# Symplectic embedding problems and infinite staircases

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joint work with Dan Cristofaro-Gardiner, Tara Holm, and Alessia Mandini

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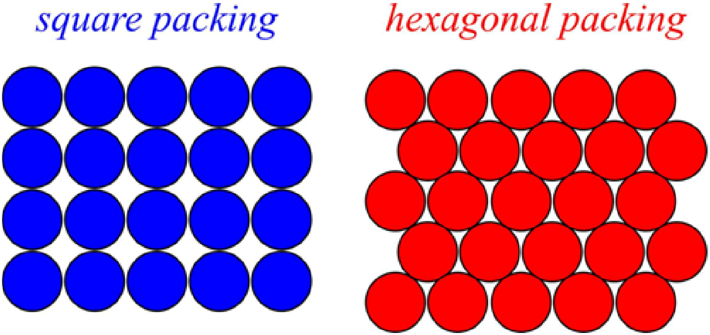
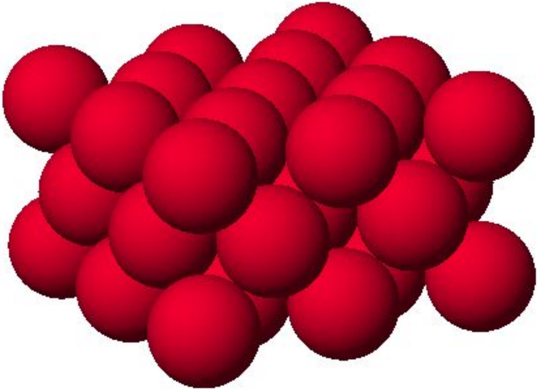
Matemáticos Portugueses no Mundo
IST Lisboa

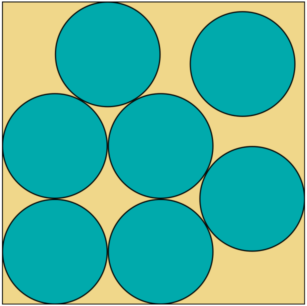


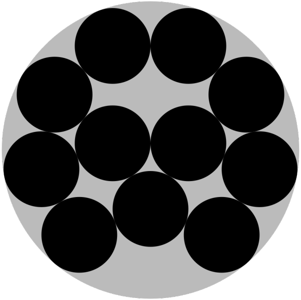

packing density $= \frac{\pi}{4} \approx 0.7854$

packing density $= \frac{\pi}{2\sqrt{3}} \approx 0.9069$



packing density $= \frac{\pi}{3\sqrt{2}} \approx 0.7405$





$$L = 4 + \sqrt{3}$$

$$p_7 = \frac{7\pi}{L^2} \approx 0.6693$$

$$r = 1 + \frac{1}{\sin \frac{\pi}{9}}$$

$$p_{11} = \frac{11}{r^2} \approx 0.7145$$

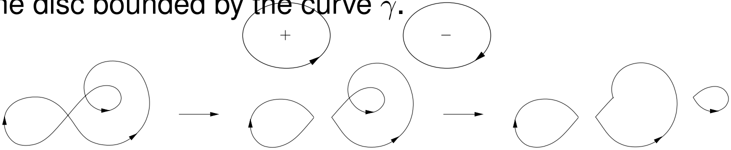
What transformations do we allow?

- Euclidean transformations: rotations and translations OR
- Volume preserving transfomations OR
- Symplectic transformations

volume preserving $\leq$ symplectic $\leq$ Euclidean

## What does "symplectic" mean?

- $\omega_0 = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$ is a symplectic form on $\mathbb{R}^{2n}$.
  For $U, V \subset \mathbb{R}^{2n}$ open, $\varphi : U \to V$ is a symplectomorphism if $\varphi^*\omega_0 = \omega_0$.
- More geometrically, in $\mathbb{R}^2 : A(\gamma) = \pm\text{area}(D)$, where $D$ is the disc bounded by the curve $\gamma$.




In $\mathbb{R}^{2n} : A(\gamma) = \sum_{i=1}^{n} A(\gamma_i)$, where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.
A symplectomorphism $\varphi : U \to V$ is a diffeomorphism that preserves the signed area of closed curves.

