# A coagulation type toy model for silicosis

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# A coagulation-fragmentation-death model for silicosis

Based on joint works with:

- Michael Drmota, Michael Grinfeld:
  - Euro. Jnl. Appl. Math., **31** (2020) 950–967.
- João Teixeira Pinto, Rafael Sasportes:
  - Nonlinear Anal. Real World Appl., **60** (2021) 103299.
- Pedro Antunes, João Teixeira Pinto, Rafael Sasportes:
  - submitted in 2021.

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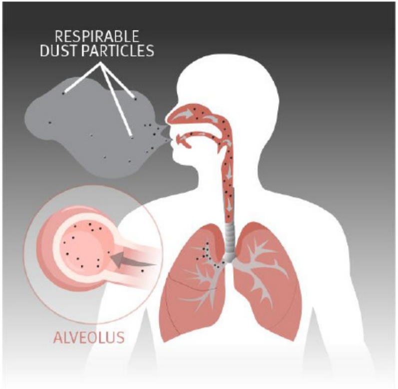
# Simplified description of the processes

Silicosis is an occupational lung disease characterized by inflammation and fibrosis of the lungs caused by the inhalation of crystalline silica dust. It is incurable, although preventable, and kills about 11 000 persons annually worldwide.



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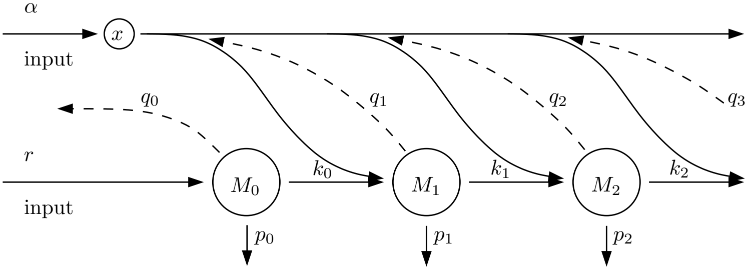
- the role of the macrophages is to expel their load of silica by travelling up the "mucociliary escalator" (lungs $\rightarrow$ trachea $\rightarrow$ pharynx) to get eliminated out of the respiratory system;
- silica particles are toxic to the macrophages and lead to their death, which can occur while they are still in the lungs, thus liberating the silica particles back into the respiratory system.

# The mathematical model




- $x(t)$, concentration of silica particles in the respiratory system;
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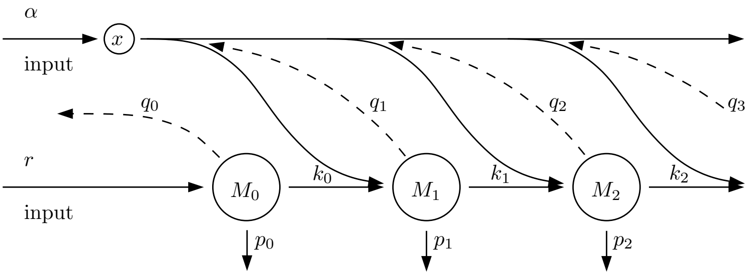


- $x(t)$, concentration of silica particles in the respiratory system;
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- $r$ input rate of new macrophages *(physiology)*;

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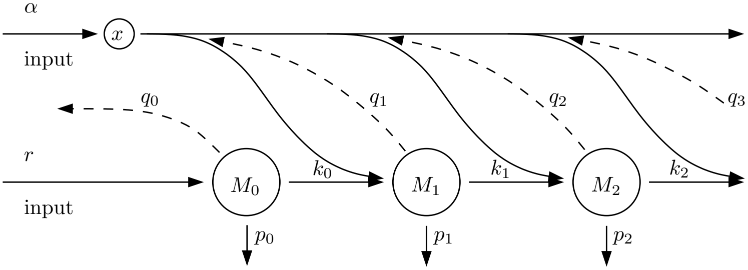


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- $M_i(t)$ concentration of macrophages with a load of $i$ silica particles;
- $\alpha$ input rate of silica particles *(environment)*;
- $r$ input rate of new macrophages *(physiology)*;
- $(q_i)_{i \in \mathbb{N}_0}$ death rate ($i$-nondecrease);
- $(p_i)_{i \in \mathbb{N}_0}$ escape rate through the mucociliary escalator ($i$-nonincrease).

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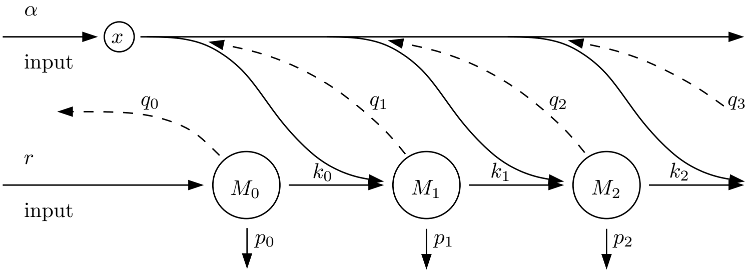


$$\begin{cases} \dot{M}_0 = r - k_0 x M_0 - (p_0 + q_0) M_0 \ , \\ \dot{M}_i = k_{i-1} x M_{i-1} - k_i x M_i - (p_i + q_i) M_i \ , \quad i \geqslant 1 \, , \\ \dot{x} = \alpha - x \sum_{i=0}^{\infty} k_i M_i + \sum_{i=0}^{\infty} i q_i M_i \ , \\ x(0) = x_0, \quad M_i(0) = M_{0i}, \quad i = 0, 1, 2, \ldots \ . \end{cases} \tag{1}$$

