YAMABE SYSTEMS, OPTIMAL PARTITIONS AND NODAL SOLUTIONS TO THE YAMABE EQUATION

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Yamabe systems, optimal partitions, and nodal solutions to the Yamabe equation
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Main points:

- Optimal Partition Problems on manifolds: shape optimization problems involving partitions on manifolds
- The energy functional is related to the Yamabe equation
- Explore a connection with elliptic semilinear systems
- Sign-changing solutions to the Yamabe equation

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Keywords:

- A priori estimates for elliptic systems
- Variational Methods, compactness conditions
- Monotonicity formula and free boundary regularity for equations and systems with variable coefficients.

1. Background:
   1.1 What is an Optimal Partition Problem?
   1.2 The Yamabe Equation
2. Statement of our results
3. Outline of some proofs

BACKGROUND

- Class of admissible sets: $\mathcal{A}(\Omega)$
- Cost Function: $\Phi : \mathcal{A}(\Omega)^\ell \to \mathbb{R}$

Minimization problem:

$$\inf \{\Phi(\omega_1, \ldots, \omega_\ell) : \ \omega_i \in \mathcal{A}(\Omega), \ \omega_i \cap \omega_j = \emptyset \ \forall i \neq j\}$$

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Examples

1) (Spectral minimal partitions)
$\Omega \subset \mathbb{R}^m$, $\mathcal{A}(\Omega) = \{\omega \subset \Omega \text{ open}\}$, $k \in \mathbb{N}$:

$$\Phi\left(\omega_1, \ldots, \omega_l\right) = \sum_{i=1}^{l} \lambda_k\left(w_i\right)$$

[Bucur, Buttazzo, Henrot, 1998], [Conti-Terracini-Verzini, 2005], [Caffarelli-Lin, 2007], [Boundin-Bucur-Oudet, 2010], [Ramos-T. Terracini, 2016],...

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Examples

2) (Nonlinear eigenvalues) $\Omega \subset \mathbb{R}^m$, $\mathcal{A}(\Omega) = \{\omega \subset \Omega \text{ open}\}$, $\lambda \in (-\lambda_1(\Omega), \infty)$,

$$-\Delta u + \lambda u = |u|^{p-2}u, \ u \in H_0^1(w_i), \quad 2 < p < 2^*$$

$$w_i \mapsto c(w_i) \ \text{ least energy level,}$$

$$\Phi(\omega_1, \ldots, \omega_l) = \sum_{i=1}^{l} c(w_i)$$

[Conti-Terracini -Verzini, 2002, 2003, 2005], [Tavares-Terracini, 2012],. . .

$l = 2 \longrightarrow$ least energy nodal solution

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Examples

3) (Nonlinear eigenvalues - critical) $\lambda \in (-\lambda_1(\Omega), 0)$,

$$-\Delta u + \lambda u = |u|^{2^*-2} u, \ u \in H_0^1(w_i),$$
$$w_i \mapsto c(w_i) \ \text{least energy level},$$

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[Chen-Zou, 2012, 2015], [T.-You, 2020], [T.-You-Zou, 2022],...

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Examples

4) (Manifolds - sphere)

[Szulkin-Clapp-Saldaña, 2020]:

optimal partition problems related to the Yamabe equation on the sphere + symmetries

$$-\Delta_g u + \kappa_m S_g u = |u|^{2^*-2} u \qquad \text{on } M, \tag{Yamabe}$$

where:

- (M, g) is a closed Riemannian manifold of dimension $m \geq 3$;
- S_g scalar curvature;
- $\Delta_g := \mathrm{div}_g \nabla_g$ is the Laplace-Beltrami operator
- $\kappa_m := \frac{m-2}{4(m-1)}$
- $2^* := \frac{2m}{m-2}$ is the critical Sobolev exponent.

