Deep neural networks, universal approximation, and nonlinear geometric control.

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Autonomy and deep learning

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Here is one example from the robotics domain.

The Qualcomm Robotics RB5 Platform supports the development of next generation of high-compute, AI-enabled, low power robots and drones for the consumer, enterprise, defense, industrial and professional service sectors that can be connected by 5G.

The platform’s Qualcomm QRB5165 processor, customized for robotics applications, offers a powerful heterogeneous computing architecture coupled with the leading 5th generation Qualcomm® Artificial Intelligence (AI) Engine delivering 15 Trillion Operations Per Second (TOPS) of AI performance to efficiently run complex AI and deep learning workloads and on-device edge inferencing while using lower power, on-device machine learning, and accurate edge inferencing. The processor also offers a powerful image signal processor (ISP) with support for seven concurrent cameras, a dedicated computer
Autonomy and deep learning

- Clearly, industry is ahead of academia.
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- Formal guarantees when deep learning is used within a control loop?
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  6. Outlook
Residual Neural Networks
What are ResNets?

- A diagrammatic depiction of a neural network:

```
Input
Layer 1 Layer 2 Layer 3
Layer ℓ
Output
```

- Let us denote by $x(k) \in \mathbb{R}^4$ the state of each layer with $k = 1, 2, \ldots, \ell$. 

What are ResNets?

- A diagrammatic depiction of a neural network:

Let us denote by \( x(k) \in \mathbb{R}^4 \) the state of each layer with \( k = 1, 2, \ldots, \ell \).

The state of layer \( k + 1 \) is computed from the state of layer \( k \) according to:

\[
x(k + 1) = \Sigma(W(k)x(k) + b(k)),
\]

where \((W, b)\) are the weights of the connections (arrows) and \( \Sigma \) is of the form:

\[
\Sigma(x) = (\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_n)),
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for an activation function \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \).
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![Diagram of a neural network]

- Let us denote by $x(k) \in \mathbb{R}^4$ the state of each layer with $k = 1, 2, \ldots, \ell$.

- For ResNets, the state of layer $k + 1$ is computed from the state of layer $k$ according to:

$$x(k + 1) = x(k) + S(k)\Sigma(W(k)x(k) + b(k)),$$

where $(S, W, b)$ are the weights of the connections (arrows) and $\Sigma$ is of the form:

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Paulo Tabuada (CyPhyLab - UCLA)
It was observed in the last 4 years\(^1\) that the equation:

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x(k + 1) = x(k) + s(k)\Sigma(W(k)x(k) + b(k)),
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is remarkably similar to the forward Euler discretization of the continuous-time control system:

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\dot{x} = s\Sigma(Wx + b),
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with state \(x \in \mathbb{R}^n\) and where \((s, W, b) \in \mathbb{R} \times \mathbb{R}^{n \times n} \times \mathbb{R}^n\) are regarded as control inputs.

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\(^1\)A proposal on machine learning via dynamical systems

Stable architectures for deep neural networks

Beyond finite layer neural networks: Bridging deep architectures and numerical differential equations
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Properties of (2) approximately transfer to (1) by time-discretizing solutions.

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\(^1\) A proposal on machine learning via dynamical systems

**Stable architectures for deep neural networks**

**Beyond finite layer neural networks: Bridging deep architectures and numerical differential equations**
Memorization Capabilities of Residual Neural Networks
Memorization capabilities of ResNets

Problem (Memorization)

Given:

