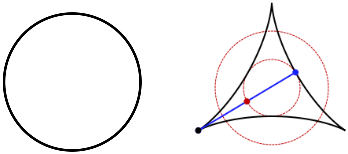
From Kakeya to Restriction, and how to make it sharp?

Diogo Oliveira e Silva, HCM Bonn

Global Portuguese Mathematicians
IST Lisboa, July 14, 2017

The Kakeya Problem

What is the smallest area which is required to rotate a unit line segment by 180 degrees in the plane?

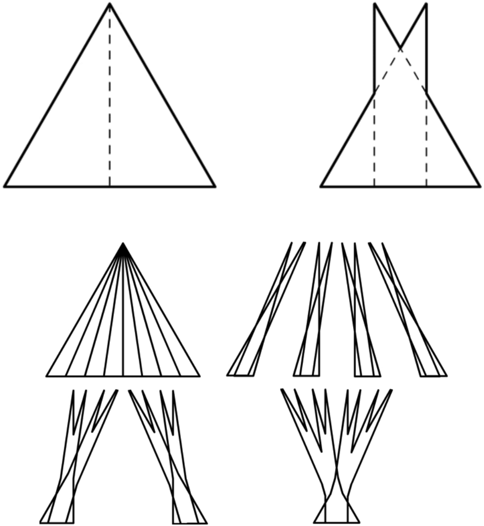


A **Kakeya set** is a compact subset $K \subset \mathbb{R}^d$ which contains a unit line segment in every direction.

Theorem (Besicovitch, 1920)

There exists a Kakeya set in $\mathbb{R}^d$ with zero Lebesgue measure if $d \geq 2$.

Really?



A monster with many arms and a tiny heart

Kakeya Set Conjecture

If $K \subset \mathbb{R}^d$ is a Kakeya set, then $\dim_H(K) = d$.

- Known for $d = 2$ (Davies 1972)
- Open for $d \geq 3$ despite significant partial progress by Bourgain, Wolff, Tao, $\cdots$, Katz–Zahl (April 2017)

Kakeya Maximal function:

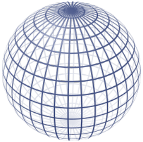
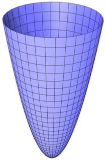
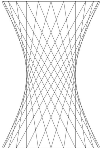
$$\boxed{f_\delta^*(\omega) = \sup_{a \in \mathbb{R}^d} \frac{1}{|T^\delta|} \int_{T_\omega^\delta(a)} |f|}$$

When does an inequality

$$\forall \varepsilon > 0, \exists C_\varepsilon < \infty : \quad \|f_\delta^*\|_{L^p(\mathbb{S}^{d-1})} \leq C_\varepsilon \delta^{-\varepsilon} \|f\|_{L^p(\mathbb{R}^d)}$$

hold, for some $p < \infty$? This requires $p \geq d$.

Fourier Restriction Theory

For $d \geq 2$, consider $(\mathbb{M}, \sigma) \subset \mathbb{R}^d$, a smooth compact hypersurface equipped with surface measure. The **restriction** operator

$$\begin{aligned} T : L^p(\mathbb{R}^d) &\to L^q(\mathbb{M}, \sigma) \\ f &\mapsto \widehat{f}\,|_{\mathbb{M}} \end{aligned}$$

is the adjoint of the **extension** operator

$$\begin{aligned} T^* : L^{q'}(\mathbb{M}, \sigma) &\to L^{p'}(\mathbb{R}^d) \\ f &\mapsto \widehat{f\sigma} \end{aligned}$$

If $q = q' = 2$, then $(T^* \circ T)(f) = f * K$ with $K(x) = \widehat{\sigma}(-x)$.