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If we write $\rho = u^{\frac{4}{m-2}}$ for $u > 0$ smooth, and let

- S_g - scalar curvature of (M, g);
- $S_{\widetilde{g}}$ - scalar curvature of $(M, \widetilde{g})$;

then

$$-\frac{4(m-2)}{2(m-1)}\Delta_g u + S_g u = S_{\widetilde{g}} u^{\frac{m+2}{m-2}} \iff -\Delta_g u + \kappa_m S_g u = \kappa_m S_{\widetilde{g}} u^{\frac{m+2}{m-2}}.$$

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So, if we ask:

*Given (M, g) with scalar curvature S_g, is there a conformal metric with **constant** scalar curvature?*

This amounts to solve:

$$\underbrace{-\Delta_g u + \kappa_m S_g u}_{=:\mathscr{L}_g u} = \kappa |u|^{2^*-2} u \qquad \text{on } M$$

Associated to

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we have the Yamabe invariant:

$$Y_M := \inf_{u \in H_g^1(M) \setminus \{0\}} \frac{Q(u)}{|u|_{g,2^*}^2},$$

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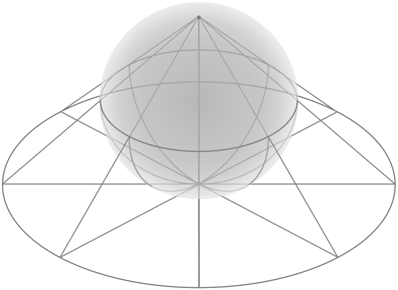
where

$$Q(u) = \int_M |\nabla_g u|^2 + \kappa_m S_g |u|^2 \, \mathrm{d}\mu_g$$

$$|u|_{g,2^*} = \left(\int_M |u|^{2^*} \, \mathrm{d}\mu_g \right)^{1/2^*}$$

If $M = \mathbb{S}^m \subset \mathbb{R}^{m+1}$ with the standard metric:



$$Y_{\mathbb{S}^m} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^m) \setminus \{0\}} \frac{\displaystyle\int_{\mathbb{R}^m} |\nabla u|^2 \, dx}{\left(\displaystyle\int_{\mathbb{R}^m} |u|^2 \, dx\right)^{2/2^*}}$$

Equation and Bubble:

$$-\Delta U = U^{2^*-1} \text{ in } \mathbb{R}^m, \quad U_{\delta,\xi}(x) = \alpha_m \left(\frac{\delta}{\delta^2 + |x-\xi|^2}\right)^{\frac{m-2}{2}}$$

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A BIT OF HISTORY

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Recommended reading:

Lee and Parker, The Yamabe problem. Bull. Amer. Math Soc. (1987).

Some key ideas:

1. work with the Green function of $\mathcal{L}_g$, and understand its expansion;
2. use *conformal normal coordinates* around some special points;
3. consider a very spiked bubble.

$$\mathcal{L}_g u = \kappa |u|^{2^*-2} u, \qquad Y_M = \inf_{u \in H_g^1(M) \setminus \{0\}} \frac{\|u\|_g^2}{|u|_{g,2^*}^2} > 0$$

Another point of view:

$$\begin{aligned} J(u) := & \frac{1}{2}\|u\|_g^2 - \frac{1}{2^*}|u|_{g,2^*}^{2^*} \\ &= \frac{1}{2}\int_M (|\nabla_g u|_g^2 + \kappa_m S_g u^2)\, \mathrm{d}\mu_g - \frac{1}{2^*}\int_M |u|^{2^*}\, \mathrm{d}\mu_g. \end{aligned}$$

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Nehari manifold:

$$\mathcal{N} := \{u \in H^1_g(M) : u \neq 0 \text{ and } J'(u)u = 0\}.$$

and

$$c = \inf_{\mathcal{N}} J.$$

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Aubin's compactness condition:

$$c < \frac{1}{m} Y_{\mathbb{S}^m}^{m/2}$$

BACK TO PARTITION PROBLEMS

Ω is an open subset of M, $Y_M > 0$.

$$\begin{cases} -\Delta_g u + \kappa_m S_g u = |u|^{2^*-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

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Energy functional: $J_\Omega : H^1_{g,0}(\Omega) \to \mathbb{R}$

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We have a map:

$$\Omega \mapsto c_\Omega := \inf_{u \in \mathcal{N}_\Omega} J_\Omega(u).$$

Given $\ell \geq 2$, we consider the optimal ℓ-partition problem

$$\inf_{\{\Omega_1,\ldots,\Omega_\ell\}\in\mathcal{P}_\ell} \sum_{i=1}^{\ell} c_{\Omega_i}, \tag{OPP}$$

where $\mathcal{P}_\ell := \{\{\Omega_1,\ldots,\Omega_\ell\} : \Omega_i \neq \emptyset \text{ is open in } M \text{ and } \Omega_i \cap \Omega_j = \emptyset \text{ if } i \neq j\}$.