- a function $f : E \rightarrow \mathbb{R}^n$ defined on a compact set $E \subset \mathbb{R}^n$,
- a finite set $E_{\text{samples}} \subset E$,
- the evaluation of $f$ on $E_{\text{samples}}$, i.e., $f(x)$ for each $x \in E_{\text{samples}},$

does there exist a ResNet outputing $f(x)$ for each input $x \in E_{\text{samples}}$?
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Does there exist a time $\tau \in \mathbb{R}_0^+$ and an input $(s, W, b) : [0, \tau] \to \mathbb{R} \times \mathbb{R}^{n \times n} \times \mathbb{R}^n$ so that the solution $\xi$ of:

$$\dot{x} = s\Sigma(Wx + b),$$

satisfies $\xi(0) = x$ and $\xi(\tau) = f(x)$ for every $x \in E_{\text{samples}}$. 
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- Is this a controllability problem?
Let’s consider the case where $E_{\text{samples}} = \{x^1, x^2\}$. 
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The same input needs to control two copies of the same system.
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- Controllability on \( \mathbb{R}^n \times \mathbb{R}^n \)?
Given a finite set of samples $E_{samples} = \{x^1, x^2, \ldots, x^d\}$ we consider the ensemble control system:

$$\dot{X} = \left[ s\Sigma(WX_{\bullet 1} + b)|s\Sigma(WX_{\bullet 2} + b)| \ldots |s\Sigma(WX_{\bullet d} + b) \right], \quad (3)$$

where the state $X(t) \in \mathbb{R}^{n \times d}$ is the matrix:

$$X(t) = [X_{\bullet 1}(t)|X_{\bullet 2}(t)| \ldots |X_{\bullet d}(t)].$$
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$$\dot{X} = \begin{bmatrix} s\Sigma(WX_1 + b) | s\Sigma(WX_2 + b) | \ldots | s\Sigma(WX_d + b) \end{bmatrix},$$  \hspace{1cm} (3)

where the state $X(t) \in \mathbb{R}^{n \times d}$ is the matrix:

$$X(t) = \begin{bmatrix} X_1(t) | X_2(t) | \ldots | X_d(t) \end{bmatrix}.$$

- We can now ask: does there exist an input $(s, W, b) : [0, \tau] \rightarrow \mathbb{R} \times \mathbb{R}^{n \times n} \times \mathbb{R}^n$ so that the solution $X$ of (3) satisfies:

$$X(0) = \begin{bmatrix} x^1 | x^2 | \ldots | x^d \end{bmatrix} \text{ and } X(\tau) = \begin{bmatrix} f(x^1) | f(x^2) | \ldots | f(x^d) \end{bmatrix}?$$
As typically done in geometric control theory, we work with piecewise constant control inputs so that for each choice of input we obtain a vector field:

\[ \mathcal{F} = \{Z_1, Z_2, \ldots, Z_k\}. \]

---

2. **Orbits of families of vector fields and integrability of distributions.**

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- What is the orbit (the reachable space) of the family \( \mathcal{F} \) of vector fields?

- Difficult problem, in general, that has a simpler answer\(^2\) when \( \mathcal{F} \) is symmetric, i.e.:

\[ Z \in \mathcal{F} \implies -Z \in \mathcal{F}. \]

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The answer is then given by the Lie algebra rank condition:

the dimension of the Lie algebra generated by \( \mathcal{F} \) equals \( nd \) at every \( X \in \mathbb{R}^{n \times d} \).

---

The Lie algebra generated by $\mathcal{F}$ is the smallest vector subspace of $T_{\mathbb{R}^{n \times d}}$ containing $\mathcal{F}$ and closed under Lie brackets:

$$[Z_1, Z_2] = \frac{\partial Z_2}{\partial A} Z_1 - \frac{\partial Z_1}{\partial A} Z_2.$$
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- Choose the inputs $W$ and $b$ so that the vector fields and their Lie brackets only contain:

  \[
  \sigma, D\sigma, D^2\sigma, \ldots
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- Since the activation function is not known, these brackets are not known.