A classical result

Theorem (Tomas–Stein, 1975)

Let $d \geq 2$ and $p' \geq \frac{2d+2}{d-1}$. Then, for every $f \in L^2(\mathbb{S}^{d-1})$,

$$\|\widehat{f\sigma}\|_{L^{p'}(\mathbb{R}^d)} \lesssim_{d,p} \|f\|_{L^2(\mathbb{S}^{d-1})}$$

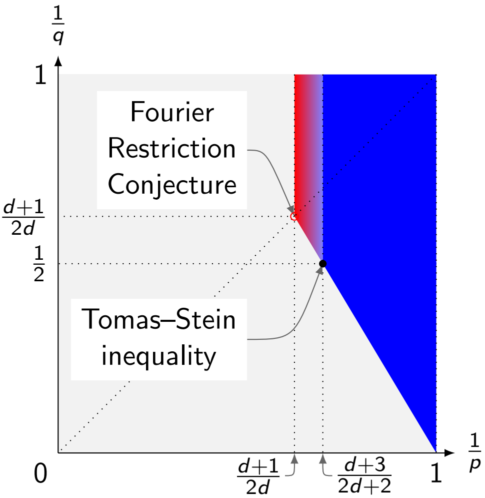
- Explicitly:

$$\widehat{f\sigma}(x) = \int_{\mathbb{S}^{d-1}} f(\omega) e^{-ix\cdot\omega}\,\mathrm{d}\sigma_\omega$$

- Range of exponents is best possible for L^2 densities.
- **Curvature** plays a role: Any smooth compact hypersurface of nonvanishing Gaussian curvature will do.

$$|\widehat{\sigma}(x)| = |x|^{-\frac{d-2}{2}} |J_{\frac{d-2}{2}}(|x|)| \lesssim_d (1+|x|)^{-\frac{d-1}{2}}$$

The Fourier Restriction Conjecture



Kakeya vs. Restriction

Kakeya Maximal Function Conjecture

$$\forall \varepsilon > 0, \exists C_\varepsilon < \infty : \quad \|f_\delta^*\|_{L^d(\mathbb{S}^{d-1})} \leq C_\varepsilon \delta^{-\varepsilon} \|f\|_{L^d(\mathbb{R}^d)}$$

- Known for $d = 2$ (Córdoba 1977), open for $d \geq 3$
- Implies Kakeya Set Conjecture
- Implied by Fourier Restriction Conjecture, via

Uncertainty Principle

If $\widehat{f}$ is supported in a ball of radius R, then f is "essentially constant" at scale R^{-1}.

Low dimensional Tomas–Stein

If $d = 2$, then the endpoint exponent $p' = \frac{2\cdot 2+2}{2-1} = 6$, and $\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leq \mathbf{C}_{2,6}\|f\|_{L^2(\mathbb{S}^1)}$ is equivalent to

$$\boxed{\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^2)} \lesssim \mathbf{C}_{2,6}^3\|f\|_{L^2(\mathbb{S}^1)}^3}$$

If $d = 3$, then the endpoint exponent $p' = \frac{2\cdot 3+2}{3-1} = 4$, and $\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leq \mathbf{C}_{3,4}\|f\|_{L^2(\mathbb{S}^2)}$ is equivalent to

$$\boxed{\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \lesssim \mathbf{C}_{3,4}^2\|f\|_{L^2(\mathbb{S}^2)}^2}$$

- Case $d = 3$: Christ–Shao (2012), Foschi (2015).
- No such reduction is possible in higher dimensions but, for $4 \leq d \leq 7$, a sharp $L^2(\mathbb{S}^{d-1}) \to L^4$ extension inequality was established in **Carneiro–OS** (2015).

Strichartz estimates (1977)

- For the homogeneous Schrödinger equation $iu_t = \Delta u$ with initial datum $u(x,0) = f(x)$:

$$\boxed{\|u\|_{L^{2+\frac{4}{d}}(\mathbb{R}^{d+1})} \leq \mathbf{S}_d \|f\|_{L^2(\mathbb{R}^d)}}$$

  Restriction theory on the **paraboloid**.

