Order Preserving Semiflows Revisited

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CAMGSD - Instituto Superior Técnico

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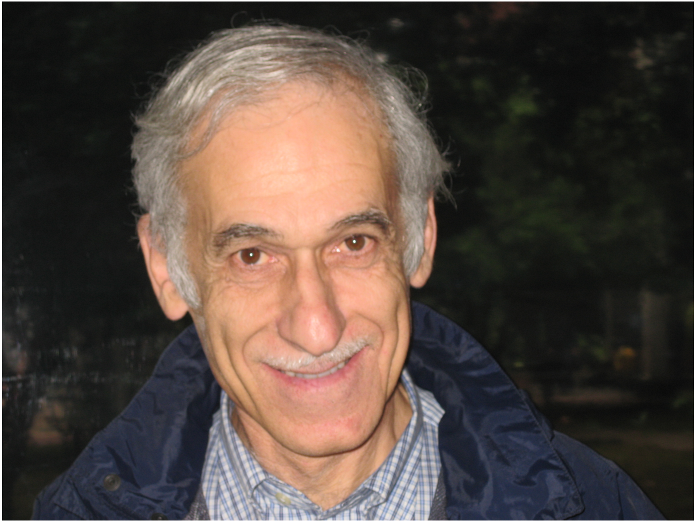
In honor of Giorgio Fusco

To Giorgio Fusco



Introduction

Linear scalar parabolic equations exhibit a natural discrete Liapunov functional (the zero crossing number):

The number of zeros of a solution is a nonincreasing function of time.

This is at the root of a most complete description of the geometric properties of nonlinear semiflows generated by semilinear scalar parabolic equations.

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(ii) Do such functionals determine a class of problems?

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Outline

- The zero number in linear parabolic equations
- Semiflows with discrete Lyapunov functions
- Linear operators with the zero number property
- Order structures preserved by semigroups
- A class of order preserving semiflows

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Linear scalar parabolic equations

For a positive, consider the linear problem

$$(P) \qquad u_t = a(x)u_{xx} + b(t,x)u_x + c(t,x)u,\ 0 < x < 1,\ u_x(0) = u_x(1) = 0\ ,$$

and let

$$C_n^1([0,1]) = \{\varphi \in C^1([0,1]) : \varphi'(0) = \varphi'(1) = 0\}\ .$$

Then (P) defines an injective solution operator

$$S(t,t_0) : C_n^1([0,1]) \to C_n^1([0,1]), t \geq t_0\ .$$

If $b = b(x)$, $c = c(x)$, (P) generates a C_0-semigroup $\{T(t) = S(t,0)\}_{t\geq 0}$.

Define the zero number $z : C_n^1([0,1]) \setminus \{0\} \to \mathbb{N}_0 \cup \{\infty\}$:

$$z(\varphi) = \#\{\text{ strict sign changes of } \varphi \text{ in } [0,1]\}$$

$z(S(t,t_0)\varphi)$ is a monotone nonincreasing function of $t > t_0$,

[Sturm, Niquel, Matano, Henry, Angenent, . . .]

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Let $\mathcal{N} \subset C^1_n([0,1])$ denote the dense subset of functions with all zeros nondegenerate (i.e. $\varphi(x) = 0 \Rightarrow \varphi'(x) \neq 0$).

Theorem [Angenent '1988]

For $\psi \in C^1_n([0,1])$, $\psi \neq 0$,

(i) the set $\Theta = \{t \in (t_0, +\infty) : S(t, t_0)\psi \notin \mathcal{N}\}$ is finite;