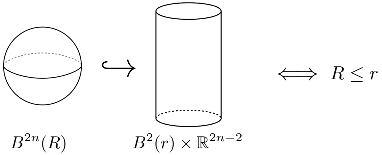
$$\int_D \omega_0 = \int_D \left( \sum_{i=1}^{n} dx_i \wedge dy_i \right) = \sum_{i=1}^{n} \left( \int_{D_i} dx_i \wedge dy_i \right) = \sum_{i=1}^{n} A(\gamma_i) = A(\gamma)$$

$\varphi$ symplectic $\implies$ $\varphi$ volume preserving

$$\varphi^*\Omega = \varphi^*(\omega^n) = \varphi^*(\omega \wedge \ldots \wedge \omega) = (\varphi^*\omega)\wedge\ldots\wedge(\varphi^*\omega) = \omega\wedge\ldots\wedge\omega = \Omega,$$

but symplectic is much more special:

- (Gromov 1985) Nonsqueezing theorem




- (Biran 1996) Symplectic packing density
  $p_k$ = percentage of volume of $B^4 \subset \mathbb{R}^4$ that can be symplectically filled by $k$ disjoint equal balls

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\geq 9$ |
|---|---|---|---|---|---|---|---|---|---|
| $p_k$ | 1 | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 | $\frac{20}{25}$ | $\frac{24}{25}$ | $\frac{63}{64}$ | $\frac{288}{289}$ | 1 |

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What are these numbers?

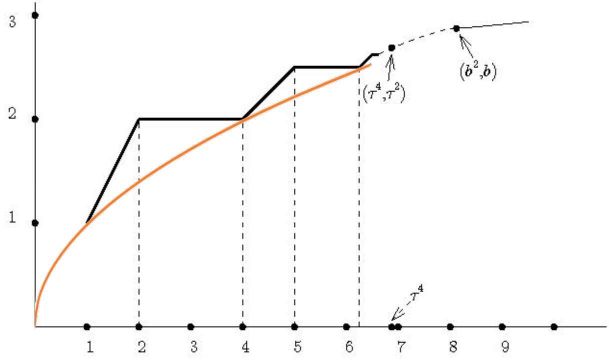
- (McDuff 2008)

$$\coprod_k B^4(1) \hookrightarrow B^4(R) \iff E(1,k) \hookrightarrow B^4(R),$$

  where $E(a,b) = \{(z_1, z_2) \in \mathbb{C}^2 : \pi\frac{|z_1|^2}{a} + \pi\frac{|z_2|^2}{b} < 1\}$ is an ellipsoid.
- (McDuff-Schlenk 2009) Study the embedding capacity function:

$$f_{B^4}(a) = \inf\{\lambda | E(1,a) \hookrightarrow B^4(\lambda)\}.$$

$$f_{B^4}(a) = \inf\{\lambda | E(1,a) \hookrightarrow B^4(\lambda)\} \geq \sqrt{a}$$




| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\geq 9$ |
|---|---|---|---|---|---|---|---|---|---|
| $p_k = \frac{k}{(f_{B^4}(k))^2}$ | 1 | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 | $\frac{20}{25}$ | $\frac{24}{25}$ | $\frac{63}{64}$ | $\frac{288}{289}$ | 1 |

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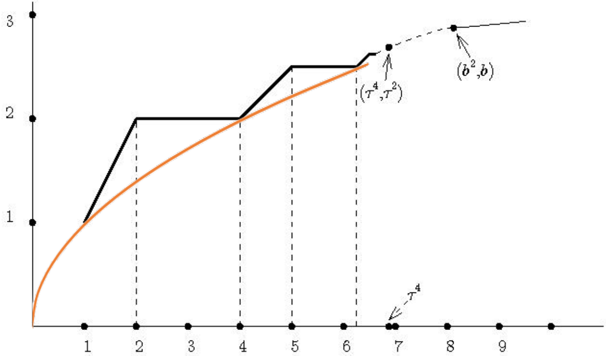


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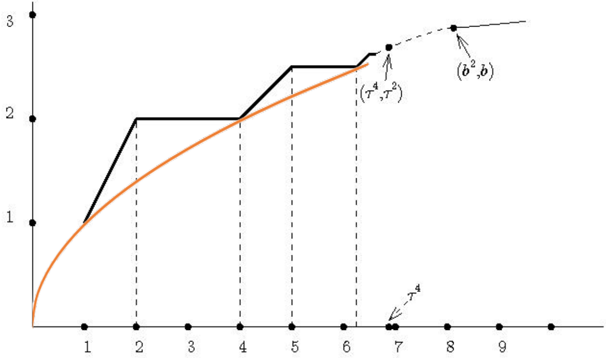
| $f_{B^4}(k)$ | 1 | 2 | 2 | 2 | $\frac{5}{2}$ | $\frac{5}{2}$ | $\frac{8}{3}$ | $\frac{17}{6}$ | $\sqrt{k}$ |

$$f_X(a) = \inf\{\lambda | E(1,a) \hookrightarrow \lambda X\} \geq \frac{\sqrt{a}}{\mathrm{vol}(X)}$$

