Weyl Law for the Volume Spectrum

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Summary

Part 1: Non-linear Spectrum

1. Volume spectrum;
2. Weyl Law for volume spectrum.

Part 2: Connection with other problems

1. Existence of minimal hypersurfaces;
2. Volume of nodal sets.

Background

- Given a $n \times n$ symmetric matrix A, the p-th eigenvalue λ_p is computed as

$$\lambda_p = \min_{\{p\text{-plane } P \subset \mathbb{R}^n\}} \max_{v \in P - \{0\}} \frac{\langle v, A(v) \rangle}{|v|^2}.$$

- From the variational property one finds $u \in \mathbb{R}^n$ with $A(u) = \lambda_p u$.

Background

- Given a closed manifold (M, g), consider the Hilbert space

$$W^{1,2}(M) = \left\{\text{all functions } f \text{ with } \int_M (f^2 + |\nabla f|^2) < \infty\right\}.$$

The Laplacian Δ is a symmetric operator and the quadratic form becomes $\langle -\Delta f, f\rangle = \int_M |\nabla f|^2$.

- Like before the p-th eigenvalue is given by

$$\lambda_p(M) = \inf_{\{(p+1)\text{-plane } P \subset W^{1,2}\}} \sup_{f \in P - \{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2}.$$

Using the variational property one obtains a smooth function f such that $\Delta f = -\lambda_p f$.

Background

Eigenvalues for the Laplacian appear in many natural contexts in Mathematical Physics. Lorentz in 1910 gave a lecture in Gottingen in which he asked

“It is here that there arises the mathematical problem to prove that the number of sufficiently high overtones which lie between ν an $\nu + d\nu$ is independent of the shape of the enclosure and is simply proportional to its volume. For many shapes for which calculations can be carried out, this theorem has been verified.”

Weyl Law, 1911: *The asymptotic behavior of* $\{\lambda_p\}_{p\in\mathbb{N}}$ *depends only on the volume of* M*:*

$$\lim_{p\to\infty} \lambda_p(M) p^{-\frac{2}{n+1}} = a(n) vol(M)^{-\frac{2}{n+1}},$$

where $a(n) = 4\pi^2 vol(B)^{-\frac{2}{n+1}}$ *and* B *is the unit ball in* $\mathbb{R}^{n+1}$*.*

Minakshisundaram-Pleijel, 1949: *The Weyl Law also holds for compact Riemannian manifolds* (M, g)*.*

Another point of view

• On $W^{1,2}(M)$, identify a function f with all its constant multiples and denote the space of all such equivalence classes $[f]$ by $\mathcal{P}$.

• The space $\mathcal{P}$ is homeomorphic to $\mathbb{RP}^\infty$ and a $(p+1)$–plane in $W^{1,2}(M)$ becomes a p-projective space $\mathbb{RP}^p$ in $\mathcal{P}$.

• The Raleigh quotient $R([f]) = \frac{\int_M |\nabla f|^2}{\int_M f^2}$ is well defined on $\mathcal{P}$ because it is scale invariant.

• We have $\lambda_p(M) = \inf_{\{p\text{–projective space } Q \subset \mathcal{P}\}} \sup_{[f] \in Q} R([f])$.

Another point of view

$\lambda_p(M) = \inf_{\{p\text{–projective space } Q \subset \mathcal{P}\}} \sup_{[f]\in Q} R([f]).$

Advantage: Linear structure is gone! One only needs

- A space $\mathcal{Z}$ that is (weakly) homeomorphic to $\mathbb{RP}^\infty$.
- A functional $F : \mathcal{Z} \to [0, \infty]$;

The cohomology ring of $\mathcal{Z}$ is generated by $\lambda \in H^1(\mathcal{Z}; \mathbb{Z}_2)$ and we replace $\mathbb{RP}^p \subset \mathbb{RP}^\infty$ by sets $Q \subset \mathcal{Z}$ such that λ^p doesn't vanish on Q.

The p-th (nonlinear) eigenvalue is defined as

$$\lambda_p = \inf_{\{Q \subset \mathcal{Z} \text{ that detect } \lambda^p\}} \sup_{x \in Q} F(x).$$

Volume Spectrum

(M^{n+1}, g) closed compact Riemannian n-manifold, $2 \leq n \leq 6$.

- The space will be $\mathcal{Z}_n(M; \mathbb{Z}_2) =$ "{all compact hypersurfaces in M}" $=$ {integral mod 2 cycles}.

- The functional will be the volume functional $vol : \mathcal{Z}_n(M; \mathbb{Z}_2) \to [0, \infty]$.

- Using the fact that a nontrivial element in $\pi_k(\mathcal{Z}_n(M; \mathbb{Z}_2), \{0\})$ gives rise to a nontrivial element in $H_{n+k}(M; \mathbb{Z}_2)$, Almgren in the 60's showed that $\mathcal{Z}_n(M; \mathbb{Z}_2)$ is weakly homotopic to $\mathbb{RP}^\infty$.