# The mathematical model

Amount of quartz particles: $\mathcal{X} := x + \sum_{i=0}^{\infty} iM_i$ ,

Amount of macrophages: $\mathcal{M} := \sum_{i=0}^{\infty} M_i$ ,

Total amount of matter: $\mathcal{U} := \mathcal{X} + \mathcal{M} = x + \sum_{i=0}^{\infty}(i+1)M_i$ .

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Formally: $\dot{\mathcal{X}} = \alpha - \sum_{i=1}^{\infty} ip_iM_i \leqslant \alpha \,, \quad \dot{\mathcal{M}} = r - \sum_{i=0}^{\infty}(p_i+q_i)M_i \leqslant r \,,$

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$$\dot{\mathcal{U}} = (r+\alpha) - \sum_{i=0}^{\infty}\big((i+1)p_i + q_i\big)M_i \leqslant r+\alpha .$$

Banach space $X$ of sequences $y = (y_i) = (x, M_0, M_1, M_2, \dots)$ with

$$\|y\| := |x| + \|((i+1)M_i)_{i=0,1,2,\dots}\|_{\ell_1}$$

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(i) $\forall t \in [0, T) \quad y(t) \geqslant 0$;

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(iv) $M_0(t) = M_{00} + rt - \int_0^t \left[x(s)k_0 M_0(s) - (p_0 + q_0) M_0(s)\right]$,

$$M_i(t) = M_{0i} + \int_0^t \left[x(s)k_{i-1}M_{i-1}(s) - x(s)k_i M_i(s) - (p_i + q_i)M_i(s)\right],$$

$$x(t) = x_0 + \alpha t + \int_0^t \left[-x(s)\sum_{i=0}^{\infty} k_i M_i(s) + \sum_{i=0}^{\infty} iq_i M_i(s)\right].$$

# Existence

## Theorem (existence of solution)

*Let* $(g_i)_{i\in\mathbb{N}_0}$ *and* $(k_i)_{i\in\mathbb{N}_0}$ *be nonnegative sequences such that, for some* $\delta > 0$, $g_{i+1} - g_i \geqslant \delta$, $i = 0, 1, 2, \ldots$, *and furthermore,*

$$(g_{i+1} - g_i)k_i = O(g_i), \quad \text{for} \quad i = 0, 1, 2, \ldots.$$

*Let* $(p_i)_{i\in\mathbb{N}_0}$ *and* $(q_i)_{i\in\mathbb{N}_0}$ *be two arbitrary nonnegative sequences.*

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*Let* $(p_i)_{i\in\mathbb{N}_0}$ *and* $(q_i)_{i\in\mathbb{N}_0}$ *be two arbitrary nonnegative sequences.*

*Then, for* $y_0 = (x_0, M_{00}, M_{01}, \ldots) \in X_+$*, such that,* $\sum_{i=0}^{\infty} g_i M_{0i} < \infty$*, there exists a solution* $y = (x, M_0, M_1, \ldots)$ *of* (1) *on* $[0, +\infty)$ *with* $y(0) = y_0$*, that satisfies, for any* $T \in (0, \infty)$*,*

$$\sup_{t\in[0,T]} \sum_{i=0}^{\infty} g_i M_i(t) < \infty, \qquad \int_0^T \sum_{i=1}^{\infty} g_i(p_i + q_i)M_i(s)\, ds < \infty\,. \tag{2}$$

# Existence

The proof is adapted from the existence proof in (Ball, Carr & Penrose, 1986) for the Becker-Döring, and consists in:

- defining finite $n$-dimensional truncations of (1), with solutions $y^n = (x^n, M_0^n, \dots, M_n^n)$;
- getting *a priori* estimates $|x^n(t)| \leqslant \|y^n(t)\| \leqslant \|y_0\| + (\alpha + r)t$, and uniform in $n$ estimates for $|\frac{d}{dt} M_i^n(t)|$;
- using the bound on $x^n$ to get $x^n \overset{*}{\rightharpoonup} x$ in $L^\infty(0, T)$ for some non-negative $x \in L^\infty(0, T)$, and some subsequence (not relabelled), meaning:

$$\forall \phi \in L^1(0, T), \int_0^T \big(x^n(s) - x(s)\big)\phi(s)ds \to 0 \text{ as } n \to \infty;$$