(ii) for $t \in \Theta$ there exists an ε_0 such that for $0 < \varepsilon < \varepsilon_0$,

$$z(S(t+\varepsilon, t_0)\psi) < z(S(t-\varepsilon, t_0)\psi)\,.$$

Discrete Lyapunov functions

Semilinear reaction-diffusion equations

$$u_t = u_{xx} + f(x, u, u_x),\ 0 < x < 1,\ u_x(0) = u_x(1) = 0\ ,$$

with $f \in C^2([0,1] \times \mathbb{R}^2, \mathbb{R})$, define semiflows in $X = H^1(0,1)$. The zero number z provides a discrete Lyapunov function for the difference of any two solutions u_1, u_2

$$z(u_1(t,\cdot) - u_2(t,\cdot)) \text{ is nonincreasing for } t > 0\ .$$

A different example is provided by differential delay equations

$$\dot{x}(t) = h(x(t), x(t-1))$$

with $h \in C^2(\mathbb{R}^2, \mathbb{R})$ and monotone feedback conditions $h_v(u,v) \lessgtr 0$, defining semiflows in $X = C^0[-1, 0]$.

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For the $(\pm)$ feedback conditions, let

$$V^+(\varphi) = 2\lfloor \frac{z(\varphi)+1}{2} \rfloor \quad , \quad V^-(\varphi) = 2\lfloor \frac{z(\varphi)}{2} \rfloor + 1 \ ,$$

notice that $V^+(\varphi)$ is even and $V^-(\varphi)$ is odd.

Then, in each feedback case, $V^\pm$ provides a discrete Lyapunov function for the difference of any two solutions $x_t^j(\theta) = x^j(t+\theta), \theta \in [-1, 0], j = 1, 2,$

$V^\pm(x_t^1 - x_t^2)$ is nonincreasing for $t > 0$.

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[Myschkis, Mallet-Paret, Sell, ...]

Generators of semiflows with zero number decay

Theorem

Let A be the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$, $T(t) : C^1_n([0,1]) \to C^1_n([0,1])$, such that

(i) the set $\Theta = \{t \in (t_0, +\infty) : T(t)\psi \notin \mathcal{N}\}$ is finite;

(ii) $z(T(t_1)\varphi) \geq z(T(t_2)\varphi)$ for all $0 < t_1 \leq t_2$.

If $D(A) = C^3([0,1]) \cap C^1_n([0,1])$, then there exist $\alpha, \beta, \gamma \in C^1(0,1)$, with α nonnegative, such that for all $\varphi \in D(A)$ we have

$$(A\varphi)(x) = \alpha(x)\varphi_{xx}(x) + \beta(x)\varphi_x(x) + \gamma(x)\varphi(x),\ 0 < x < 1.$$

Sketch of the proof

Step 1: Use the Taylor expansion

Introduce families of functions $a^\xi, b^\xi, c^\xi : \xi \in [0,1] \mapsto D(A)$ such that

$$\begin{aligned} &a^\xi(\xi) = a^\xi_x(\xi) = b^\xi(\xi) = 0 \ , \ \xi \in [0,1] \\ &a^\xi_{xx}(\xi) = c^\xi(\xi) = 1 \ , \ \xi \in [0,1] \\ &b^\xi_x(\xi) = 1 \ , \ \xi \in (0,1) \\ &a^\xi_x(x)(x-\xi) > 0 \ , \ x \in (0,1) \setminus \{\xi\} \end{aligned}$$

Given $\varphi \in D(A)$ we can uniquely decompose φ in the form

$$\varphi = \varphi(x)c^x + \varphi_x(x)b^x + \varphi_{xx}(x)a^x + \psi^{x,\varphi}$$

where $\psi^{x,\varphi} \in D(A)$ satisfies $\psi^{x,\varphi}(x) = \psi^{x,\varphi}_x(x) = \psi^{x,\varphi}_{xx}(x) = 0.$

Then, if $\left(A\psi^{x,\varphi}\right)(x) = 0$, we obtain

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Step 2: Use $\psi^\xi = \psi^{\xi,\varphi}$ to perturb all degenerate zeros $x \neq \xi$

Let $\quad \varphi^{\xi,\lambda} = a^\xi + \lambda\psi^\xi$. $\quad$ Notice that a^ξ has a degenerate zero at $x = \xi$.