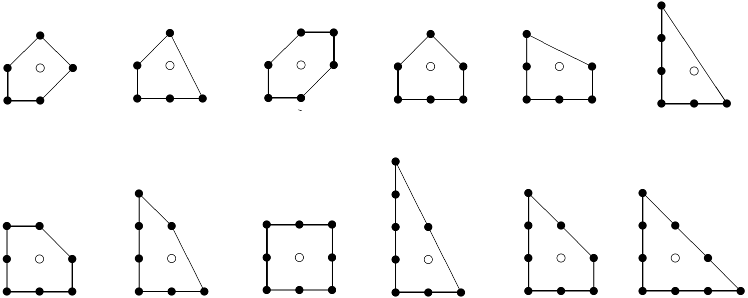



- Ball: $X = B^4(1)$, Fibonacci staircase (McDuff–Schlenk 2009)
- Polydisk: $X = B^2(1) \times B^2(1)$, Pell staircase (Frenkel–Müller 2012)
- A particular ellipsoid: $X = E(1, \frac{3}{2})$,

Region $\Omega \subset \mathbb{R}^2 \rightsquigarrow$ toric domain
$X = \{(z_1, z_2) \in \mathbb{C}^2 | \pi(|z_1|^2, |z_2|^2) \in \Omega\} \subset \mathbb{C}^2$.

**Theorem (Cristofaro-Gardiner–Holm–Mandini–P.)**

*For the toric domains corresponding to these twelve regions, the embedding capacity function* $f_X(a) = \inf\{\lambda | E(1, a) \hookrightarrow \lambda X\}$ *has an infinite staircase.*



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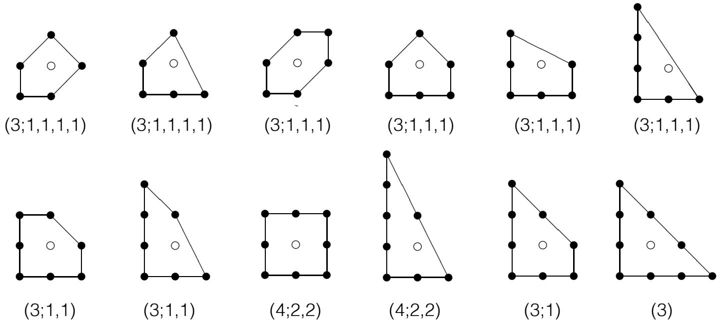

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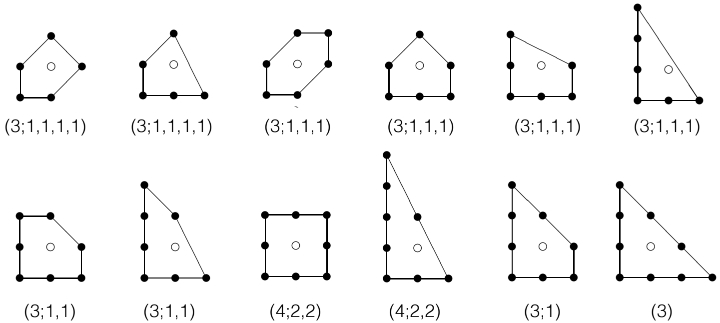



## Conjecture (Cristofaro-Gardiner–Holm–Mandini–P.)

*Among the rational convex toric domains, only for these twelve (and their scalings) does* $f_X(a)$ *have an infinite staircase.*

- $(N_k(a,b))_{k\geq 0}$: sequence formed by arranging the linear combinations $ma+nb$ with $m,n\geq 0$ in nondecreasing order, with repetitions.