# Existence

- using Ascoli-Arzelà and a diagonalization procedure to get $M_i^n \to M_i$, uniformly on $[0, T]$, for some continuous $M_i$, for some subsequence (not relabelled), which satisfy the equations in (iv);
- using Gronwall's lemma in the integral equation for the evolution of $\sum_{i=1}^n g_i M_i^n(t)$, and the monotone convergence theorem to take limits as $n \to \infty$, obtain the *a priori* bound

$$\sum_{i=1}^{\infty} g_i M_i(t) + \int_0^t \sum_{i=1}^{\infty} g_i(p_i + q_i) M_i(s)\, ds \leqslant C_1 e^{C_2 t},$$

for constants $C_1, C_2$ independent of $t$ and $n$;
- and also getting the *a priori* bound, for all $\varepsilon > 0$, and sufficiently large $m$ and $n$,

$$\int_0^t \sum_{i=m}^{\infty} g_i M_i^n(s) ds + \frac{t}{2} \int_0^{t/2} \sum_{i=m}^{\infty} g_i(p_i + q_i) M_i^n(s)\, ds \leqslant \varepsilon(2 + K_1^{-1}) e^{K_1 t},$$

for constant $K_1$ independent of $t$ and $n$;

# Existence

- with all the above ingredients we can prove that $(x^n)$ is a Cauchy sequence in the uniform convergence norm and that the limit function $x$ satisfies the equation in (iv), and this completes the proof of the existence theorem.

# Regularity

Under some reasonably general conditions we conclude

- the moment's evolution

$$\sum_{i=m}^{\infty} g_i M_i(t_2) - \sum_{i=m}^{\infty} g_i M_i(t_1) + \int_{t_1}^{t_2} \sum_{i=m}^{\infty} g_i (p_i + q_i) M_i(s)\, ds$$
$$= \int_{t_1}^{t_2} g_m x(s) k_{m-1} M_{m-1}(s)\, ds + \int_{t_1}^{t_2} \sum_{i=m}^{\infty} (g_{i+1} - g_i) x(s) k_i M_i(s)\, ds \,.$$

- from which evolution equations for $\mathcal{X}$, $\mathcal{M}$ and $\mathcal{U}$ are obtained;
- and with more restrictive conditions (e.g.: $k_i = O(i)$, $q_i = O(1)$) we conclude that $x$ and $M_i$ are $C^1([0, T))$ functions and satisfy (1) in its differential form, in $[0, T)$.

# Uniqueness

## Theorem (uniqueness of solution)

*Let $(k_i)_{i\in\mathbb{N}_0}$ and $(g_i)_{i\in\mathbb{N}_0}$ be as in the hypothesis of the existence theorem. Furthermore, assume that $k_i = O(i)$ and $ik_i = O(g_i)$, and let $(p_i), (q_i)$ be nonnegative sequences such that $p_i = O(1)$ and $q_i = O(i)$.*

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*Then, for each* $y_0 = (x_0, M_{00}, M_{01}, \dots) \geqslant 0$*, satisfying* $\sum_{i=0}^{\infty} g_i y_{0i} < \infty$*, there is exactly one solution* $y = (x, M_0, M_1, \dots)$ *of* $(1)$ *on* $[0, \infty)$ *satisfying* $y(0) = y_0$*.*

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*Then, for each $y_0 = (x_0, M_{00}, M_{01}, \dots) \geqslant 0$, satisfying $\sum_{i=0}^{\infty} g_i y_{0i} < \infty$, there is exactly one solution $y = (x, M_0, M_1, \dots)$ of (1) on $[0, \infty)$ satisfying $y(0) = y_0$.*

## Corollary

*Let $(p_i)$, $(q_i)$, $(k_i)$ be nonnegative sequences satisfying $p_i = O(1)$, $q_i = O(i)$, and $k_i = O(i^\gamma)$ for some $\gamma \in [0, 1]$.*

*Then, for each $y_0 \in X_+^{1+\gamma}$ there is a unique solution $y$ of (1) satisfying $y(0) = y_0$, and, moreover, $y(t) \in X_+^{1+\gamma}$ for all $t > 0$.*

# Semigroup property

Under the same assumptions of the previous corollary, one can prove a semigroup property of the set of solutions:

## Theorem ($C_0$-semigroup)

*Let $(p_i)$, $(q_i)$, $(k_i)$ be nonnegative sequences satisfying $p_i = O(1)$, $q_i = O(i)$, and $k_i = O(i^\gamma)$ for some $\gamma \in [0,1]$. Let $y_0 \in X_+^{1+\gamma}$. Denote by $T(\cdot)y_0$ the unique solution $y(\cdot\,; y_0)$ of (1) satisfying the initial condition $y(0; y_0) = y_0$.*

*Then, $\{T(t) : X_+^{1+\gamma} \to X_+^{1+\gamma} \mid t \geqslant 0\}$ is a $C_0$-semigroup, i.e.:*

(i) $T(0) = id$, *the identity operator;*

(ii) $T(t+s) = T(t)T(s)$, *for all* $t, s \geqslant 0$;

(iii) $(t, y_0) \mapsto T(t)y_0$ *is a continuous map from* $[0,\infty) \times X_+^{1+\gamma}$ *into* $X_+^{1+\gamma}$.