At a degenerate zero $x \neq \xi$ of $\varphi^{\xi,\lambda}$ we have

$$\begin{cases} \varphi^{\xi,\lambda}(x) &= a^\xi(x) + \lambda\psi^\xi(x) &= 0\,, \\ \varphi_x^{\xi,\lambda}(x) &= a_x^\xi(x) + \lambda\psi_x^\xi(x) &= 0\,. \end{cases}$$

Then

$$\Psi(x) = \frac{\psi^\xi(x)}{a^\xi(x)} = \frac{1}{\lambda} \quad , \quad \Psi_x(x) = \frac{\psi_x^\xi(x)a^\xi(x) - \psi^\xi(x)a_x^\xi(x)}{(a^\xi(x))^2} = 0\,,$$

and $x \neq \xi$ is a degenerate zero of $\varphi^{\xi,\lambda}$ if and only if λ^{-1} is a critical value of $\Psi : [0,1] \setminus \{\xi\} \to \mathbb{R}$. By Sard the set of $\lambda \in \mathbb{R}$ such that $\varphi^{\xi,\lambda} : [0,1] \setminus \{\xi\} \to \mathbb{R}$ has a degenerate zero has Lebesgue measure zero.

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Step 3: Show that $\left(A\psi^{x,\varphi}\right)(x) = 0$

Again let $\varphi^{\xi,\lambda} = a^{\xi} + \lambda\psi^{\xi}$ with $\psi^{\xi} = \psi^{\xi,\varphi}$. For small $t > 0$ we have

$$\left(T(t)\varphi^{\xi,\lambda}\right)(\xi) = \left(T(t)\varphi^{\xi,\lambda}\right)(\xi) - \varphi^{\xi,\lambda}(\xi)$$

$$= \left(\left(Aa^{\xi}\right)(\xi) + \lambda\left(A\psi^{\xi}\right)(\xi)\right) t + (o(t))(\xi) + \lambda(o(t))(\xi).$$

If $\left(A\psi^{\xi}\right)(\xi) \neq 0$, there is $\lambda_0 \in \mathbb{R}$ (suficiently large) such that:

$$\left(T(t)\varphi^{\xi,\lambda_0}\right)(\xi) < 0\ ,\ 0 < t \leq t_0;$$

Moreover, φ^{ξ,λ_0} has exactly one degenerate zero in $[\xi - \delta, \xi + \delta]$ and a finite number of nondegenerate zeros in $[0,1] \setminus (\xi - \delta, \xi + \delta)$.

Then, there is $\phi \in D(A)$ (close to φ^{ξ,λ_0}) such that we obtain the contradiction

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Order structures preserved by semigroups

The discrete Liapunov functional z endows X with an order structure preserved by the semigroup $T(t)$ [Fusco+Lunel '1997]:

- z defines on X an *order structure* $(\mathcal{N}, \sim, <)$ if:

  (a) z defines in $\mathcal{N}$ an *equivalence relation* $\sim$

  $$\text{for } \varphi, \psi \in \mathcal{N} \qquad \varphi \sim \psi \text{ iff } z(\varphi) = z(\psi)$$

  (b) z defines in $\mathcal{N}/\sim$ a *total order* $<$

  $$\text{for } [\varphi], [\psi] \in \mathcal{N}/\sim \qquad [\varphi] < [\psi] \text{ iff } z(\varphi) = z(\psi)$$

- $T(t)$ *preserves the order structure* $(\mathcal{N}, \sim, <)$ if:

  (a) $z(T(t)\varphi)$ is defined for all $t \in \Phi = \mathbb{R}_+ \setminus \{$ discrete set $\}$

  (b) $z(T(t')\varphi) \leq z(T(t)\varphi)$ for all $t, t' \in \Phi$, $t' > t$

  (c) $(\mathcal{N}/\sim)$ is complete $\qquad [\varphi]_1 \leftarrow [\varphi]_2 \leftarrow [\varphi]_3 \leftarrow \dots$

Order structures preserved by semigroups

The discrete Liapunov functional z endows X with an order structure preserved by the semigroup $T(t)$ [Fusco+Lunel '1997]:

- z defines on X an *order structure* $(\mathcal{N}, \sim, <)$ if:

