Flexibility of distributions

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How? We will use a family of tools, popularised by Gromov, called the h*-principle*.

Why? For most types of distributions little is known. But the one case we know (Contact Topology) is pretty amazing.

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Some questions with the “flavour” we care about may be:

- Fix a manifold M and a dimension n. Can M be immersed into $\mathbb{R}^n$?
- How many connected components does the space of symplectic structures on M have?
- Is there a non-trivial loop of positive scalar curvature metrics on M?

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- Any distribution of rank r is locally of this form.

Typical example

Consider the standard contact structure

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we have that

$$[\phi^t_{\partial_y}, \phi^t_{\partial_x - y\partial_z}] = \phi^{t^2}_{\partial_z}\ (+h.o.t.)$$

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Main Question

What is the homotopy type of the space of the non-degenerate distributions of type (k, n)?

One case at a time

Dimension 3

Lemma

(M^3, ξ^2) is non-degenerate iff it is *contact*

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Theorem (Eliashberg; 1989)

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\begin{array}{ccc}
\mathrm{Dist}_{\mathrm{nd}}(M^3, 2) & \xrightarrow{\pi_k-\text{surjection}} & \mathrm{Dist}_f(M^3, 2) \\
\uparrow & \nearrow_{\pi_0-\text{bijection}} & \\
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Theorem (Bennequin; 1982)

There are contact structures that are not in $\mathrm{Dist}_{\mathrm{OT}}(M, 2)$.

Dimension 3 (continued)

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$$\mathrm{Dist}_{\mathrm{OT}}(M^3, 2) \xrightarrow{\pi_0-\text{surjection}} \mathrm{Dist}_f(M^3, 2)$$

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Theorem (Borman-Eliashberg-Murphy; 2014)

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Theorem (Murphy; 2012)

In dimension at least $2n+1 \geq 5$:

$$\begin{array}{ccc} \mathrm{Emb}_{\mathrm{leg}}(L; M^{2n+1}, \xi) & \xrightarrow{\pi_k-\text{surjection}} & \mathrm{Emb}_f(L; M^{2n+1}, \xi) \\ \uparrow & \nearrow_{\pi_0-\text{bijection}} & \\ \mathrm{Emb}_{\mathrm{loose}}(L; M^{2n+1}, \xi) & & \end{array}$$

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Theorem (Casals-Murphy-Presas; 2015)

(M, ξ) is overtwisted iff the unknot $\mathbb{S}^n \subset M$ is loose.

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There are still interesting questions about these structures, but they are more *geometric* in nature (Pia; 2018).

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Theorem (Casals-dP-Perez-Presas-Vogel; 2016-2020)

$$\begin{array}{ccc} \mathrm{Dist}_{\mathrm{loose}}(M^4,2) & & \\ \downarrow & \overset{\pi_0\text{—bijection}}{\searrow} & \\ \mathrm{Dist}_{\mathrm{nd}}(M^4,2) & \xrightarrow{\pi_k\text{—surjection}} & \mathrm{Dist}_f(M^4,2) \\ \uparrow & \underset{\pi_0\text{—bijection}}{\nearrow} & \\ \mathrm{Dist}_{\mathrm{OT}}(M^4,2) & & \end{array}$$

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Conjecture/theorem (Fokma-Martinez Aguinaga-dP; 2023?)

There is a π_k-surjection for surfaces transverse to Engel structures.
Same result for so-called $(1, 1)$-surfaces.

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One can define overtwisted $(2,3,5)$ distributions much like in the contact and Engel cases.

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Question

What about distributions of rank 2 in arbitrary dimension?

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Hyperbolic distributions of rank 4 in dimension 6 are flexible.

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Theorem (Jovanovik-Martinez Aguinaga-dP-Zelenko; 2022?)

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What about distributions of corank 2 in even dimensions?

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Transverse embeddings of codimension 2 satisfy π_k-surjectivity.

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