# Equilibria

The equilibria of (1) are the solutions $(x, M_0, M_1, \ldots) \in X_+$ of

$$\begin{cases} 0 = r - k_0 x M_0 - (p_0 + q_0) M_0 \ , \\ 0 = k_{i-1} x M_{i-1} - k_i x M_i - (p_i + q_i) M_i \ , \quad i \geqslant 1 \, , \\ 0 = \alpha - x \displaystyle\sum_{i=0}^{\infty} k_i M_i + \sum_{i=0}^{\infty} i q_i M_i \end{cases}$$

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The equations for $M_0$ and $M_i, i \geqslant 1$, can be solved in terms of $x$ to get the following equations for the equilibria:

$$\begin{cases} M_i = \dfrac{r x^i}{k_i \displaystyle\prod_{j=0}^{i} (x + d_j)}, \quad i \geqslant 0, \qquad \text{(where } d_j = (p_j + q_j)/k_j) \\ \alpha - x \displaystyle\sum_{i=0}^{\infty} k_i M_i + \sum_{i=0}^{\infty} q_i i M_i = 0 \end{cases} \tag{3}$$

# Equilibria

To be able to solve

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Imposing conditions compatible with the biological requirements that $p_i$ is nonincreasing and $q_i$ is nondecreasing we can draw some conclusions about the structure of equilibria.

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We consider two cases: a piecewise constant, and a power law.

# Equilibria (for a class of piecewise constant coefficients)

Consider piecewise constant coefficients (for a fixed positive integer $N$)

$$k_i \equiv k, \qquad p_i = \begin{cases} 1 & \text{if } i \leqslant N, \\ 0 & \text{if } i \geqslant N+1, \end{cases} \quad \text{and} \quad q_i = \begin{cases} 0 & \text{if } i \leqslant N, \\ 1 & \text{if } i \geqslant N+1. \end{cases} \tag{4}$$

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Then, (3) becomes

$$\begin{cases} M_i = \dfrac{r/k}{x+1/k}\left(\dfrac{x}{x+1/k}\right)^i, \quad i \geqslant 0. \\ \\ \dfrac{\alpha}{r} - \mathcal{F}_{N,k}(x) = 0 \end{cases} \tag{5}$$

where

$$\mathcal{F}_{N,k}(x) := kx\left(1 - \left(\frac{x}{x+1/k}\right)^{N+1}\right) - (N+1)\left(\frac{x}{x+1/k}\right)^{N+1}$$

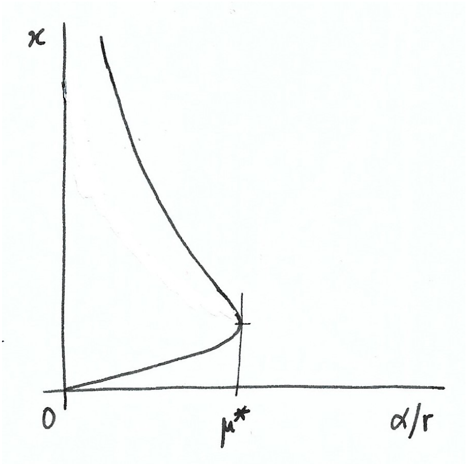
# Equilibria (for a class of piecewise constant coefficients)

**Theorem (4)**

*For all $N \in \mathbb{N}$ and $k > 0$, there exists a unique $\mu^* > 0$ such that* (1) *has:*

1. *no equilibria if $\alpha/r > \mu^*$,*
2. *exactly one equilibrium if $\alpha/r = \mu^*$,*
3. *exactly two equilibria if $\alpha/r \in (0, \mu^*)$.*

(Remark: $\mu^* := \max_{x \geqslant 0} \mathcal{F}_{N,k}(x)$)



# Equilibria (for a class of coefficients with power type relations)

Assume, now, $z := \inf_i d_i > 0$ and $q_i/k_i$ does not grow faster than a power of $i$. Then (from (3)-1)

$$\sum_{i=0}^{\infty} k_i M_i < \infty \qquad \text{and} \qquad \sum_{i=0}^{\infty} i q_i M_i < \infty,$$

and (3)-2 can be written as

$$\frac{\alpha}{r} - \mathcal{F}(x) = 0$$

with

$$\mathcal{F}(x) := \frac{x}{x+d_0} + \frac{1}{x+d_0}\sum_{i=1}^{\infty}\left(x - i\frac{q_i}{k_i}\right)\prod_{j=1}^{i}\frac{x}{x+d_j}$$

# Equilibria (for a class of coefficients with power type relations)

**Theorem**

*Let $z := \inf_i d_i > 0$ and assume $q_i/k_i$ does not grow faster than a power of $i$. Let $\rho_i := p_i/k_i$.*

1. *If $d_i = o(i\rho_i)$ as $i \to \infty$ then $\mathcal{F}(x) \to \infty$ as $x \to \infty$ and, consequently, for all $\alpha$ and $r$ there is at least one equilibrium.*
2. *If $i\rho_i = O(d_i)$ as $i \to \infty$ then $\mathcal{F}(x)$ is bounded. Thus there exists $m > 0$ such that there exists at least one equilibrium for $\alpha/r < m$ and no equilibria for $\alpha/r > m$.*
3. *If $i\rho_i = o(d_i)$ as $i \to \infty$ then $\mathcal{F}(x)$ is bounded and $\mathcal{F}(x) \to 0$ as $x \to \infty$, and there exists $m > 0$ such that there exists at least two equilibria for $\alpha/r < m$ and no equilibria for $\alpha/r > m$.*

(Remark: $m := \sup_{x \geqslant 0} \mathcal{F}(x)$)

## Equilibria (for a class of coefficients with power type relations)

Consider the case where the coefficients are given by:

$$p_i = i^{-p}, \qquad q_i = i^{q}, \qquad k_i = i^{-k},$$

for $i \in \mathbb{N}^+$ and constants $p, q, k \geqslant 0$. Let $p_0, q_0$ and $k_0 > 0$ be given.

