

Flexibility of distributions

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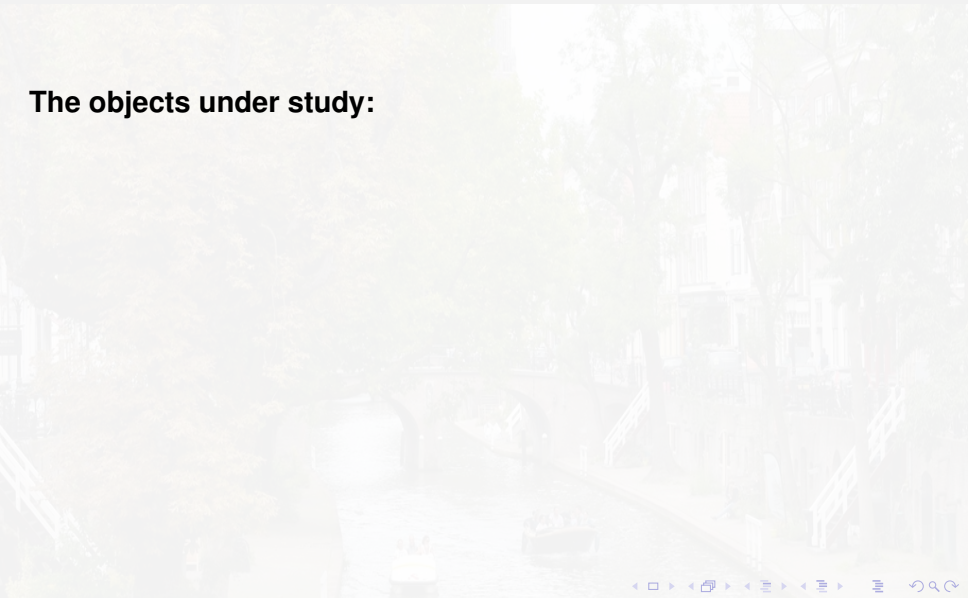
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What about them? We want to classify them up to homotopy.

How? We will use a family of tools, popularised by Gromov, called the *h-principle*.

Why? For most types of distributions little is known. But the one case we know (Contact Topology) is pretty amazing.

Other examples in the same spirit

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This story can be attempted for arbitrary geometric structures.

Some questions with the “flavour” we care about may be:

- Fix a manifold M and a dimension n . Can M be immersed into \mathbb{R}^n ?
- How many connected components does the space of symplectic structures on M have?
- Is there a non-trivial loop of positive scalar curvature metrics on M ?

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- Any distribution of rank r is locally of this form.

Typical example

Consider the standard contact structure

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we have that

$$[\phi_{\partial_y}^t, \phi_{\partial_x - y\partial_z}^t] = \phi_{\partial_z}^{t^2} (+ h.o.t.)$$

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Definition

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Main Question

What is the homotopy type of the space of the non-degenerate distributions of type (k, n) ?

One case at a time

Dimension 3

Lemma

(M^3, ξ^2) is non-degenerate iff it is *contact*

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Theorem (Eliashberg; 1989)

$$\begin{array}{ccc}
 \text{Dist}_{\text{nd}}(M^3, 2) & \xrightarrow{\pi_k\text{-surjection}} & \text{Dist}_f(M^3, 2) \\
 \uparrow & \nearrow & \\
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Theorem (Bennequin; 1982)

There are contact structures that are not in $\text{Dist}_{\text{OT}}(M, 2)$.

Dimension 3 (continued)

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Theorem (Borman-Eliashberg-Murphy; 2014)

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Fix (M^{2n+1}, ξ^{2n}) . A submanifold $L^n \subset M$ is **Legendrian** if $TL \subset \xi$.

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In dimension at least $2n + 1 \geq 5$:

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 \text{Emb}_{\text{leg}}(L; M^{2n+1}, \xi) & \xrightarrow{\pi_k\text{-surjection}} & \text{Emb}_f(L; M^{2n+1}, \xi) \\
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Theorem (Casals-Murphy-Presas; 2015)

(M, ξ) is overtwisted iff the unknot $\mathbb{S}^1 \subset M$ is loose.

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There are still interesting questions about these structures, but they are more *geometric* in nature (Pia; 2018).

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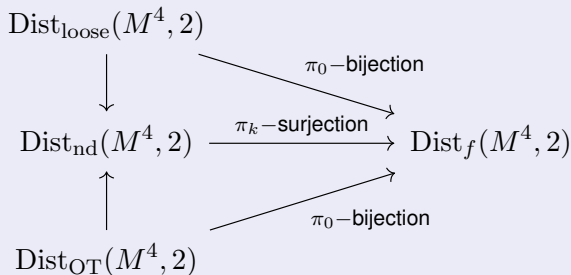
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Theorem (Casals-dP-Perez-Presas-Vogel; 2016-2020)



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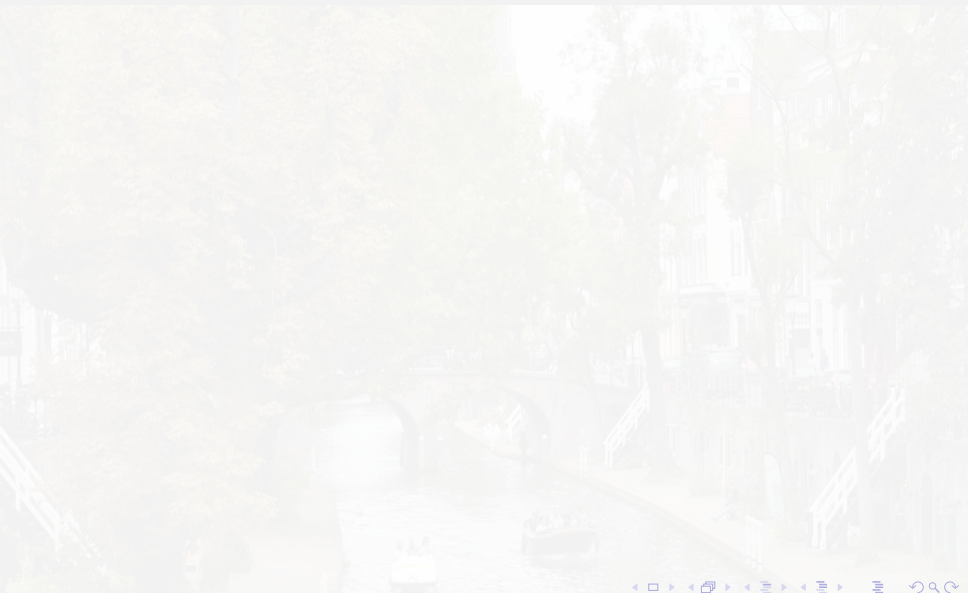
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Conjecture/theorem (Fokma-Martinez Aguinaga-dP; 2023?)

There is a π_k -surjection for surfaces transverse to Engel structures.
Same result for so-called $(1, 1)$ -surfaces.

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What about distributions of rank 2 in arbitrary dimension?

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Theorem (Jovanovik-Martinez Aguinaga-dP-Zelenko; 2022?)

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Transverse embeddings of codimension 2 satisfy π_k -surjectivity.

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