- For the homogeneous wave equation $u_{tt} = \Delta u$ with initial data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$:

$$\boxed{\|u\|_{L^{2+\frac{4}{d-1}}(\mathbb{R}^{d+1})} \leq \mathbf{W}_d \|(f,g)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)\times \dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)}}$$

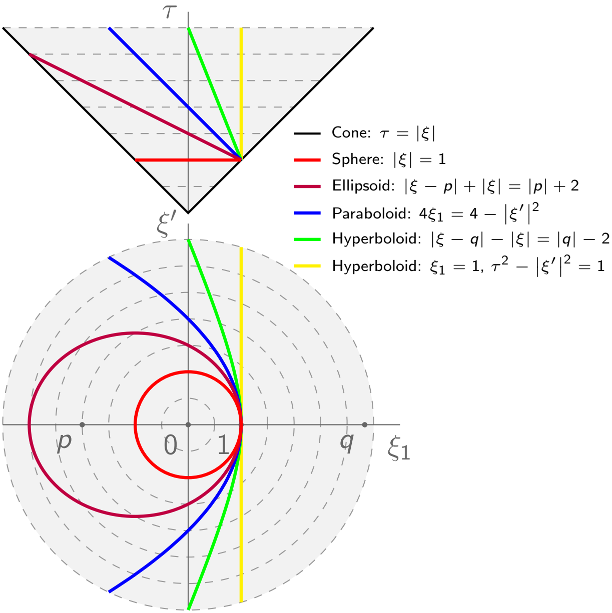
  Restriction theory on the **cone**.

- Low dimensional sharp versions?

Foschi (2007), Hundertmark–Zharnitsky (2006),

Bennett–Bez–Carbery–Hundertmark (2009),

OS–Quilodrán (2016), Gonçalves (2017).


τ
ξ'
p
0
1
q
ξ_1
Cone: $\tau = |\xi|$
Sphere: $|\xi| = 1$
Ellipsoid: $|\xi - p| + |\xi| = |p| + 2$
Paraboloid: $4\xi_1 = 4 - |\xi'|^2$
Hyperboloid: $|\xi - q| - |\xi| = |q| - 2$
Hyperboloid: $\xi_1 = 1,\ \tau^2 - |\xi'|^2 = 1$

An explicit computation

$$
\begin{aligned}
(\sigma * \sigma)(x) &= \iint_{(\mathbb{S}^{d-1})^2} \boldsymbol{\delta}\big(x - \omega - \nu\big)\, \mathrm{d}\sigma_\omega\, \mathrm{d}\sigma_\nu \\
&= \int_{\mathbb{S}^{d-1}} \boldsymbol{\delta}\big(1 - |x - \omega|^2\big)\, \mathrm{d}\sigma_\omega \\
&= \frac{1}{|x|} \int_{\mathbb{S}^{d-1}} \boldsymbol{\delta}\Big(\tfrac{|x|}{2} - \tfrac{x}{|x|} \cdot \omega\Big)\, \mathrm{d}\sigma_\omega \\
&= \frac{1}{|x|} \int_0^\pi \boldsymbol{\delta}\Big(\frac{|x|}{2} - \cos\theta\Big) (\sin\theta)^{d-2}\, \mathrm{d}\theta \\
&= \frac{1}{|x|} \int_{-1}^1 \boldsymbol{\delta}\Big(\frac{|x|}{2} - t\Big) (1 - t^2)^{\frac{d-3}{2}}\, \mathrm{d}t \\
&= \frac{1}{|x|} \Big(1 - \frac{|x|^2}{4}\Big)^{\frac{d-3}{2}} \quad \text{provided } |x| \leq 2
\end{aligned}
$$

First symmetries

Lemma (Positivity)

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^d)} \leq \||f|\sigma * |f|\sigma\|_{L^2(\mathbb{R}^d)}$$

with equality **if and only if**

$$f(\omega)f(\nu) = h(\omega+\nu)|f(\omega)f(\nu)|, \quad for \;\; a.e. \; (\omega,\nu) \in (\mathbb{S}^{d-1})^2$$

and some measurable function $h : \overline{B(2)} \to \mathbb{C}$.