Defining $a := k + q \geqslant 0$ and $b := k - p \in \mathbb{R}$, we have $d_i = i^a + i^b$ and $\rho_i = i^b$. Then, the last theorem implies the following

### Corollary

*Suppose that $a \geqslant 0$ and $b > -2$.*

1. *If $b > a - 1$, then system (1) has an equilibrium for all $\alpha/r$.*
2. *If $b = a - 1$, there exists $m > 0$ such that system (1) has an equilibrium if $\alpha/r < m$, and no equilibria if $\alpha/r > m$.*
3. *If $b < a - 1$, there is a value $m > 0$ such that system (1) has at least two equilibria if $\alpha/r < m$, and none if $\alpha/r > m$.*

# Stability of equilibria (for a class of piecewise constant coefficients)

We now present a preliminary analysis of the stability of the equilibria when

$$k_i \equiv k, \qquad p_i = \begin{cases} 1 & \text{if } i \leqslant N, \\ 0 & \text{if } i \geqslant N+1, \end{cases} \quad \text{and} \quad q_i = \begin{cases} 0 & \text{if } i \leqslant N, \\ 1 & \text{if } i \geqslant N+1, \end{cases}$$

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Introducing new variables

$$u := \sum_{i=0}^{\infty} M_i, \qquad v := \sum_{i=N+1}^{\infty} iM_i, \qquad w := \sum_{i=N}^{\infty} M_i,$$

the silicosis system (1) becomes

$$\begin{cases} \dot{x} = \alpha - kxu + v \\ \dot{u} = r - u \\ \dot{v} = -v + kxw + kNM_N \\ \dot{w} = -w + kxM_{N-1} \\ \dot{M}_0 = r - M_0 - kxM_0 \\ \dot{M}_i = -M_i - kxM_i + kxM_{i-1}, \qquad i \geqslant 1. \end{cases} \tag{6}$$

# Stability of equilibria (for a class of piecewise constant coefficients)

Observe that considering the equations for $(x, u, v, w, M_0, \dots, M_N)$ in system (6) we obtain a closed $(N+5)$-dimensional ODE system. With

## Stability of equilibria (for a class of piecewise constant coefficients)

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$$U_1 := x, \quad U_2 := u, \quad U_3 := v, \quad U_4 := w, \quad U_i := M_{i-5}, \quad 5 \leqslant i \leqslant N+5,$$

we can write that finite $(N+5)$-dimensional system in the form

$$\dot{U} = F(U) \tag{7}$$

with $F : \mathbb{R}^{N+5} \to \mathbb{R}^{N+5}$ defined by

$$F(U) := \begin{pmatrix} \alpha + U_3 - kU_1U_2 \\ r - U_2 \\ -U_3 + kU_1U_4 + NkU_1U_{N+5} \\ -U_4 + kU_1U_{N+4} \\ r - U_5 - kU_1U_5 \\ -U_6 + kU_1U_5 - kU_1U_6 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -U_{N+5} + kU_1U_{N+4} - kU_1U_{N+5} \end{pmatrix}. \tag{8}$$

# Stability of equilibria (for a class of piecewise constant coefficients)

Let $(x^{\text{eq}}, M_0^{\text{eq}}, \dots)$ be an equilibrium of (1). The corresponding equilibrium of (7) (dropping the "eq") is

$$\begin{aligned}
U_1 &= \frac{1}{k}\frac{y}{1-y},\\
U_2 &= r,\\
U_3 &= r\frac{y^{N+1}}{1-y}\big((N+1)-Ny\big),\\
U_4 &= ry^N,\\
U_{i+5} &= r(1-y)y^i, \qquad 0 \leqslant i \leqslant N.
\end{aligned}$$

where

$$y := \frac{x^{\text{eq}}}{x^{\text{eq}} + 1/k}.$$

# Stability of equilibria (for a class of piecewise constant coefficients)

The jacobian matrix of $F$ at $U = (U_1, \ldots, U_{N+5})$ is the matrix

$$DF(U) =: A(y) = \left[\begin{array}{c|c} \underset{(4\times 4)}{B} & \underset{(4\times(N+1))}{C} \\ \hline \underset{((N+1)\times 4)}{D} & \underset{((N+1)\times(N+1))}{E} \end{array}\right]$$

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where $B$ is the $4 \times 4$ matrix

$$B = \begin{bmatrix} -kr & -\dfrac{y}{1-y} & 1 & 0 \\ 0 & -1 & 0 & 0 \\ kry^N\big(1+N(1-y)\big) & 0 & -1 & \dfrac{y}{1-y} \\ kry^{N-1}(1-y) & 0 & 0 & -1 \end{bmatrix},$$