- Choose the inputs $W$ and $b$ so that the vector fields and their Lie brackets only contain:

$$\sigma, D\sigma, D^2\sigma, \ldots$$

- With such choice, and after judicious (and tedious) manipulations, the rank of the Lie algebra is $nd$ provided the rank of the following matrix is $n$:

$$
\begin{bmatrix}
1 & \sigma(A_{1\ell}) & D\sigma(A_{1\ell}) & \cdots & D^{n-2}\sigma(A_{1\ell}) \\
1 & \sigma(A_{2\ell}) & D\sigma(A_{2\ell}) & \cdots & D^{n-2}\sigma(A_{2\ell}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \sigma(A_{n\ell}) & D\sigma(A_{n\ell}) & \cdots & D^{n-2}\sigma(A_{n\ell})
\end{bmatrix}.
$$
Memorization capabilities of ResNets

Ensemble controllability

- When is the determinant of this matrix nonzero:

$$\begin{bmatrix}
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\end{bmatrix}
\]

- **Key idea:** relation to Vandermonde matrices?
Lemma

Let \( \sigma : \mathbb{R} \rightarrow \mathbb{R} \) be a function that satisfies the quadratic differential equation:

\[
D\sigma(x) = a_0 + a_1 \sigma(x) + a_2 \sigma^2(x),
\]

where \( a_0, a_1, a_2 \in \mathbb{R} \). Suppose that derivatives of \( \sigma \) of up to order \( (\ell - 2) \) exist at \( \ell \) points \( x_1, \ldots, x_\ell \in \mathbb{R} \). Then, the determinant of the matrix:

\[
L(x_1, x_2, \ldots, x_\ell) = \begin{bmatrix}
    1 & 1 & \cdots & 1 \\
    \sigma(x_1) & \sigma(x_2) & \cdots & \sigma(x_\ell) \\
    D\sigma(x_1) & D\sigma(x_2) & \cdots & D\sigma(x_\ell) \\
    \vdots & \vdots & \ddots & \vdots \\
    D^{\ell-2}\sigma(x_1) & D^{\ell-2}\sigma(x_2) & \cdots & D^{\ell-2}\sigma(x_\ell)
\end{bmatrix},
\]

is given by:

\[
\det L(x_1, x_2, \ldots, x_\ell) = \prod_{i=1}^{\ell-2} i! a_2^i \prod_{1 \leq i < j \leq \ell} (\sigma(x_i) - \sigma(x_j)).
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When is this expression non-zero:

$$\det L(x_1, x_2, \ldots, x_\ell) = \prod_{i=1}^{\ell-2} i!a_2^i \prod_{1 \leq i < j \leq \ell} (\sigma(x_i) - \sigma(x_j))$$?
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- If $a_2 \neq 0$ and $\sigma$ is injective, $\prod_{1 \leq i < j \leq \ell} (x_i - x_j) \neq 0$ implies $\det L \neq 0$. 
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- Two different ensemble elements $i$ and $j$ cannot be in states $X_{\bullet i}$ and $X_{\bullet j}$ that share an entry, i.e., for any $\ell$: $X_{\ell i} \neq X_{\ell j}$. 
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- Are there injective activation functions \( \sigma \) satisfying \( D\sigma = a_0 + a_1 \sigma + a_2 \sigma^2 \)?
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<td>( D\sigma - 1 + \sigma^2 = 0 )</td>
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<tr>
<td>Soft plus</td>
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</tr>
</tbody>
</table>

Moreover, \( \lim_{r \to \infty} \frac{1}{r} \log(1 + e^{rx}) = \text{ReLU}(x) = \max\{0, x\} \).
Memorization capabilities of ResNets

Ensemble controllability

Theorem

Let $N \subset \mathbb{R}^{n \times d}$ be the set defined by:

$$N = \left\{ A \in \mathbb{R}^{n \times d} \mid \prod_{1 \leq i < j \leq d} (A_{\ell i} - A_{\ell j}) = 0, \ \ell \in \{1, \ldots, n\} \right\}.$$  