  $(N_k(1,4))_{k\geq 0} = (0,1,2,3,4,4,5,5,6,6,7,7,8,8,8,9,9,9,\dots)$
  $(N_k(2,2))_{k\geq 0} = (0,2,2,4,4,4,6,6,6,6,8,8,8,8,8,8,10,10\dots)$

- (McDuff 2011)

$$E(a,b) \hookrightarrow E(c,d) \iff \forall_k \; N_k(a,b) \leq N_k(c,d)$$

  So for instance, $E(1,4) \hookrightarrow E(2,2) = B(2)$.

- (Hutchings 2010) $c_k(M^4)$: numerical invariants associated to a 4-dimensional symplectic manifold – ECH capacities. They obstruct symplectic embeddings:

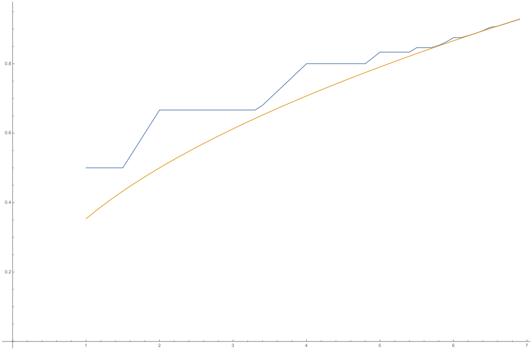
$$M_1 \hookrightarrow M_2 \implies \forall_k \; c_k(M_1) \leq c_k(M_2)$$

- (Cristofaro-Gardiner 2014) Furthermore, for certain types of $M_1$ and $M_2$, ECH capacities are sharp obstructions to symplectic embeddings:

- Combinatorial description of $f_X(a)$:

$$\text{Since } E(1,a) \hookrightarrow \lambda X \iff \forall_k \; c_k\left(E(1,a)\right) \le c_k(\lambda X)$$
$$\iff \forall_k \; N_k\left(1,a\right) \le \lambda\, c_k(X),$$

we have $f_X(a) = \sup_k \frac{N_k(1,a)}{c_k(X)}$.



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A007826 Numbered stops on the Market-Frankford rapid transit (SEPTA) railway line in Philadelphia, PA USA.

2, 5, 8, 11, 13, 15, 30, 34, 40, 46, 52, 56, 60, 63, 69 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

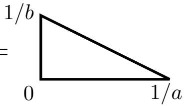
COMMENTS Formally abbreviated as The Blue Line (and known informally as 'The El'), the Market-Frankford Line extends East to West from slightly to the east of 2nd Street through the city line to the western suburbs at 63rd Street and then on to 69th Street Transportation Center, lined up almost entirely with the major dividing thoroughfare Market Street. It is actually a subway at the eastern end of this portion and through to beyond the 40th Street stop (a(1)-a(9) represent subway stops), passing under the Schuylkill River (along with trolley lines 10, 11, 13, 34 and 36) closer to 30th than to 15th Street. The only non-numbered stop on this end is suburban Milbourne between 63rd and 69th. The 'Frankford end' runs in a somewhat northeasterly direction and has all stops only with non-number names (and is entirely above ground). The semi-express A and B versions of the train both skip certain stops at peak travel times, and the only regular trains are unmarked or one of these two versions. The train is substituted for with bus service during overnight hours. - James G. Merickel, Mar 19 2014

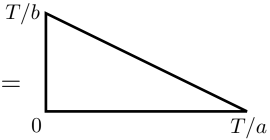
REFERENCES Ayshe Ozbekhan, Letter to N. J. A. Sloane, Oct 04, 1994.

LINKS Table of n, a(n) for n=1..15.
Wikipedia, Market-Frankford Line

FORMULA a(n) = 2 + 3 n - binomial(n, 4) + 3 binomial(n, 5) + 7 binomial(n, 6) - 66 binomial(n, 7) + 248 binomial(n, 8) - 679 binomial(n, 9) + 1554 binomial(n, 10) - 3158 binomial(n, 11) + 5897 binomial(n, 12) - 10352

- $N_k(a, b)$ as lattice point counting:

Let $\triangle_{\frac{1}{a},\frac{1}{b}} =$ 

and $T \cdot \triangle_{\frac{1}{a},\frac{1}{b}} =$ 


Note that
$\#\left(T \cdot \triangle_{\frac{1}{a},\frac{1}{b}} \bigcap \mathbb{Z}^2\right) = \#\left\{(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0 | ma + nb \leq T\right\}.$
Since $(N_k(a, b))_{k \geq 0}$ is nondecreasing,

$$N_k(a, b) = \inf\left\{T | \#\left(T \cdot \triangle_{\frac{1}{a},\frac{1}{b}} \cap \mathbb{Z}^2\right) \geq k + 1\right\}.$$