  (a) z defines in $\mathcal{N}$ an *equivalence relation* $\sim$

  $$\text{for } \varphi, \psi \in \mathcal{N} \qquad \varphi \sim \psi \text{ iff } z(\varphi) = z(\psi)$$

  (b) z defines in $\mathcal{N}/\sim$ a *total order* $<$

  $$\text{for } [\varphi], [\psi] \in \mathcal{N}/\sim \qquad [\varphi] < [\psi] \text{ iff } z(\varphi) = z(\psi)$$

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  (a) $z(T(t)\varphi)$ is defined for all $t \in \Phi = \mathbb{R}_+ \setminus \{ \text{ discrete set } \}$

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A class of order preserving semiflows

Do different order structures determine other classes of linear operators A?

For instance, the semigroup $T(t)$ with periodic boundary conditions in fact preserves the order structure defined by $V^+ : \mathcal{N} \to \{0, 2, 4, \dots\}$.

Therefore, it is natural to consider nonseparated linear boundary conditions

$$C^1_{bc}([0,1]) = \{\varphi \in C^1([0,1]) : B_0(\varphi) = B_1(\varphi) = 0\}$$

where $B_0, B_1 : C^1([0,1]) \to \mathbb{R}$ are the boundary operators

$$\begin{cases} B_0(\varphi) = \varphi'(0) + \delta_{00}\varphi(0) + \delta_{01}\varphi(1) \\ B_1(\varphi) = \varphi'(1) + \delta_{10}\varphi(0) + \delta_{11}\varphi(1) \end{cases}$$

[Coddington+Levinson]

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[Coddington+Levinson]

Theorem

Let A be the infinitesimal generator of a C_0-semigroup of bounded linear operators $\{T(t)\}_{t\geq 0}$, $T(t) : C^1_{bc}([0,1]) \to C^1_{bc}([0,1])$ that preserves the order structure $(\mathcal{N}, \sim, <)$ defined by $V^-(V^+)$.
If $D(A) = C^3([0,1]) \cap C^1_{bc}([0,1])$, then there exist $\alpha, \beta, \gamma \in C^1(0,1)$, with α nonnegative, such that for all $\varphi \in D(A)$ we have

$$(A\varphi)(x) = \alpha(x)\varphi_{xx}(x) + \beta(x)\varphi_x(x) + \gamma(x)\varphi(x), \; 0 < x < 1 \; .$$

Furthermore, the cross-boundary constants satisfy

$$\delta_{01} \leq 0 \quad , \quad \delta_{10} \geq 0 \qquad (\delta_{01} \geq 0 \quad , \quad \delta_{10} \leq 0) \; .$$

The cross-boundary conditions essentially prevent zeros to occur on the boundary when $z(\varphi)$ is even, since in this case the cross-boundary values of φ' and φ have the wrong sign:

$$\varphi'(0) = -\delta_{01}\varphi(1) \text{ if } \varphi(0) = 0 \quad \text{and} \quad \varphi'(1) = -\delta_{10}\varphi(0) \text{ if } \varphi(1) = 0.$$

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A degenerate illustration

The cross-boundary conditions play the role of negative feedback conditions.

As a highly degenerate illustration consider $\alpha = \gamma = 0$, $\beta = 1$. Then we have

$$u_t = u_x \ , \quad 0 < x < 1 \ ,$$

and the solutions $u(x,t) = \phi(t + x)$ correspond to left translations along $[0, 1]$.

Take the boundary operator B_1 with $\delta_{11} = 0$ and $\delta_{10} = \delta > 0$. This implies

$$u_x(1) = -\delta u(0) \ ,$$

and we otain the negative feedback differential delay equation

$$\dot{u}(t) = -\delta u(t-1) \ .$$

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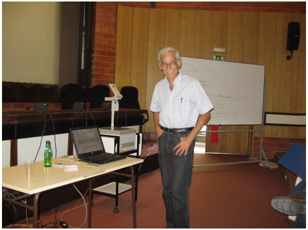
Thank you

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