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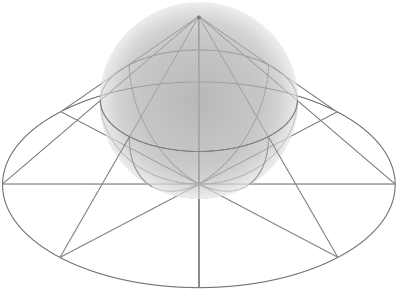
Answer: No! (for instance if $M = \mathbb{S}^m \subset \mathbb{R}^{m+1}$ with the standard metric)



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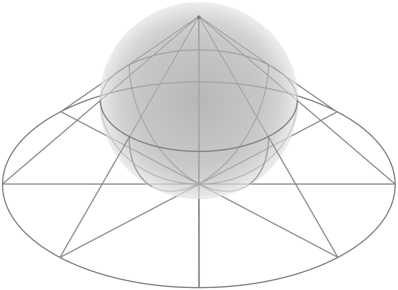

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In subsets of $\mathbb{R}^m$, the problem:

$$-\Delta u = |u|^{2^*-2}u \ \text{ in } \Sigma(\Omega), \qquad u = 0 \ \text{ on } \partial[\Sigma(\Omega)].$$

does not have a least energy solution.

1. To give conditions on (M, g) which guarantee the existence of an optimal ℓ-partition for every ℓ.
2. Characterize the optimal partition.
3. With partitions in $\ell = 2$ sets, prove new results regarding existence of least energy nodal solutions of the Yamabe equation:

$$-\Delta_g u + \kappa_m S_g u = |u|^{2^*-2} u \qquad \text{on } M.$$

How do we do it?

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$$\inf\left\{\sum_{i=1}^{\ell} J(u_i) : \ u_i \in \mathcal{N}\ \forall i, \quad u_i \cdot u_j \equiv 0\ \forall i \neq j\right\}, \qquad \text{(WOPP)}$$

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Remove the constraints $u_i\cdot u_j\equiv 0$: competition parameter $\lambda<0$,

$$\mathcal{J}(u_1,\dots,u_\ell):=\sum_{i=1}^{\ell}\left(\frac{1}{2}\|u_i\|_g^2-\frac{1}{2^*}|u_i|_{g,2^*}^{2^*}\right)-\frac{\lambda}{\gamma+1}\sum_{\substack{i,j=1\\ j\neq i}}^{\ell}\int_M |u_j|^{\gamma+1}|u_i|^{\gamma+1}\,\mathrm{d}\mu_g.$$

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1} \quad \text{on } M, \quad i = 1, \ldots, \ell,$$

where:

- $\lambda < 0$,
- $2(\gamma+1) = 2^*$.

$$\mathcal{J}_\lambda(u_1, \ldots, u_\ell) := \sum_{i=1}^{\ell} \left(\frac{1}{2} \|u_i\|_g^2 - \frac{1}{2^*} \|u_i\|_{g,2^*}^{2^*} \right) - \frac{\lambda}{\gamma+1} \sum_{\substack{i,j=1 \\ j\neq i}}^{\ell} \int_M |u_j|^{\gamma+1} |u_i|^{\gamma+1} \, d\mu_g.$$

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Define the least energy nontrivial level:

$$\widehat{c}_\lambda := \inf_{(u_1, \dots, u_\ell) \in \mathcal{N}_\lambda} \mathcal{J}_\lambda(u_1, \dots, u_\ell)$$

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MAIN RESULTS

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1} \quad \text{on } M, \quad i = 1, \dots, \ell$$

Theorem (Clapp, Pistoia, T., 2021)

Assume that one of the following two conditions holds true:

(A1) $m = 3$, $\gamma = 2$, (M, g) *is not conformal to the standard 3-sphere.*

(A2) $m \geq 9$, $2(\gamma + 1) = 2^*$, (M, g) *is not locally conformally flat.*

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1} \quad \text{on } M, \quad i = 1, \dots, \ell$$

Theorem (Clapp, Pistoia, T., 2021)
Assume that one of the following two conditions holds true:

(A1) $m = 3$, $\gamma = 2$, (M, g) *is not conformal to the standard 3-sphere.*

(A2) $m \geq 9$, $2(\gamma + 1) = 2^*$, (M, g) *is not locally conformally flat.*

Then, the system has a least energy nontrivial solution $(u_1, \dots, u_\ell)$ *such that* $u_i \in \mathcal{C}^2(M)$ *and* $u_i > 0$ *for every* $i = 1, \dots, \ell$.

