel is left open. We believe that the result holds with extra assumptions on the coupling kernel G.

Thanks

Thank you!