# Stability of equilibria (for a class of piecewise constant coefficients)

$C$ and $D$ are, respectively, the $4 \times (N+1)$ and $(N+1) \times 4$ matrices

$$C = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \dfrac{Ny}{1-y} \\ 0 & \cdots & \dfrac{y}{1-y} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -kr(1-y) & 0 & 0 & 0 \\ kr(1-y)^2 & \vdots & \vdots & \vdots \\ kr(1-y)^2 y & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ kr(1-y)^2 y^{N-1} & 0 & 0 & 0 \end{bmatrix},$$

# Stability of equilibria (for a class of piecewise constant coefficients)

and $E$ is the $(N+1)\times(N+1)$ matrix

$$E = \begin{bmatrix} -\frac{1}{1-y} & 0 & \cdots & \cdots & 0 \\ \frac{y}{1-y} & -\frac{1}{1-y} & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{y}{1-y} & -\frac{1}{1-y} \end{bmatrix}.$$

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In order to study the eigenvalues of $A$ we compute the determinant of $A - \lambda I_{N+5}$,

$$\det(A - \lambda I_{N+5}) = \det \left[ \begin{array}{c|c} B - \lambda I_4 & C \\ \hline D & E - \lambda I_{N+1} \end{array} \right].$$

# Stability of equilibria (for a class of piecewise constant coefficients)

After some lengthy manipulations we arrive at (where $\Delta := 1 + \lambda(1-y)$):

$$\det(A-\lambda I_{N+5}) = (-1)^N kr(1+\lambda)^2(1-y)^{-N-1}\Big\{(1+\lambda)\left(1+\frac{\lambda}{kr}\right)\Delta^{N+1} - \\ - y^N\Big[\Delta^{N+1} + (1-y)\Big(\big(N(1-y)(1+\lambda)+1\big)(1+\cdots+\Delta^N)-1\Big)\Big]\Big\}. \tag{9}$$

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or, separating terms with *kr* from those without:

$$\begin{aligned}\det(A-\lambda I_{N+5}) =& (-1)^N(1-y)^{-N-1}(1+\lambda)^3\lambda\Delta^{N+1} - \\ & - kr(-1)^N(1-y)^{-N-1}(1+\lambda)^2\times \\ & \times\left\{(1+\lambda)\lambda\Delta^{N+1} - y^N\left[\Delta^{N+1} + (1-y)\times\right.\right. \\ & \left.\left.\times\Big(\big(N(1-y)(1+\lambda)+1\big)(1+\cdots+\Delta^N)-1\Big)\right]\right\}\end{aligned} \tag{10}$$

# Stability of equilibria (for a class of piecewise constant coefficients)

From the expression for $\mathcal{F}_{N,k}$ in the bifurcation equation $\mathcal{F}_{N,k}(x^{\text{eq}}) = \alpha/r$, the relation between $y$ and $x^{\text{eq}}$, and using (9), one can write the characteristic equation $\det(A - \lambda I_{N+5}) = 0$ in the form

$$(1-y)^2\widetilde{\mathcal{F}}_N'(y) + \Big(2(1-y)^2\widetilde{\mathcal{F}}_N'(y) + d_1(y)\Big)\lambda + O(|\lambda|^2) = 0 \quad \text{as } \lambda \to 0,$$

where $\widetilde{\mathcal{F}}_N(y) := \mathcal{F}_{N,k}(x^{\text{eq}}(y))$ and $d_1$ is a known function.

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where $\widetilde{\mathcal{F}}_N(y) := \mathcal{F}_{N,k}(x^{\text{eq}}(y))$ and $d_1$ is a known function.

This is the crucial ingredient to establish the following:

**Proposition**

*$\lambda = 0$ is a simple eigenvalue of $A(y^*) = DF(U^*)$ at $(\alpha/r, y) = (\mu^*, y^*)$.*

(where $y^* = y(x^*)$ and $x^*$ is the value of the $U_1$-component of the unique equilibrium of (7) when $\alpha/r$ is equal to the bifurcation value $\mu^*$.)

# Stability of equilibria (for a class of piecewise constant coefficients)

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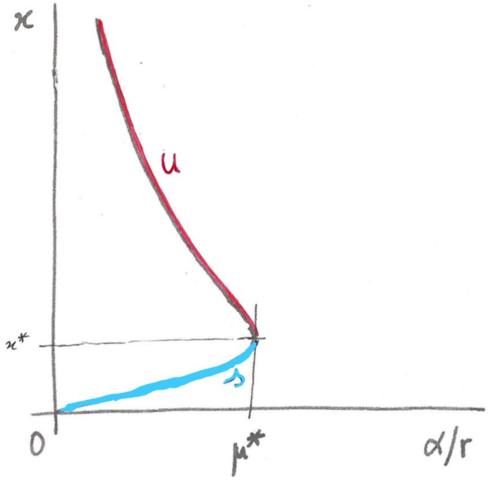

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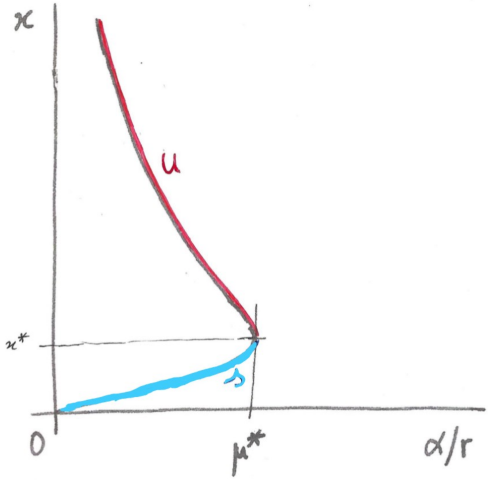
More precisely we prove. . .