- Ehrhart theory:

Ehrhart function of a polygon $P$ = number of lattice points in a scaling of $P$

$$L_P(T) = \#\left(T \cdot P \cap \mathbb{Z}^2\right), \qquad \text{for } T \in \mathbb{Z}$$

- Ehrhart theory:

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$$L_P(T) = \#\left(T \cdot P \cap \mathbb{Z}^2\right), \qquad \text{for } T \in \mathbb{Z}$$

The function $L_{\triangle_{u,v}}(T)$ is a polynomial if $u, v \in \mathbb{Z}$ and in a few more cases*,

a quasipolynomial if $u, v \in \mathbb{Q}$ and a few more cases*,

and a horrible function if $u, v \notin \mathbb{Q}$.

(* These "few more cases" of period collapse is when infinite staircases can happen!)

For example,

$$L_{\triangle_{1,1}}(T) = \tfrac{1}{2}T^2 + \tfrac{3}{2}T + 1$$

$$L_{\triangle_{\frac{1}{3},\frac{1}{3}}}(T) = \begin{cases} \frac{1}{18}\left(T^2 + 9T + 18\right) & \text{if } T \equiv 0 \\ \frac{1}{18}\left(T^2 + 7T + 10\right) & \text{if } T \equiv 1 \\ \frac{1}{18}\left(T^2 + 5T + 4\right) & \text{if } T \equiv 2 \end{cases} \pmod 3$$

...for $u, v \notin \mathbb{Q}$, define $\alpha := u + v$ and $\beta := \frac{1}{u} + \frac{1}{v}$. Then:

$$L_{\triangle_{\frac{1}{u},\frac{1}{v}}}(T) = \left(\sum_{m=0}^{\lfloor \frac{T}{\alpha} \rfloor} \lfloor \beta(T - m\alpha) \rfloor \right) + \sigma(T) + \# \left\{ 0 \le m \le \lfloor \frac{T}{\alpha} \rfloor | \{\beta(T - m\alpha)\} >\right.$$

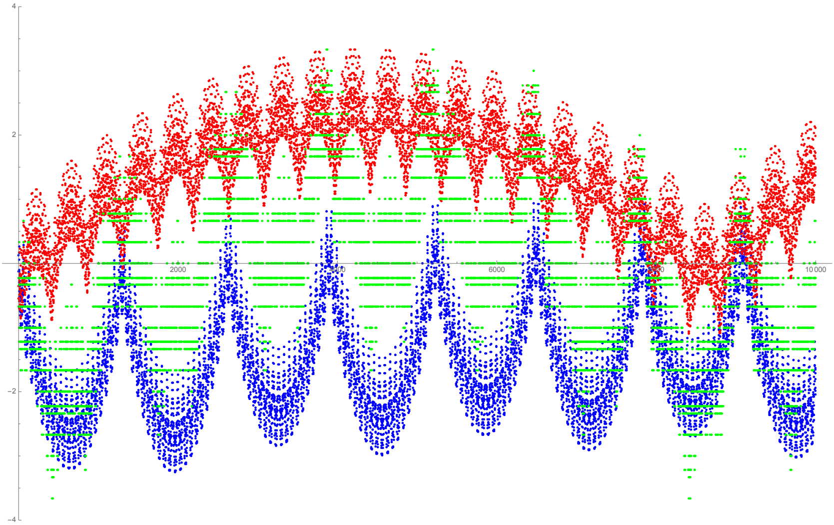
where $\sigma(T) = 1$ if $\frac{T}{\alpha} \in \mathbb{Z}$ and 0 otherwise.