- The p-width is given by

$$\omega_p(M) := \inf_{\{Q \subset \mathcal{Z}_n(M;\mathbb{Z}_2) \text{ that detect } \lambda^p\}} \sup_{T \in Q} vol(T).$$

The sequence $\{\omega_p(M)\}_{p \in \mathbb{N}}$ is called the *volume spectrum* of (M, g).

Volume Spectrum

Theorem (Gromov, 80's, Guth, '07) *There is* $C = C(M, g)$ *so that for all* $p \in \mathbb{N}$

$$C^{-1} p^{\frac{1}{n+1}} \leq \omega_p(M) \leq C p^{\frac{1}{n+1}}$$

• The upper bound follows from constructing a nice map (more on that later)

$$\Phi : \mathbb{RP}^p \to \mathcal{Z}_n(M; \mathbb{Z}_2) \quad \text{with } \Phi^* \lambda^p \neq 0$$

• The lower bound follows because if we fix p disjoint regions $\{B_i\}_{i=1}^p$ in M then any set that detects λ^p contains an element T that divides the volume of every B_i in half and so

$$vol(T) \geq \sum_{i=1}^{p} vol(T \cap B_i) \gtrsim \sum_{i=1}^{p} |B_i|^{n/(n+1)} \simeq p^{1/(n+1)}.$$

Weyl Law

- Recall the Weyl Law for the Laplacian spectrum:

$$\lim_{p\to\infty} \frac{\lambda_p(M)}{p^{\frac{2}{n+1}}} = \frac{4\pi^2}{(vol\, B^{n+1})^{\frac{2}{n+1}}}(vol\, M)^{-\frac{2}{n+1}}.$$

Inspired by it, Gromov conjectured

Conjecture (Gromov's Weyl Law, '03): *There is $\alpha(n)$ such that for all (M^{n+1}, g)*

$$\lim_{p\to\infty} \frac{\omega_p(M)}{p^{\frac{1}{n+1}}} = \alpha(n)|M|^{\frac{n}{n+1}}.$$

Theorem (Liokumovich–Marques–N, '16) *Let (M^{n+1}, g) be a compact manifold (with possible $\partial M \neq 0$). There is $\alpha(n)$ such that*

$$\lim_{p\to\infty} \omega_p(M) p^{-\frac{1}{n+1}} = \alpha(n)|M|^{\frac{n}{n+1}}.$$

Weyl Law

Weyl Law (L-M-N, '16) $\lim_{p\to\infty} \omega_p(M) p^{-\frac{1}{n+1}} = \alpha(n)|M|^{\frac{n}{n+1}}.$

- Weyl's idea was to approximate a region in space by a union of many small disjoint cubes and then use the fact the Laplacian spectrum for a disjoint union of cubes can be computed explicitly to deduce his Weyl Law.

- Unlike the Laplacian spectrum, the volume spectrum, due to being a non-linear problem, is not known on any single example.

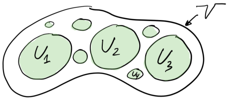
- We believe our proof can be used to prove Weyl Law's for other non-linear spectrums, like the spectrum for the p-Laplacian (as conjectured by Friedlander)

Weyl Law

Weyl Law (L-M-N, '16) $\lim_{p\to\infty} \omega_p(M) p^{-\frac{1}{n+1}} = \alpha(n)|M|^{\frac{n}{n+1}}$.

• The key new ingredient we showed was the following Supperaditivity inequality. Set $\tilde{\omega}_p(\Omega) = \omega_p(\Omega) p^{-\frac{1}{n+1}}$

Let U, V be two unit volume regions of $\mathbb{R}^{n+1}$ and assume that $\{U_i\}_{i=1}^N \subset V$ are disjoint regions all similar to U.



Then, for all $p \in \mathbb{N}$,
$\tilde{\omega}_p(V) \gtrsim \sum_{i=1}^N vol(U_i)\tilde{\omega}_{p_i}(U),$ *where $p_i = [p vol(U_i)]$.*

• Using this, we were able to prove a Weyl Law without knowing the value of $\alpha(n)$ when M is a region of space. Using cut and paste arguments we were able to deduce the case of a general closed manifold from the previous case.

Minimal surfaces

• In the same way that eigenvalues are realized by eigenfunctions we have this great connection to the theory of minimal hypersurfaces:

Theorem (Pitts, '81, Schoen–Simon, '82) *For all* $p \in \mathbb{N}$ *there is an embedded minimal hypersurface* Σ_p *(with multiplicities) so that* $\omega_p(M) = vol(\Sigma_p)$.

Beware: Σ_p can be, for instance, 3Σ where Σ is a embedded minimal hypersurface.

• Computing $\omega_p(S^3)$ is an important but hard problem. It is simple to see that $\omega_1(S^3) = \ldots = \omega_4(S^3) = 4\pi$ and the minimal surface is the equator.