Let now $\lambda = \lambda_n \to -\infty$. What happens to the solutions?

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda_n u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1} \quad \text{on } M, \qquad i = 1, \ldots, \ell,$$

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We assume that

(A3) (M, g) is not locally conformally flat and $m \geq 10$.
Moreover, if $m = 10$, also ask that

$$|S_g(q)|^2 < \frac{5}{28} |W_g(q)|_g^2 \qquad \forall q \in M,$$

where $W_g(q)$ is the Weyl tensor of (M, g) at q.

Theorem (Clapp-Pistoia-T., 2021)

Assume (A3), let $\lambda_n \to -\infty$ *and take* $(u_{n,1}, \dots, u_{n,\ell})$ *to be a least energy nontrivial solution to the system*

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(iii) ***Optimal Regularity of the limit:*** *$u_{\infty,i} \in \mathcal{C}^{0,1}(M)$*

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In particular, $M = \cup_{i=1}^{\ell} \overline{\Omega}_i$.

In the case $\ell = 2$, we have the following:

Theorem (Clapp-Pistoia-T., 2021)

Assume (A3). Then $w := u_{\infty,1} - u_{\infty,2}$ *is a least energy sign-changing solution to the Yamabe equation*

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For any $\ell \geq 3$, let $u = u_{\infty,1} + \dots u_{\infty,\ell}$:

- $u > 0$ and regular in $\cup_{i=1}^{\ell} \Omega_i = M \setminus \Gamma$.

$\widetilde{g} = u^{2^*-2} g$ is a generalized metric conformal to g.

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Multiplicity Results on the sphere:

[Ding, CMP 1986]: Existence of infinitely many sign-changing solutions to the Yamabe equation

[Fernández and Petean, JDE 2020]: Given $\ell \geq 2$, there exists a solution with ℓ-nodal domains

SOME IDEAS ABOUT THE PROOFS

1. Existence of nontrivial solutions for systems:

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1} \quad \text{on } M, \quad i = 1, \dots, \ell,$$

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3. Regularity results for limiting profiles.

1. EXISTENCE RESULTS FOR SYSTEMS: THE COMPACTNESS CONDITION

Recall that for the Yamabe equation:

$$-\Delta_g u + \kappa_m S_g u = |u|^{2^*-2}u \qquad \text{on } M, \tag{2}$$

a least energy solution exists when $0 < Y_M < Y_{\mathbb{S}^m}$.

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Taking the point of view of the Euler-Lagrange functional:

$$J(u) := \frac{1}{2}\|u\|_g^2 - \frac{1}{2^*}|u|_{g,2^*}^{2^*}, \quad u \in H_g^1(M)$$

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Nehari manifold:

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Then the compactness condition reads:

$$c < \frac{1}{m} Y_{\mathbb{S}^m}^{m/2}.$$

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1} \quad \text{on } M, \qquad i = 1, \dots, \ell,$$

$$\mathcal{J}(u_1, \dots, u_\ell) := \sum_{i=1}^{\ell} \left(\frac{1}{2} \|u_i\|_g^2 - \frac{1}{2^*} |u_i|_{g,2^*}^{2^*} \right) - \frac{\lambda}{\gamma+1} \sum_{\substack{i,j=1 \\ j\neq i}}^{\ell} \int_M |u_j|^{\gamma+1} |u_i|^{\gamma+1} \, d\mu_g.$$

We need to not only prevent blowup of minimizing sequences, but also minimizers with zero components.

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1} \quad \text{on } M, \qquad i = 1, \dots, \ell,$$

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For each $Z \subset \{1, \dots, \ell\}$, take the system of $|Z|$ equations

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda u_i |u_i|^{\gamma-1} \sum_{\substack{j\in Z \\ j\neq i}} |u_j|^{\gamma+1}, \quad i \in Z \quad (\mathscr{S}_Z)$$

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Least energy level:

$$\widehat{c}_Z := \inf_{u\in\mathcal{N}_Z}\mathcal{J}_Z(u).$$

Proposition

Assume that

$$\widehat{c} < \min\left\{\widehat{c}_Z + (\ell - |Z|)\frac{(Y_{\mathbb{S}^m})^{m/2}}{m} \;:\; Z \subsetneq \{1, \ldots, \ell\}\right\}.$$

Then $\widehat{c}$ is attained at a solution $(u_1, \ldots, u_\ell)$ with $u_i \neq 0$ $\forall i$.

