Ω^1_X big and hyperbolicity

$\mathbb{P}^1$, E and $\mathbb{C}$ do not have herm. metrics with curv $< -\varepsilon$, $\varepsilon > 0$.

Ω^1_X big $\Leftrightarrow$ negative curv. properties $\Rightarrow$ restrictions on curves as $\mathbb{P}^1, E, \mathbb{C}$ in X.

$\Updownarrow$

$$\limsup_{m\to\infty} \frac{h^0(X, S^m \Omega^1_X)}{m^{2\dim X - 1}} \neq 0. \quad (\Leftrightarrow \text{highest order of growth possible})$$

$\mathbb{P}^1, E$ $\qquad\qquad$ $\mathbb{C}$

- <u>Thm</u> (Bogomolov 78, McQuillan 96) A surface of gen. type X with big Ω^1_X satisfies the Green-Griffiths conj:

A variety of gen. type Y has a proper subvar. $Z \subset Y$ s.t. $f : \mathbb{C} \to Y$ nonconstant (entire curve) must have $f(\mathbb{C}) \subset Z$.

$\Rightarrow$ If X a surf. of gen. type, then X has only finitely many rational or elliptic curves.

Criterions for Ω^1_X big

X surf of gen. type

- Ω^1_X big is a birational property

  $P^S_m(X) = h^0(S^m \Omega^1_X)$ are birat. inv.

- Is bigness of Ω^1_X Topological?

- $h^0(S^m\Omega^1_X) - h^1(S^m\Omega^1_X) + h^2(S^m\Omega^1_X) = \frac{s_2(X)}{3!} m^3 + O(m^2)$

  $s_2(X) := c_1^2 - c_2$ 2nd Segre number

- Bog. vanish $h^2(S^m\Omega^1_X) = 0$, $m \geq 3$

Topological criterion: $s_2(X) > 0 \Rightarrow \Omega^1_X$ big.

$\not\Leftarrow$

b/c $s_2(Bl_p X) = s_2(X) - 2$

Bigness of Ω^1_X not preserved under deformations $\Rightarrow$ not a Topol. prop.

- <u>Full Bigness criterion</u>:

$$\Omega^1_X \text{ big} \iff \underbrace{\frac{s_2(X)}{3!} + \limsup_{m\to\infty} \frac{h^1(X, S^m\Omega^1_X)}{m^3}}_{(*)} > 0$$

not easy to find
(depends on cx structure)

Our idea is to bound $(*)$ from below by local info from the singularities of the canonical model of X

$$X \xrightarrow[(-1)]{\text{contracts}} X_{min} \xrightarrow[(-2)]{\text{contracts}} X_{can}$$

gen type

K_X nef
K_X big
smooth

K_X ample
Canonical sing
(A, D, E sing)

- <u>Canonical sing criterion</u> (–, Weiss 19)

$$\frac{s_2(X_{min})}{3!} + \sum_{x\in Sing(X_{can})} \bar{h}^1(x) > 0 \implies \Omega^1_X \text{ big}$$

$\bar{h}^1(x)$ inv. of the sing.

Surface canonical sing. germ (X,x)

$$\begin{tikzcd}
(\tilde{X},E) \arrow[dr, "\sigma"'] & & (\mathbb{C}^2,0) \arrow[dl, "\pi"] \arrow[loop right, "G_x"] \\
& (X,x) = (\mathbb{C}^2,0)/G_x &
\end{tikzcd}$$

resolution minimal

π smoothing

G_x loc. fund. gp, $G_x \subset SL(2,\mathbb{C})$ finite.

Set $\quad h^1(x,m) := h^1(\tilde{X}, S^m\Omega^1_{\tilde{X}})$

Then: $\quad \bar{h}^1(x) = \lim_{m\to\infty} \dfrac{h^1(x,m)}{m^3}$

How do we find $h^1(x,m)$ and $\bar{h}^1(x)$?

- X proj. orbifold surf (i.e. with quotient sing)

$\tilde{X} \xrightarrow{\sigma} X$ min. resol ; $E \subset \tilde{X}$ excep. div.