• Nurser showed that $\omega_5(S^3) = \ldots = \omega_7(S^3) = 2\pi^2$ and the minimal surface is the Clifford torus. He had to use the proof of the Willmore Conjecture.

• He also showed that $2\pi^2 < \omega_9(S^3) < 8\pi$. Which minimal surface is it?

Minimal surfaces

- Birkhoff, (1917) *Every* (S^2, g) *admits a closed geodesic.*
- Lusternik–Schnirelmann, (1929–1947) *Every* (S^2, g) *has three distinct simple closed geodesics.*
- Pitts (1981), Schoen–Simon, (1982) *Every compact manifold* (M, g) *admits an embedded minimal hypersurface smooth outside a set of codimension 7.*
- Franks (1992), Bangert (1993), Hingston (1993) *Every* (S^2, g) *has an infinite number of closed geodesics.*

Yau's Conjecture '82 *Every compact* 3*-dimensional manifold admits an infinite number of immersed minimal surfaces.*

Theorem (Marques–N., '13) *Assume* (M^{n+1}, g) *has positive Ricci curvature.*
Then M *admits an infinite number of distinct embedded minimal hypersurfaces.*

Minimal surfaces

• Uhlenbeck showed that, for a generic set of metrics, the eigenvalues $\lambda_p(M)$ are simple and the eigenfunction f_p has Morse index p for all $p \in \mathbb{N}$.

Multiplicity One Conjecture (Marques–N, '15) *For bumpy metrics (M, g) (generic condition), unstable components of minimal hypersurfaces Σ_p with $\omega_p(M) = vol(\Sigma_p)$ have multiplicity one.*

• We confirmed this conjecture when $p = 1$. Assuming the Multiplicity One Conjecture we can show that:

For every $p \in \mathbb{N}$, the minimal hypersurface Σ_p with $vol(\Sigma_p) = \omega_p(M)$ has Morse index p and so they are all genuinely different.

• In particular this would imply that (M^{n+1}, g) would have an infinite number of distinct minimal hypersurfaces for bumpy metrics and thus solve a stronger version of Yau's conjecture in this case.

Volume of Nodal sets

• Let $\phi_0, \dots, \phi_p$ be the first $(p+1)$-eigenfunctions for the Laplacian in (M, g). The zero set $\{\phi_p = 0\}$ is called a nodal set.

• Nodal sets give us a way to construct maps into $\mathcal{Z}_n(M; \mathbb{Z}_2)$ thss making a connection between them and the volume spectrum. More precisely, there is a "natural" map that detects λ^p

$$\Phi_p : \mathbb{RP}^p \to \mathcal{Z}_n(M; \mathbb{Z}_2),$$
$$\Phi_p([a_0, \dots, a_p]) = \{x \in M : a_0\phi_0(x) + \dots + a_p\phi_p(x) = 0\}.$$

• On round S^3 we can use Crofton formula to estimate $\sup_{z \in \mathbb{RP}^p} vol(\Phi_p(z))$ and conclude that $\alpha(2) \leq (48/\pi)^{1/3}$. I would expect this to be sharp.

Volume of Nodal sets

Recall the map

$$\Phi_p : \mathbb{RP}^p \to \mathcal{Z}_n(M; \mathbb{Z}_2),$$
$$\Phi_p([a_0, \ldots, a_p]) = \{x \in M : a_0\phi_0(x) + \ldots + a_p\phi_p(x) = 0\}.$$

Yau Conjecture *The volume of the nodal sets* $\{\phi_p = 0\}$ *grows like* $p^{1/(n+1)}$*.*

Proven by Donnelly–Fefferman in the analytic case and the lower bound has been recently proven by Logunov for general metrics.

• Weyl Law implies a universal lower bound for $\lim_{p\to\infty} \sup_{z\in\mathbb{RP}^p} \frac{vol(\Phi_p(z))}{p^{1/(n+1)}}$.

• Is $\sup_{x\in\mathbb{RP}^p} vol(\Phi_p(x))$ asymptotically optimal, as I would expect? If true, $vol(\{\phi_p = 0\})p^{-1/(n+1)}$ would be bounded from above and Yau's conjecture would follow.

Conclusion

This is an exciting moment with lots of activity by young people:

- X. Zhou studied one parameter min-max for positive Ricci curvature;
- Montezuma constructed min-max hypersurfaces intersecting a concave set;
- Liokumovich and Chambers showed that minimal hypersurfaces always exist on complete manifolds with finite volume;
- Liokumovich and Glynn-Adey found universal bounds for the k-widths;
- Ketover and Zhou studied min-max for self-shrinkers;
- Ketover studied genus estimates for min-max in the surface case;
- Guaraco did min-max for Allen-Cahn equation; Guaraco-Gaspar studied non-linear spectrum for Allen-Cahn equation;
- Song showed that the least area minimal surface is always embedded;
- Compactness properties of minimal hypersurfaces with bounded index: Sharp, Buzano–Sharp, Carlotto, Chodosh–Ketover–Maximo, Li-Zhou;

Thank You!