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Idea to check this inequality: choice of a suitable test function V.

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If $(u_1,\dots,u_k)$ is a least energy solution of a system with $k < \ell$ equations $\longrightarrow$ use $(u_1,\dots,u_k,V,\dots,V)$.

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Choice of Test Function: idea by Lee-Parker (1987)

1. Conformal normal coordinates at p:

$$\det \widetilde{g}_{ij} = 1 + O(r^m), \quad r = |x|^m$$

$$m \geq 5 \implies S = O(r^2), \ \Delta S = \frac{1}{6}|W|^2 \text{ at } p$$

2. Asymptotic expansion of the Green function of $\mathcal{L}_g$
3. Use the bubble: $U_{\delta,\xi}(x) = \alpha_m \left(\frac{\delta}{\delta^2 + |x - \xi|^2} \right)^{\frac{m-2}{2}}$
4. Construct a suitable test function V and use estimates from Esposito-Pistoia-Vetois (2014)

2. ASYMPTOTIC BEHAVIOR AS $\lambda_n \to -\infty$

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda_n u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1} \quad \text{on } M, \qquad i = 1, \ldots, \ell,$$

Let $(u_{n,1}, \ldots, u_{n,\ell})$ be (nonnegative) nontrivial energy solution.

Weak convergence as $\lambda_n \to -\infty$:

$$u_{n,i} \rightharpoonup \bar{u}_i \quad \text{in } H_g^1(M).$$

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Weak convergence as $\lambda_n \to -\infty$:

$$u_{n,i} \rightharpoonup \bar{u}_i \quad \text{in } H^1_g(M).$$

Challenges:

- Is it true that $\bar{u}_i \not\equiv 0$ for every i?
- Is $(\bar{u}_1, \dots, \bar{u}_\ell)$ a solution of the weak formulation of the optimal partition problem?
- Are the limiting profiles continuous?

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda_n u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1} \quad \text{on } M, \qquad i = 1, \dots, \ell,$$

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Compactness property + test function argument implies:

- strong $H_g^1(M)$ convergence
- $\bar{u}_i \not\equiv 0$

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- $\bar{u}_i \not\equiv 0$

Classical iterative procedure implies:

- Uniform bounds in L^∞ norm:

$$|u_{i,n}|_{g,\infty} \leq C \quad \forall n$$

Proposition

For $\lambda_n < 0$ such that $\lambda_n \to -\infty$, let $u_n = (u_{n,1}, \dots, u_{n,\ell})$ be a nonnegative solution to the system

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In local coordinates:

$$-\operatorname{div}(A(x)\nabla u_i) = f_i(x, u_i) + a(x) \sum_{\substack{j=1 \\ j\neq i}}^{\ell} \lambda |u_j|^{\gamma+1} |u_i|^{\gamma-1} u_i \quad \text{in } \Omega \subset \mathbb{R}^m$$

- $a \in \mathcal{C}^0(\Omega)$ and $a > 0$ in Ω,
- $A \in \mathcal{C}^1$ and $\langle A(x)\xi, \xi \rangle \geq \theta |\xi|^2$
- f_i are continuous and $|f_i(x, s)| = o(s)$ as $s \to 0$

$$-\operatorname{div}(A(x)\nabla u_i) = f_i(x,u_i) + a(x)\sum_{\substack{j=1\\ j\neq i}}^{\ell} \lambda |u_j|^{\gamma+1}|u_i|^{\gamma-1}u_i \quad \text{in } \Omega, \quad i=1,\dots,\ell.$$

Theorem

Let Ω be an open subset of $\mathbb{R}^m$, $\gamma > 0$.

For each $\lambda < 0$ let $(u_{\lambda,1},\dots,u_{\lambda,\ell})$ be a nonnegative solution to the system such that $\{u_{\lambda,i} : \lambda < 0\}$ is uniformly bounded in $L^\infty(\Omega)$.