Given $f : \mathbb{S}^{d-1} \to \mathbb{R}^+$, define $f_\star$ via $f_\star(\omega) := \sqrt{\frac{f(\omega)^2+f(-\omega)^2}{2}}$.

Lemma (Antipodality)

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^d)} \leq \|f_\star\sigma * f_\star\sigma\|_{L^2(\mathbb{R}^d)}$$

with equality **if and only if** $f = f_\star$ *(σ-a.e.).*

Keeping the analysis global

$$\|f\sigma * f\sigma\|^2_{L^2(\mathbb{R}^d)} = Q(f,f,f,f) =$$
$$= \int_{(\mathbb{S}^{d-1})^4} f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)\, \boldsymbol{\delta}\big(\omega_1+\omega_2+\omega_3+\omega_4\big)\, \mathrm{d}\sigma_{\vec{\omega}}$$

where the 4-linear form Q is given by

$$Q(f_1,f_2,f_3,f_4) := \int_{(\mathbb{S}^{d-1})^4} f_1(\omega_1)f_2(\omega_2)f_3(\omega_3)f_4(\omega_4)\, \mathrm{d}\Sigma_{\vec{\omega}}$$

and the singular measure Σ on $(\mathbb{S}^{d-1})^4$ is given by

$$\mathrm{d}\Sigma_{\vec{\omega}} = \boldsymbol{\delta}\big(\omega_1+\omega_2+\omega_3+\omega_4\big)\, \mathrm{d}\sigma_{\omega_1}\, \mathrm{d}\sigma_{\omega_2}\, \mathrm{d}\sigma_{\omega_3}\, \mathrm{d}\sigma_{\omega_4}$$

and supported on $\Gamma_0 := \{\vec{\omega} \in (\mathbb{S}^{d-1})^4 : \omega_1+\omega_2+\omega_3+\omega_4 = 0\}$.

Almost sharp

Take four vectors $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{S}^{d-1}$ such that

$$\omega_1 + \omega_2 + \omega_3 + \omega_4 = 0$$

In this case, we have that

$$|\omega_1 + \omega_2||\omega_3 + \omega_4| + |\omega_2 + \omega_3||\omega_1 + \omega_4| + |\omega_3 + \omega_1||\omega_2 + \omega_4| = 4$$

(Think about $|\omega_1 + \omega_2|^2 + |\omega_2 + \omega_3|^2 + |\omega_3 + \omega_1|^2$ instead)

$$\begin{aligned} Q(f,f,f,f) &= \int_{(\mathbb{S}^{d-1})^4} f(\omega_1)f(\omega_2)f(\omega_3)f(\omega_4)\,\mathrm{d}\Sigma_{\vec{\omega}} \\ &= \frac{3}{4}\int_{(\mathbb{S}^{d-1})^4} f(\omega_1)f(\omega_2)|\omega_1+\omega_2| f(\omega_3)f(\omega_4)|\omega_3+\omega_4|\,\mathrm{d}\Sigma_{\vec{\omega}} \\ &\lesssim_d \iint_{(\mathbb{S}^{d-1})^2} f(\omega_1)^2 f(\omega_2)^2 |\omega_1+\omega_2|^2 \underbrace{\sigma * \sigma(\omega_1+\omega_2)}_{=\frac{(4-|\omega_1+\omega_2|^2)^{\frac{d-3}{2}}}{|\omega_1+\omega_2|}}\,\mathrm{d}\sigma_{\omega_1}\,\mathrm{d}\sigma_{\omega_2} \end{aligned}$$