$\hat{S}^m \Omega^1_X := (\sigma_* S^m \Omega^1_{\tilde{X}})^{\vee\vee}$ is an orbifold v.b. on X.

- $\chi(X, \hat{S}^m \Omega^1_X) = \chi(\tilde{X}, S^m \Omega^1_{\tilde{X}}) + \sum_{x \in Sing(X)} \left(h^0(x,m) + h^1(x,m) \right)$

$h^0(x,m) = \dim \left(\dfrac{H^0(\tilde{U}_x \setminus E_x, S^m \Omega^1_{\tilde{X}})}{H^0(\tilde{U}_x, S^m \Omega^1_{\tilde{X}})} \right)$, $(\tilde{U}_x, E_x)$ min res of (X,x)

R.R for orb. v.b.

(= 0 if X smooth)

Correction Terms

- $\underbrace{\chi(X, \hat{S}^m \Omega^1_X)}_{\chi_{orb}(X, \hat{S}^m \Omega^1_X)} = \int ch_{orb}(\hat{S}^m \Omega^1_X)\, Td_{orb}(X) + \sum_{x \in Sing(X)} \mu(x, \hat{S}^m \Omega_X)$

$\chi_{orb}(X, \hat{S}^m \Omega^1_X) = \dfrac{c_1^2 - c_2}{3!} m^3 - \frac{1}{2} c_2 m^2 - \frac{1}{12}(c_1^2 + c_2) m + \frac{1}{12}(c_1^2 + c_2)$

c_1^2 and c_2 orb. Chern numbers of X.

(Same expression with local Chern #s)

local. orb. E.C

$$- \quad \chi(\tilde{X}, S^m\Omega^1_{\tilde{X}})$$
$$\|$$
$$\chi_{orb}(\tilde{X}, S^m\Omega^1_{\tilde{X}}) = \chi_{orb}(X, \hat{S}^m\Omega^1_X) + \sum_{x \in Sing(X)} \chi_{orb}(x, S^m\Omega^1_{\tilde{X}})$$

We get from this the following expression for $h^1(x,m)$

$$h^1(x,m) = \mu(x,m) - \chi_{orb}(x,m) - h^0(x,m)$$
$$\mu(x,m) = \mu(x, \hat{S}^m\Omega^1_X) \qquad \chi_{orb}(x,m) = \chi_{orb}(x, S^m\Omega^1_{\tilde{X}})$$

$$\bar{h}^0(x) = \lim_{m\to\infty} \frac{h^0(x,m)}{m^3}$$

$$\bar{\chi}_{orb}(x) = \lim_{m\to\infty} \frac{\chi_{orb}(x,m)}{m^3} = \frac{s_{2,orb}(x)}{3!}$$

$$\bar{\mu}(x) = \lim_{m\to\infty} \frac{\mu(x,m)}{m^3} = 0$$

$$\bar{h}^1(x) + \bar{h}^0(x) = -\frac{s_{2,orb}(x)}{3!}$$

Notation: $\bar{h}^0(A_n)$, $\bar{h}^1(A_n)$ are resp $\bar{h}^0(x)$ and $\bar{h}^1(x)$ if (X,x) germ of A_n sing.

Rousseau – Rollenske Bigness criterion :

$$\frac{s_2(X_{min})}{3!} + \sum_{x \in Sing(X_{can})} \frac{\bar{h}^0(x) + \bar{h}^1(x)}{2} > 0$$

Prop : (X, x) canonical surf sing. $\quad h^0(x, m) \leq h^1(x, m)$

In case of A_n sing. $\quad \frac{\bar{h}^0(A_n) + \bar{h}^1(A_n)}{2} = r(n) \; \bar{h}^1(A_n)$