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Then, given a compact subset $\mathcal{K}$ of Ω and $\alpha \in (0,1)$, there exists $C = C(\alpha,\mathcal{K}) > 0$ such that

$$\|u_{\lambda,i}\|_{\mathcal{C}^{0,\alpha}(\mathcal{K})} \leq C.$$

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Proof of this theorem: adaptation of

Soave, T., Terracini and Zilio, Hölder bounds and regularity of emerging free boundaries for strongly competing Schrödinger equations with nontrivial grouping. Nonlinear Analysis (2016)

(which deals with the Laplacian operator)

Proof of uniform Hölder bounds:

- Contradiction argument and blowup analysis
- Liouville type results applied to blowup limits
- Almgren's monotonicity formula

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with A a symmetric, positive definite (constant!) matrix.

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At the blowup limit, we find the operator:

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with A a symmetric, positive definite (constant!) matrix.

A natural change of variables leads to

$$\Delta u$$

(here we were inspired by [Soave, Terracini, JEMS 2022])

Theorem (Dias-T., ongoing)
Optimal uniform bounds:

$$|u_\lambda|_{L^\infty(M)} \leq C \implies \|u_\lambda\|_{\mathcal{C}^{0,1}(M)} \leq C_1$$

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Uniform bounds in $C^{0,1}$-norm are much harder to get because Liouville-type results like:

$$u \text{ harmonic in } \mathbb{R}^m,\ [u]_{C^{0,\alpha}(\mathbb{R}^m)} := \sup_{\substack{x,y\in\mathbb{R}^m \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \implies u \equiv C$$

are false when $\alpha = 1$!

3. REGULARITY RESULTS FOR LIMITING PROFILES.

$$-\Delta_g u_i + \kappa_m S_g u_i = |u_i|^{2^*-2} u_i + \lambda u_i |u_i|^{\gamma-1} \sum_{\substack{j=1 \\ j\neq i}}^{\ell} |u_j|^{\gamma+1}$$

$$\Big\downarrow \lambda \to -\infty$$

$$\inf_{\{\Omega_1,\dots,\Omega_\ell\}\in\mathcal{P}_\ell} \sum_{i=1}^{\ell} c_{\Omega_i},$$

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Once again: local coordinates

$$-\operatorname{div}(A(x)\nabla u_i) = f_i(x, u_i) + a(x) \sum_{\substack{j=1 \\ j \neq i}}^{\ell} \lambda |u_j|^{\gamma+1} |u_i|^{\gamma-1} u_i \quad \text{in } \Omega, \quad i = 1, \dots, \ell.$$

Theorem

Let Ω *be an open subset of* $\mathbb{R}^m$ *and* $\gamma > 0$*. For each* $\lambda < 0$*, let* $(u_{\lambda,1}, \ldots, u_{\lambda,\ell})$ *be a nonnegative solution to the system satisfying*

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- $\int_\Omega \lambda |u_{\lambda,i}|^{\gamma+1} |u_{\lambda,j}|^{\gamma+1} \to 0$ *whenever* $i \neq j$;
- $-\operatorname{div}(A(x)\nabla u_i) = f_i(x, u_i)$ *in the open set* $\{x \in \Omega : \ u_i(x) > 0\}$.

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Then, the following statements hold true:

(a) u_i *is Lipschitz continuous for every* $i = 1, \ldots, \ell$.

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$$\lim_{p \to x_0^+} \langle A(x)\nabla u_i, \nabla u_i \rangle = \lim_{p \to x_0^-} \langle A(x)\nabla u_j, \nabla u_j \rangle \neq 0,$$

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- $\mathscr{S}$ *is a closed subset of* M *with Hausdorff measure* $\leq m-2$.

$$\lim_{p \to x_0} \langle A(x)\nabla u_i, \nabla u_i \rangle = 0$$

Proof of the Lipschitz continuity goes along the lines of:

[Noris-T.-Terracini-Verzini, CPAM 2010]

[Soave-T.-Tavares-Zilio, Nonl. Anal. 2016]

while the regularity of the nodal set follows

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but therein the **differential operator is the Laplacian.**

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but therein the **differential operator is the Laplacian.**