# Stability of equilibria (for a class of piecewise constant coefficients)

## Proposition

*Let $\mu^*$ be as before, and let $x^*$ be the value of the $U_1$ component of the unique equilibrium of (7) when $\alpha/r = \mu^*$ (see previous figure.) Then, for every $\alpha/r \in (0, \mu^*)$ and all $kr > 0$, the equilibrium solution $U^{1*}$ with $U_1^{1*} < x^*$ is locally exponentially asymptotically stable, and the equilibrium solution $U^{2*}$, with $U_1^{2*} > x^*$, is unstable.*

# Stability of equilibria (for a class of piecewise constant coefficients)

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## Conjecture

*$\forall 0 < \alpha/r < \mu^*$, and $kr > 0$, the branch of $U^{2*}$ solutions has a 1-dimensional unstable manifold.*

# Stability of equilibria (for a class of piecewise constant coefficients)

Some numerical evidence supporting the conjecture

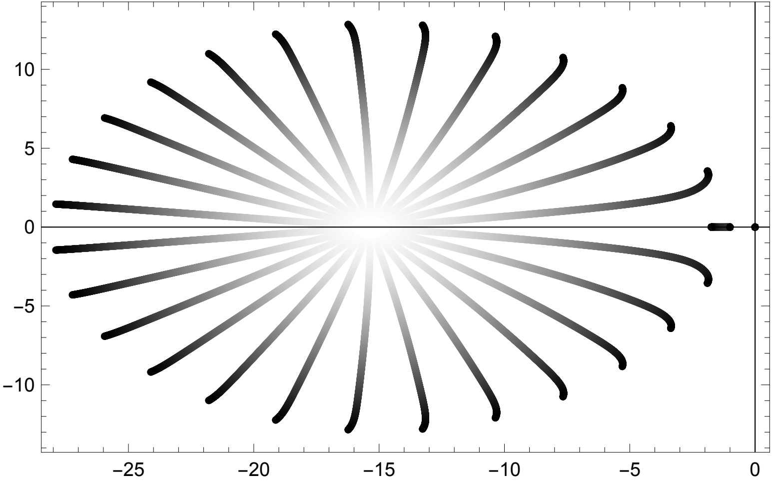


Figure: Spectra of $DF(U^*)$ with $N = 25$, $kr$ from 0 (light gray) to 10 (black).

# Stability of equilibria (for a class of piecewise constant coefficients)

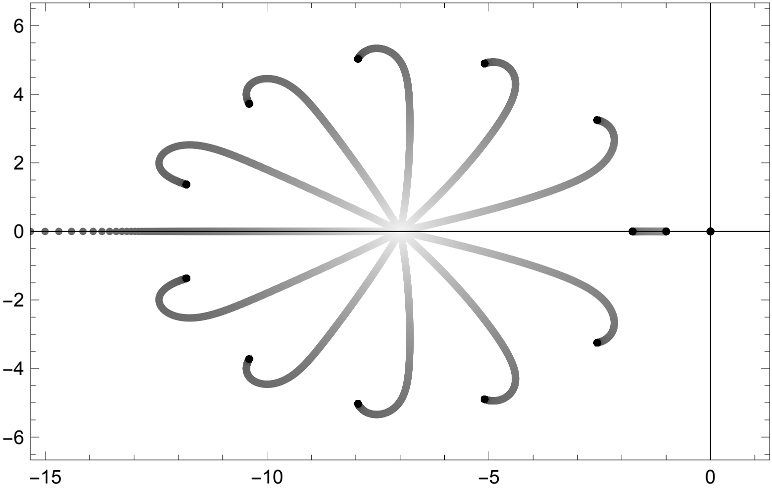


Figure: Spectra of $DF(U^*)$ with $N = 10$, $kr$ from 0 (light gray) to $10^5$ (black).

# Stability of equilibria (for a class of piecewise constant coefficients)

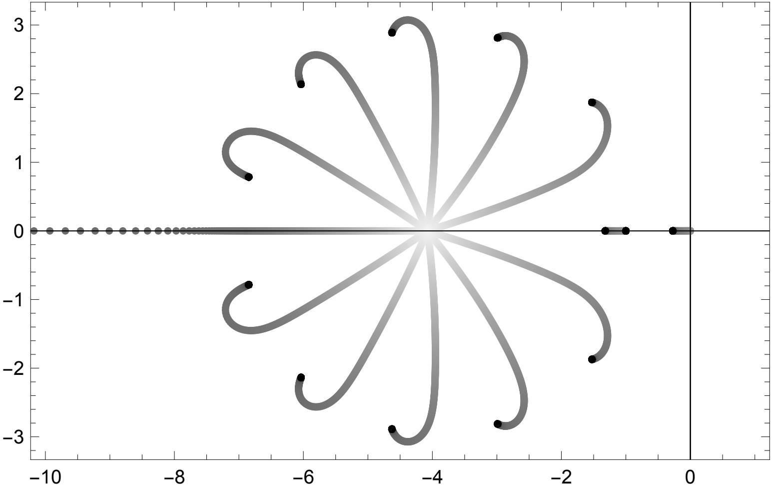


Figure: Spectra of $DF(U^{1*})$ with $N = 10$, $\alpha/r = 2.44$, $kr$ from 0 (light gray) to $10^5$ (black). Eigenvalue close to the origin starts at the origin when $kr = 0$ and moves slowly to $\mathrm{Re}(\lambda) < 0$ as $kr$ increases.