This turns out to be:

$$L_{\triangle_{\frac{1}{u},\frac{1}{v}}}(T) = \frac{1}{2uv} T^2 + \frac{1}{2}\left(\frac{1}{u} + \frac{1}{v}\right) T + o(T)$$

and under certain conditions:

$$L_{\triangle_{\frac{1}{u},\frac{1}{v}}}(T) = \frac{1}{2uv} T^2 + \frac{1}{2}\left(\frac{1}{u} + \frac{1}{v}\right) T + 1 + \sum_{m=1}^{\frac{T}{\alpha}} \left(\{m\frac{u}{v}\} - \frac{1}{2}\right) + \sum_{m=1}^{\frac{T}{\alpha}} \left(\{m\right.$$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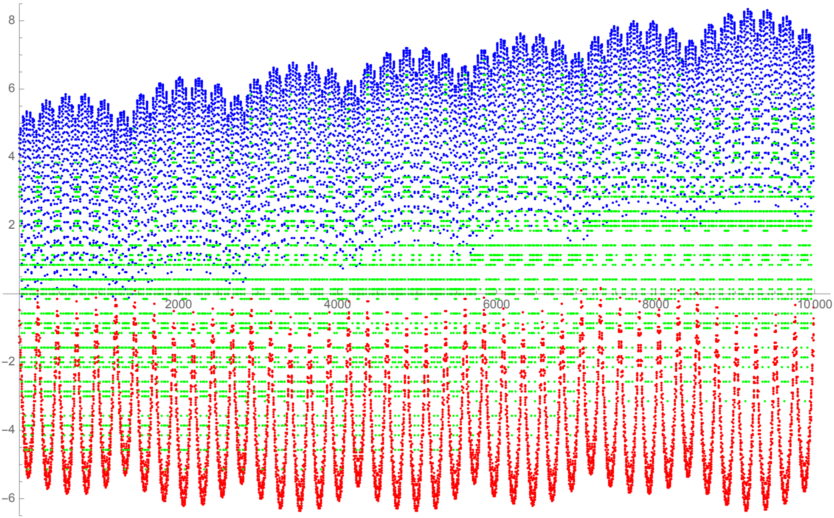
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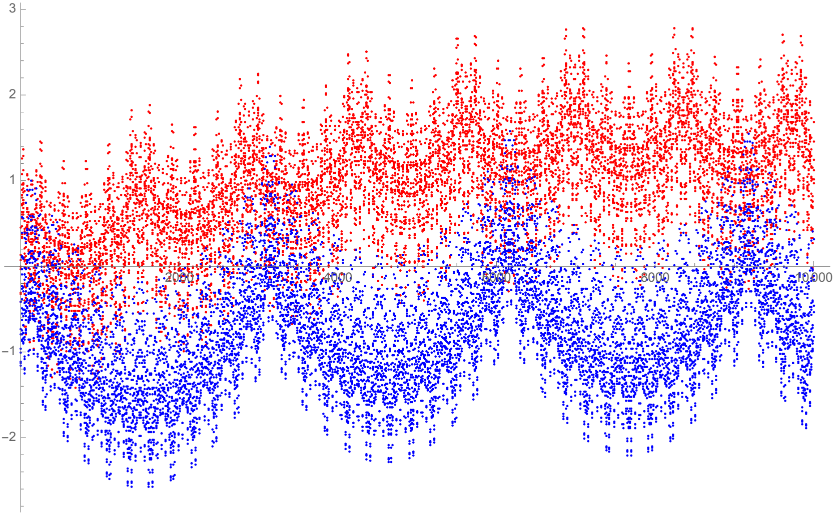
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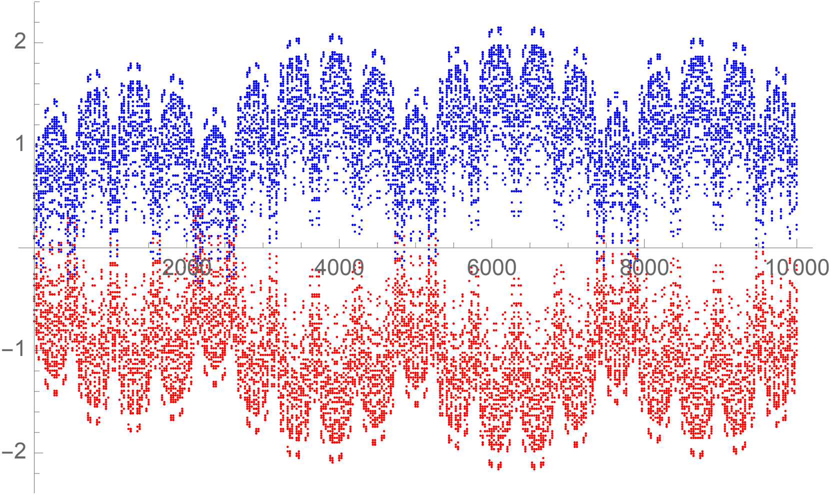
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Thank you.