A key fact is the proof of local Pohozaev type identities:

$$
\begin{aligned}
r\sum_{i=1}^{\ell}\int_{\partial B_r(x_0)}\left(2(\partial_\nu u_i)^2-|\nabla u_i|^2\right) =& 2\sum_{i=1}^{\ell}\int_{B_r(x_0)} f_i(x,u_i)\langle\nabla u_i, x-x_0\rangle \\
&+(2-m)\sum_{i=1}^{\ell}\int_{B_r(x_0)}|\nabla u_i|^2
\end{aligned}
$$

$$E(r) = \frac{1}{r^{m-2}} \sum_{i=1}^{d} \int_{B_r(x_0)} (|\nabla u_i|^2 - f_i(x, u_i)u_i)$$

$$H(r) = \frac{1}{r^{m-1}} \sum_{i=1}^{d} \int_{\partial B_r(x_0)} u_i^2$$

Almgren's quotient:

$$N(r) = \frac{E(r)}{H(r)}$$

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Almgren's quotient:

$$N(r) = \frac{E(r)}{H(r)}$$

Then

$$N'(r) \geq -2Cr(N(r) + 1).$$

In particular,

$e^{Cr^2}(N(r) + 1)$ is a non decreasing function for $r \sim 0$

$N(0^+) := \lim_{r \to 0^+} N(r)$ exists and is finite.

Also

$$(\log H(r))' = \frac{2}{r} N(r)$$

Adapting to divergence form operators with nonconstant matrices is not straightforward at all!

Adapting to divergence form operators with nonconstant matrices is not straightforward at all!

Use ideas from

[Kukavica, Duke 2008]

[Garofalo-Smit Vega Garcia, Advances Math. 2014]

[Garofalo-Petrosyan-Smit Vega Garcia, JMPA 2016]

[Soave-Weth, SIAM Math. Anal. 2018]

Reduction to the case $A(0) = Id$:

$$T_{x_0} x := x_0 + A(x_0)^{\frac{1}{2}} x,$$
$$A_{x_0}(x) := A(x_0)^{-\frac{1}{2}} A(T_{x_0} x) A(x_0)^{-\frac{1}{2}}$$

Advantages:

- Reduction to a case locally close to the Laplacian case.
- Good control as a function of x_0.

In the case $A(0) = Id$:

$$\mu(x) := \left\langle A(x)\frac{x}{|x|}, \frac{x}{|x|} \right\rangle \sim 1, \qquad Z(x) := \frac{A(x)x}{\mu(x)} \sim x.$$

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Local Pohozaev Identities:

$$r\int_{\partial B_r} \langle A\nabla u_i, \nabla u_i\rangle - 2\int_{\partial B_r} \langle Z, \nabla u_i\rangle\langle A\nabla u_i, \nu\rangle = 2\int_{B_r} f_i(x,u_i)\langle \nabla u_i, Z\rangle$$
$$+\int_{B_r} \langle Z, \nabla a_{hl}\rangle \frac{\partial u_i}{\partial x_h}\frac{\partial u_i}{\partial x_l} - 2\int_{B_r} a_{hl}\frac{\partial Z_j}{\partial x_h}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_l} + \int_{B_r} \operatorname{div} Z\langle A\nabla u_i, \nabla u_i\rangle$$

Almgren quotient:

$$E(r) := \frac{1}{r^{m-2}}\sum_{i=1}^{\ell}\int_{B_r} \left(\langle A(x)\nabla u_i, \nabla u_i\rangle - f_i(x,u_i)u_i\right) dx$$

$$H(r) := \frac{1}{r^{m-1}}\sum_{i=1}^{\ell}\int_{\partial B_r} \mu(x) u_i^2 \, d\sigma$$

$$N(r) := \frac{E(r)}{H(r)}$$

Monotonicity formula, case $A(0) = Id$:

$$N'(r) \geq -C(N(r) + 1).$$

In particular:

$e^{Cr}(N(r) + 1)$ is a non decreasing function for $r \sim 0$,

$N(0^+) := \lim_{r \to 0^+} N(r)$ exists and is finite.

Moreover,

$$\left| (\log H(r))' - \frac{2}{r} N(r) \right| \leq C \qquad \text{for } r \sim 0.$$

M. Clapp, A. Pistoia, H. T.
Yamabe systems, optimal partitions, and nodal solutions to the Yamabe equation
arXiv:2106.00579 (2021)

Main points:

- Consider shape optimization problems involving partitions on manifolds
- The energy functional is related to the Yamabe equation
- Explore a connection with elliptic systems
- Sign-changing solutions to the Yamabe equation

Keywords:

- A priori estimates for elliptic systems
- Variational Methods
- Monotonicity formula and free boundary regularity for equations and systems with variable coefficients.

Thank you for your attention.