Let $n > 1$ and assume the activation function $\sigma$ is injective, non-negative, and satisfies $D\sigma = a_0 + a_1 \sigma + a_2 \sigma^2$ for some $a_2 \neq 0$. Then the ensemble control system is controllable on the submanifold $M = \mathbb{R}^{n \times d} \setminus N$. 
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- When $n > 1$, $M$ is connected, open, and dense in $\mathbb{R}^{n \times d}$.
- If $E_{\text{samples}}$ and $f(E_{\text{samples}})$ are subsets of $M$, a ResNet can memorize them exactly.
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- When \( n > 1 \), \( M \) is connected, open, and dense in \( \mathbb{R}^{n \times d} \).
- If \( E_{\text{samples}} \) and \( f(E_{\text{samples}}) \) are subsets of \( M \), a ResNet can memorize them exactly.
- Otherwise we can perturb \( E_{\text{samples}} \) or/and \( f(E_{\text{samples}}) \) to make them subsets of \( M \).
Approximation Capabilities of Residual Neural Networks
We established controllability on the finite dimensional state space $\mathbb{R}^{n \times d}$.

But we would really like to establish controllability on some infinite dimensional space of functions.\(^3\)

Approximation capabilities of ResNets

- We established controllability on the finite dimensional state space $\mathbb{R}^{n \times d}$.
- But we would really like to establish controllability on some infinite dimensional space of functions\(^3\).
- Let $\phi^t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the flow defined by the solution of the control system:
  $$\dot{x} = s \Sigma (Wx + b),$$
  i.e., $\phi^t(x) = \xi(t)$ where $\xi$ is the solution satisfying $\xi(0) = x$.

\(^3\)Control on the manifold of mappings as a setting for deep learning.
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Do there exist inputs $(s, W, b) : [0, \tau] \rightarrow \mathbb{R} \times \mathbb{R}^{n \times n} \times \mathbb{R}^n$ resulting in a flow $\phi^t$ satisfying:

$$\phi^0(x) = x \text{ and } \phi^\tau(x) = f(x)?$$

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- Can we use the previous controllability result as a stepping stone?
  - If we map finitely many points to the right location, can things go wrong for the points we leave out?

---

Consider the function interpolation problem.
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How to control the behavior of the interpolating function between the interpolation points?
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How to control the behavior of the interpolating function between the interpolation points?

Key idea: monotonicity.
What is monotonicity?

Define the ordering $\preceq$ on $\mathbb{R}^n$ by $x \preceq x'$ iff $x_i \leq x'_i$ for all $i = 1, 2, \ldots, n$. 

A flow $\phi: \mathbb{R}^n \to \mathbb{R}^n$ is monotone if:

$x \preceq x' \implies \phi(x) \preceq \phi(x')$. 

Paulo Tabuada (CyPhyLab - UCLA)
Approximation capabilities of ResNets

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Main result

- When the function $f$ to be learned is monotone, we can construct a monotone flow $\phi^t$, by using the previous controllability result, approximating $f$ on $E_{\text{samples}}$.

Theorem

Let $n > 1$ and assume the activation function $\sigma$ is injective, non-negative, and satisfies $D\sigma = a_0 + a_1\sigma + a_2\sigma^2$ for some $a_2 \neq 0$. Then, for every monotone analytic function $f : \mathbb{R}^n \to \mathbb{R}^n$, for every compact set $E \subset \mathbb{R}^n$, and for every $\varepsilon \in \mathbb{R}^+$ there exist a time $\tau \in \mathbb{R}^+$ and an input $(s, W, b) : [0, \tau] \to \mathbb{R} \times \mathbb{R}^{n \times n} \times \mathbb{R}^n$ so that the flow $\phi^\tau : \mathbb{R}^n \to \mathbb{R}^n$ defined by the solution of $\dot{x} = s\Sigma(Wx + b)$ under the said input satisfies:

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- What happens when $f$ is not monotone?
When the function $f$ to be learned is monotone, we can construct a monotone flow $\phi^t$, by using the previous controllability result, approximating $f$ on $E_{\text{samples}}$.