One last ingredient

Consider the (real-valued, continuous) functional on $L^1(\mathbb{S}^{d-1})$:

$$H(g) := \iint_{(\mathbb{S}^{d-1})^2} \overline{g(\omega)} g(\nu) |\omega - \nu| (4 - |\omega - \nu|^2)^{\frac{d-3}{2}} \, \mathrm{d}\sigma_\omega \, \mathrm{d}\sigma_\nu$$

Lemma (Monotonicity of H)

Let $3 \leq d \leq 7$. Let $g \in L^1(\mathbb{S}^{d-1})$ be an even function with average μ. Then

$$H(g) \leq H(\mu \mathbf{1}) = |\mu|^2 H(\mathbf{1})$$

with equality **if and only if** *g is a constant function.*

Two possible approaches: heat flow, spectral decomposition.

Spherical harmonics and the Funk–Hecke formula

If $g \in L^{2}(\mathbb{S}^{d-1})$, can decompose $g = \sum_{k\geq 0} Y_k$. Then:

$$H(g) = \sum_{k,j\geq 0} \int_{\mathbb{S}^{d-1}} \overline{Y_k(\omega)} \underbrace{\Big(\int_{\mathbb{S}^{d-1}} Y_j(\nu)|\omega-\nu|(4-|\omega-\nu|^2)^{\frac{d-3}{2}} \,\mathrm{d}\sigma_\nu \Big)}_{=\lambda_j Y_j(\omega)} \mathrm{d}\sigma_\omega$$

$$= \sum_{k\geq 0} \lambda_k \|Y_k\|^2_{L^2(\mathbb{S}^{d-1})} \leq \lambda_0 \|Y_0\|^2_{L^2(\mathbb{S}^{d-1})}?$$

Compute the (signs of the) coefficients λ_k via Funk–Hecke:

$$\lambda_k = \omega_{d-2} \int_{-1}^{1} \frac{C_k^{\frac{d-2}{2}}(t)}{C_k^{\frac{d-2}{2}}(1)} \phi(t)(1-t^2)^{\frac{d-3}{2}} \,\mathrm{d}t$$

And the result is...

Signs of the coefficients λ_k

	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	$\cdots$
$d = 3$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$\cdots$
$d = 4$	$+$	0	$-$	0	$-$	0	$-$	$\cdots$
$d = 5, 6, 7$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$\cdots$
$d \geq 8$	$+$	$+$	$+$	$\star$	$\star$	$\star$	$\star$	$\cdots$

2D Paraboloids via convolution estimates

$$\mathbb{P}^2 = \{(\xi,\tau)\in\mathbb{R}^{2+1} : \tau = |\xi|^2\}$$
$$\mu(\xi,\tau) = \boldsymbol{\delta}\big(\tau - |\xi|^2\big)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

Strichartz for Schrödinger in $\mathbb{R}^{2+1}$ $\simeq$ $L^2(\mu)\to L^4$ extension ineq.

$$\begin{aligned}
(\mu*\mu)(\xi,\tau) &= \int_{\mathbb{R}^{2+2}} \boldsymbol{\delta}\begin{pmatrix}\xi-\eta-\zeta\\ \tau-|\eta|^2-|\zeta|^2\end{pmatrix}\mathrm{d}\eta\,\mathrm{d}\zeta\\
&= \int_{\mathbb{R}^2} \boldsymbol{\delta}\big(\tau-|\eta|^2-|\xi-\eta|^2\big)\,\mathrm{d}\eta\\
&= \int_{\mathbb{R}^2} \boldsymbol{\delta}\Big(\tau-\tfrac{|\xi|^2}{2}-2|\eta|^2\Big)\,\mathrm{d}\eta\\
&= 2\pi\int_0^\infty \boldsymbol{\delta}\Big(\tau-\tfrac{|\xi|^2}{2}-2r^2\Big)r\,\mathrm{d}r\\
&= \frac{\pi}{2}\int_0^\infty \boldsymbol{\delta}\Big(\tau-\tfrac{|\xi|^2}{2}-s\Big)\,\mathrm{d}s = \frac{\pi}{2}\chi\Big(\tau\geq\frac{|\xi|^2}{2}\Big)
\end{aligned}$$