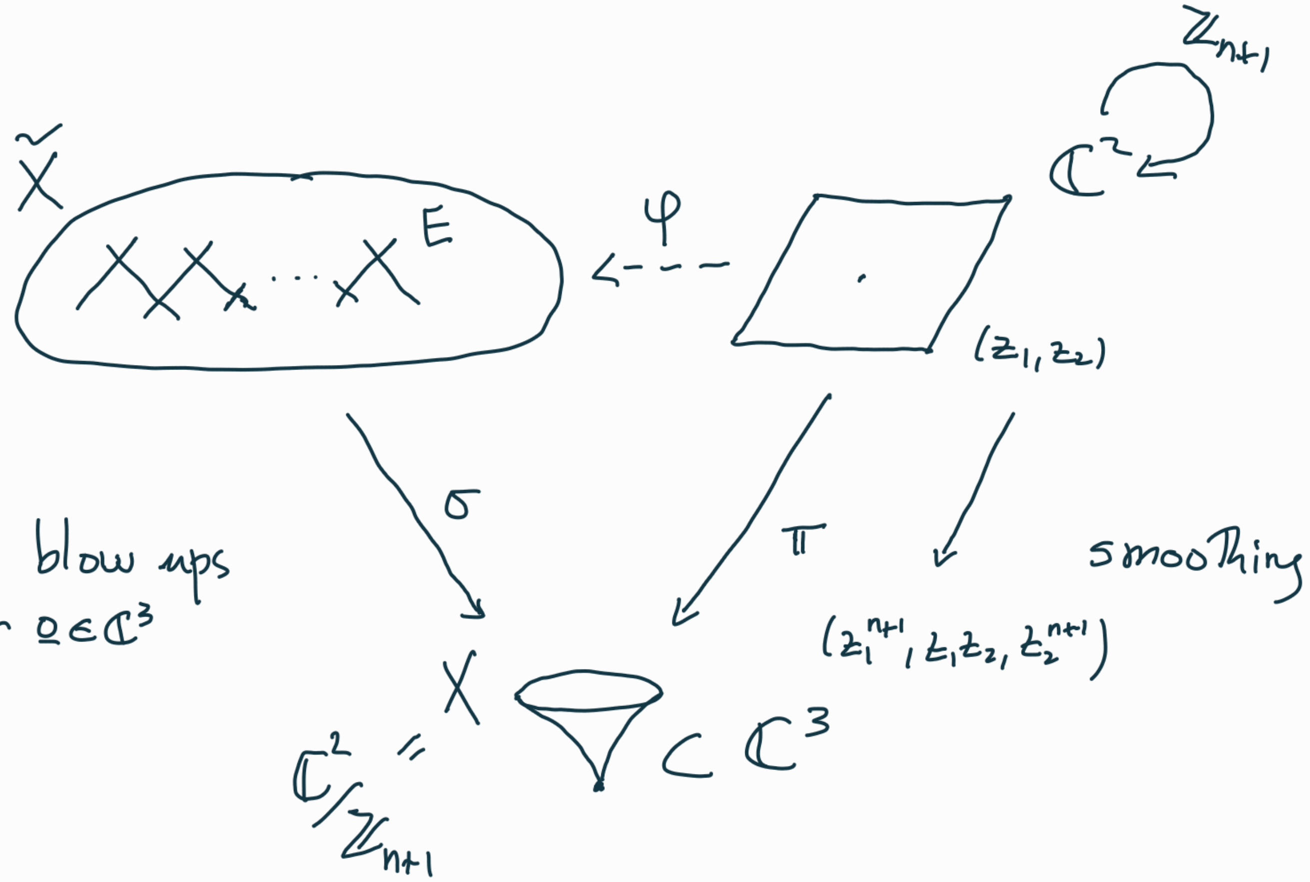
$$r(1) \approx \frac{27}{32} \; ; \; r(2) = \frac{48}{67} \; ; \; \dots \; ; \; \lim_{n \to \infty} r(n) = \frac{1}{2}$$