# Stability of equilibria (for a class of piecewise constant coefficients)

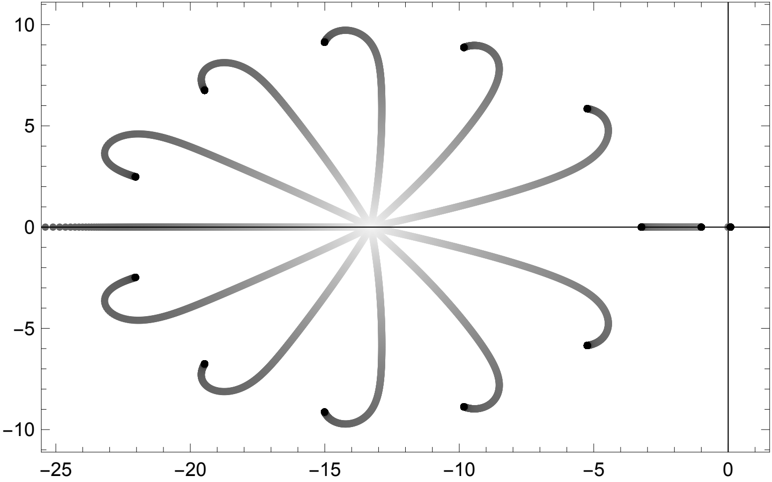


Figure: Spectra of $DF(U^{2*})$ with $N = 10$, $\alpha/r = 2.44$, $kr$ from 0 (light gray) to $10^5$ (black). Eigenvalue close to the origin starts at the origin when $kr = 0$ and moves slowly to $\mathrm{Re}(\lambda) > 0$ as $kr$ increases.

## Stability of equilibria (for a class of piecewise constant coefficients)

Independently of the truthfullness of the conjecture, the result of the last proposition can be used to prove the following local stability result for the full infinite-dimensional system (1)

### Theorem

*Let $\alpha/r < \mu^*$ and let $\widetilde{U}^{eq} = (x^{eq}, M_0^{eq}, M_1^{eq}, \ldots)$ be an equilibrium solution of (1) such that the corresponding equilibrium of the $(N+5)$-dimensional system (7), $U^{eq} = (U_1^{eq}, \ldots, U_{N+5}^{eq})$, is locally exponentially asymptotically stable.*
*Then, $\widetilde{U}^{eq}$ is a locally exponentially asymptotically stable solution of (1) in the strong topology of $X$.*

# Stability of equilibria (for a class of piecewise constant coefficients)

The proof of the asymptotic stability in the strong topology of $X$ is based on:

- $\|\widetilde{U}(t)\| = x(t) + u(t) + v(t) + \sum_{i=0}^{N} iM_i(t), \quad$ for $\widetilde{U} \in X_+$
- the locally asymptotic stability of the solution to the finite dimensional system, which implies that $\|\widetilde{U}(t)\| \to \|\widetilde{U}^{\mathrm{eq}}\|$
- the general result valid in $X$ guaranteeing that if $\widetilde{U}_j(t) \to \widetilde{U}_j^{\mathrm{eq}}$ and $\|\widetilde{U}(t)\| \to \|\widetilde{U}^{\mathrm{eq}}\|$, then $\|\widetilde{U}(t) - \widetilde{U}^{\mathrm{eq}}\| \to 0$.

# Stability of equilibria (for a class of piecewise constant coefficients)

The proof that the convergence to zero of $\|\widetilde{U}(t) - \widetilde{U}^{\text{eq}}\|$ is exponential entails:

- using $\|\boldsymbol{\delta}(t)\| = |x(t) - x^{\text{eq}}| + \|\boldsymbol{\delta}(t)\|_{\ell^1} + \|\boldsymbol{\xi}(t)\|_{\ell^1}$, where $\boldsymbol{\delta}(t) = (\delta_j(t)) := (M_j(t) - M_j^{\text{eq}})$ and $\boldsymbol{\xi}(t) := (j\delta_j(t))$
- proving that $\boldsymbol{\delta}(t)$ converges locally exponentially to zero in $\ell_1$
- proving that $\boldsymbol{\xi}(t)$ converges locally exponentially to zero in $\ell_1$
- the last two steps are achieved because the evolutions of $\boldsymbol{\delta}(t)$ and $\boldsymbol{\xi}(t)$ are governed by equations for which the **variation of constants formula** together with appropriate estimates on the intervenient functions can be used.

Thank you!