Cauchy–Schwarz implies:

$$|(f\mu * f\mu)(\xi,\tau)|^2 \leq (\mu*\mu)(\xi,\tau)\cdot(|f|^2\mu * |f|^2\mu)(\xi,\tau)$$

Integrate:

$$\|f\mu * f\mu\|^2_{L^2(\mathbb{R}^3)} \leq \int_{\mathbb{R}^{2+1}} (\mu*\mu)(\xi,\tau)\cdot(|f|^2\mu * |f|^2\mu)(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

Hölder implies:

$$\boxed{\|f\mu * f\mu\|^2_{L^2(\mathbb{R}^3)} \leq \frac{\pi}{2}\|f\|^4_{L^2(\mathbb{R}^2)}}$$

Both inequalities simultaneously become equalities if

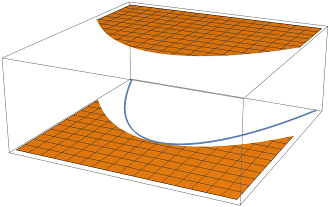
$$f(\eta)f(\zeta) = F(\eta+\zeta, |\eta|^2+|\zeta|^2)$$

(for some complex-valued F defined on $\mathrm{supp}(\mu*\mu)$, and almost every $(\eta,\zeta)\in\mathbb{R}^2\times\mathbb{R}^2$)

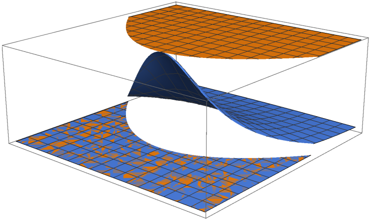
Sharp inequality, **Gaussians** are unique extremizers.

What about perturbations?

If $\tau = |\xi|^2$, then the convolution $\mu * \mu$ is **constant** in its support:



If $\tau = |\xi|^2 + |\xi|^4$, then we instead have that:



Consequences to PDE

Family of fourth order Schrödinger equations in $\mathbb{R}^{2+1}$: For $\mu \geq 0$,

$$\begin{cases} iu_t + \Delta^2 u - \mu \Delta u = 0 \\ u(\cdot, 0) = f \in L^2_x(\mathbb{R}^2) \end{cases}$$

Jiang–Shao–Stovall (2014): Either extremizers for

$$\|(\mu + |\nabla|^2)^{\frac{1}{4}} e^{it(\Delta^2 - \mu\Delta)} f\|_{L^4_{t,x}(\mathbb{R}^3)} \lesssim \|f\|_{L^2_x(\mathbb{R}^2)}$$

exist, or "they exhibit classical Schrödinger behavior".

- Sharpened Strichartz inequality:

$$\|S_\mu(t) D_\mu^{\frac{1}{2}} f\|_{L^4_{t,x}} \lesssim \sup_\kappa \left(|\kappa|^{-\frac{3}{22}} \|S_\mu(t) D_\mu^{\frac{4}{11}} f_\kappa\|_{L^{\frac{11}{2}}_{t,x}} \right)^{\frac{1}{8}} \|f\|_{L^2_x}^{\frac{7}{8}}$$

- Linear profile decomposition

Our methods imply that extremizers **exist** if $\mu = 0$, and that extremizers **do not exist** if $\mu = 1$.

Three natural questions

- How to treat non-even integers?
  - **Do Gaussians extremize the endpoint extension inequality on the paraboloid in all dimensions?**
  - **Do Constants extremize the endpoint extension inequality on the sphere in all dimensions?**
- Common proof in the Lorentz invariant case?
- How to sharpen Kakeya?

Thank you very much