A_n model

$$X = \{ xz - y^{n+1} = 0 \} \subset \mathbb{C}^3$$

quotient of $\mathbb{C}^2$ by $\mathbb{Z}_{n+1}$ via $\rho : \mathbb{Z}_{n+1} \to SL(2, \mathbb{C})$

$$1 \longmapsto \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^n \end{bmatrix} \quad \varepsilon \text{ prim } n+1\text{-root}$$



$$E = E_1 \cup \ldots \cup E_n \qquad E_i \text{ } (-2)\text{-curve}$$

Extension results (A_n sing)

Goal: find $h^0(A_n, m)$ = codim of $H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}})$ in $H^0(\tilde{X}\setminus E, S^m\Omega^1_{\tilde{X}})$

$$H^0(\tilde{X}\setminus E, S^m\Omega^1_{\tilde{X}}) \xrightarrow[\cong]{\varphi^*} H^0(\mathbb{C}^2, S^m\Omega^1_{\mathbb{C}^2})^{\mathbb{Z}_{n+1}}$$

$$\omega \longrightarrow \varphi^*\omega$$

Characteristics of $\varphi^*\omega$ are hence also of ω.

$$(+) \quad \varphi^*\omega = \sum_{m_1+m_2=m} c_{i_1 i_2 m_1 m_2}\, z_1^{i_1} z_2^{i_2}\, dz_1^{m_1}\, dz_2^{m_2}$$

$$\text{order } \omega := \min\{i_1+i_2 \text{ in } (+)\}$$

* Thm (A, -, U): If order $\omega \geq n \deg \omega$, then $\omega \in H^0(\tilde{X}, S^m\Omega^1_{\tilde{X}})$

($\Rightarrow h^0(A_n, m)$ finite)

Poles along components of exceptional divisor.

(Miyaoka, 84) $(\tilde{X}, E)$ a resol. of quotient sing.

$$H^0(\tilde{X} \setminus E, S^m \Omega^1_{\tilde{X}}) = H^0(\tilde{X}, S^m \Omega^1_{\tilde{X}}(\log E))$$

$\uparrow$ logarithmic poles

but for the special case of A_n sing. the poles are substantially milder:

* <u>Thm</u>: (Asega, –, Weiss) $(\tilde{X}, E)$ resol. of A_n sing.

$$H^0(\tilde{X} \setminus E, S^m \Omega^1_{\tilde{X}}) = H^0(\tilde{X}, S^m \Omega^1_{\tilde{X}}(\log \tfrac{n}{n+1} E))$$

$E \cap U = \{z_1 = 0\}$ $\quad \omega|_U = \sum c_{i_1 i_2 m_1 m_2} z_1^{i_1} z_2^{i_2} dz_1^{m_1} dz_2^{m_2}$

$i_1 \geq -\frac{n}{n+1} m_1$, $i_2 \geq 0$ (vs. $i_1 \geq -m_1$ above)

* $\underline{\text{Thm}}$ (Asega, –, Weiss)

$\exists$ modulus $\mu(n)$ s.t.

$$h^0(A_n, m) = \overline{h^0}(A_n)\, m^3 + 3\,\overline{h^0}(A_n)\, m^2 + c_{n,r}\, m + d_{n,r}$$

if $m \equiv r \mod \mu(n)$.

$$\overline{h^0}(A_n) = \frac{4}{3} \sum_{k=1}^{n} \frac{1}{k^2} - \frac{12n^4 + 65n^3 + 117n^2 + 72n}{6(n+1)^2(n+2)^2}.$$

$$\lim_{n \to \infty} \overline{h^0}(A_n) = \frac{2\pi^2}{9} - 2$$

$$\boxed{\overline{h^1}(A_n) = \frac{n^5 + 19n^4 + 83n^3 + 137n^2 + 80n}{6(n+1)^2(n+2)^2} - \frac{4}{3}\sum_{k=1}^{n} \frac{1}{k^2}.}$$

$$\lim_{n \to \infty} \overline{h^1}(A_n) = \infty$$

Ex: A_2 ; we have that $\mu(2)=6$.