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- What happens when $f$ is not monotone?
- Key idea: monotone embedding.
Main result

We seek:

- a linear injection $\alpha : \mathbb{R}^n \to \mathbb{R}^{n+1}$,
- a linear projection $\beta : \mathbb{R}^{n+1} \to \mathbb{R}^n$,
- a monotone function $\tilde{f} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

so that:

$$f = \beta \circ \tilde{f} \circ \alpha.$$
Approximation capabilities of ResNets

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Approximation capabilities of ResNets

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- This can be accomplished with:

  - $\alpha(x) = (x, 1^T x) = (x, x_1 + x_2 + \ldots + x_n)$,
  - $\beta(x, y) = x - \kappa y$,
  - $\tilde{f}(x, y) = (f(x) + \kappa 1 y, y)$. 

Approximation capabilities of ResNets

Main result

Corollary

Let $n > 1$ and assume the activation function $\sigma$ is injective, non-negative, and satisfies $D\sigma = a_0 + a_1 \sigma + a_2 \sigma^2$ for some $a_2 \neq 0$. Then, for every continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$, for every compact set $E \subset \mathbb{R}^n$, and for every $\varepsilon \in \mathbb{R}^+$ there exist a time $\tau \in \mathbb{R}^+$, an injection $\alpha : \mathbb{R}^n \to \mathbb{R}^{n+1}$, a projection $\beta : \mathbb{R}^{n+1} \to \mathbb{R}^n$, and an input $(s, W, b) : [0, \tau] \to \mathbb{R} \times \mathbb{R}^{(n+1) \times (n+1)} \times \mathbb{R}^{n+1}$ so that the flow $\phi^\tau : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by the solution of $\dot{x} = s \Sigma(Wx + b)$ under the said input satisfies:

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A Deterministic Generalization Bound
A deterministic generalization bound

**Lemma**

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous map defined on a compact set $E \subset \mathbb{R}^n$. Suppose $E_{\text{samples}} \subset \mathbb{R}^n$ is a finite set satisfying:

$$\forall x \in E \quad \exists \underline{x}, \overline{x} \in E_{\text{samples}}, \quad |\underline{x} - \overline{x}|_{\infty} \leq \delta \land \underline{x} \preceq x \preceq \overline{x},$$

with $\delta \in \mathbb{R}^+$. For any monotone map $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ we have:

$$\|f - \phi\|_{L^{\infty}(E)} \leq 2\omega_f(\delta) + 3\|f - \phi\|_{L^{\infty}(E_{\text{samples}})},$$

where $\omega_f$ is the modulus\(^a\) of continuity of $f$.

\(^a\)Note that $f$, being continuous, is uniformly continuous on any compact set.
Lemma

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where $\omega_f$ is the modulus\(^a\) of continuity of $f$.

\(^a\)Note that $f$, being continuous, is uniformly continuous on any compact set.

- Can we use such bound in a control context?
Outlook
Consider a closed-loop system with a ResNet in the perception pipeline.

\[
u(t) = k(\hat{x}(t)) = k(x(t) + e(t))
\]

Assume the controller renders the closed-loop system ISS with respect to estimation errors \(e(t)\), i.e.:

\[
\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|e\|_{L_\infty}).
\]

Train a ResNet to learn the map from output measurements \(y\) to the state \(x\), i.e., to act as an observer \(\phi(y) = \hat{x}\).

By the generalization lemma, \(e = \hat{x} - x\) is bounded by a constant \(c \in \mathbb{R}^+\) and we directly obtain practical stability:

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Assume the controller $u = k(\hat{x}) = k(x + e)$ renders the closed-loop system ISS with respect to estimation errors $e$, i.e.:

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Ideas from control can play a big role in understanding deep neural networks.
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Approximation and generalization guarantees are possible in deterministic settings.
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Approximation and generalization guarantees are possible in deterministic settings.

We can start to imagine control loops with learning components designed to satisfy formal safety and performance guarantees.
However, there are still many challenges.

- Learning observers from vision/LiDAR data require us to address extrapolation (the generalization lemma is about interpolation).
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For more information:
http://www.cyphylab.ee.ucla.edu/
http://www.ee.ucla.edu/~tabuada