$$h^0(A_2,m) = \begin{cases} \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{12}m & m\equiv 0 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{8}m - \frac{143}{216} & m\equiv 1 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{7}{36}m - \frac{2}{27} & m\equiv 2 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{8}m + \frac{3}{8} & m\equiv 3 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{1}{12}m - \frac{10}{27} & m\equiv 4 \pmod 6 \\ \frac{29}{216}m^3 + \frac{29}{72}m^2 + \frac{17}{72}m - \frac{7}{216} & m\equiv 5 \pmod 6 \end{cases}$$

Also $\mu(1)=6$ (ABT) ; $\mu(3)=60$, while $\mu(4)=30$.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$\overline{h}^1(A_n)=$	$\frac{4}{27}$	$\frac{67}{216}$	$\frac{1283}{2700}$	$\frac{577}{900}$	$\frac{106819}{132300}$	$\frac{1030727}{1058400}$

Deformations of hypersurf. in $\mathbb{P}^3$ with big Ω^1_X

$X_d \subset \mathbb{P}^3$ hypersurf of degree d

If X_d smooth, K_X ample if $d \geq 5$

gen. Type.

$$s_2(X_d) = d(10-4d) < 0$$

Bruckmann 70's $P^S_m(X_d) = 0$ for $m \geq 1$.

- Are There deformations of X_d smooth with big cotangent bdl?

$X_{d,[n,\ell]} \subset \mathbb{P}^3$ hypersurf of degree d

with ℓ A_n sing.

Brierskorn simulT resol. result

$\Downarrow$

$\widetilde{X}_{d,[n,\ell]}$ is a deform. of sm. X_d.

Hence $s_2(\tilde{X}_{d,[n,\ell]}) < 0$

Apply our criterion to see if the singularities can help.

Essential: How many A_n sing. can exist on an hypersurf of degree n?

Theoretical upper bounds:

Miyaoka 84: $\#A_n$ sing on degree d $\leq \frac{2}{3} \frac{n+1}{n(n+2)} d(d-1)^2$

There are also bounds by Varschenko 83.

lower bounds via constructions: (Labs, for more info)

variations of Chmutov's contruction gives the examples with most A_n sing.

Thm $(A, \sim, W)$: A resol. $\tilde{X}_{d,[n,\ell]}$ has big cotangent bdl if

$$\ell > \frac{d(10-4d)(n+1)^2(n+2)^2}{n^5+19n^4+83n^3+137n^2+80n-8(n+1)^2(n+2)^2\sum_{k=1}^{n}\frac{1}{k^2}}$$

Using existing constructions and theorectical bounds:

i) For $d \geqslant 8$, $\exists$ deformations of smooth $X_d \subset \mathbb{P}^3$ with big cotangent bundle.

ii) If theorectical upper bounds true, i) also true for $d \geqslant 6$ ($d=5$ not possible!)

$d=6$ would occur theorectically (with 3 A_{17})

About $\exists$ of sym. diff. on $X_{d,[n,e]}$ of low degree

Sing.	lowest d with resol. with sym. diff	lowest degree of sym appearing for lowest d.
	Const / Theor	Constr / Theor
A_1	10/10	160/52
A_2	9/8	284/58
A_3	8/8	57/15
A_4	10/7	135/20
A_5	11/7	41/18
A_6	13/7	17/11
$\vdots$		
A_{10}	11/7	8/14

Also relevant.

best constructed low degree m, $m=5$ $(d=24, A_{11})$

best theoretical low degree m, $m=3$ $(d=11, A_9)$

$$A_1 = \frac{4}{27}$$

$$A_2 = \frac{67}{216}$$

$$A_3 = \frac{1283}{2700}$$

$$A_4 = \frac{577}{900}$$

$$A_5 = \frac{106819}{132300}$$

$$A_6 = \frac{1030727}{1058400}$$

$$A_7 = \frac{5431459}{4762800}$$

$$A_8 = \frac{6224867}{4762800}$$

$$A_9 = \frac{169845391}{115259760}$$

$$A_{10} = \frac{189050941}{115259760}$$