The Toda lattice and the Viterbo conjecture

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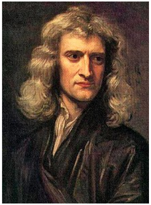

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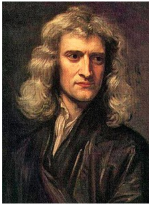

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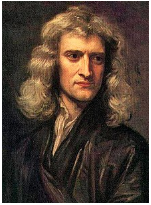

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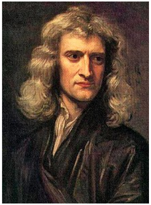

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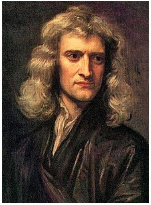

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If it exists, we write $X_1 \overset{s}{\hookrightarrow} X_2$.

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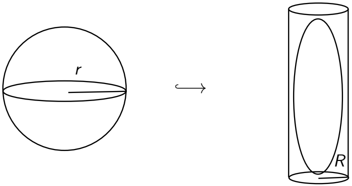

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Gromov's nonsqueezing theorem, 1985

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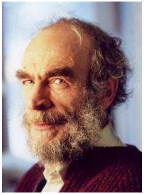

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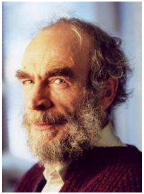

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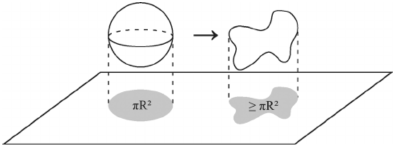

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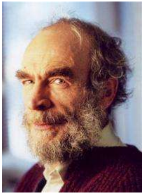

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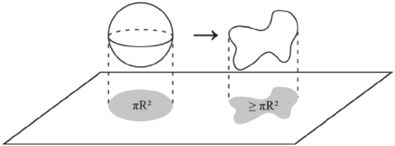

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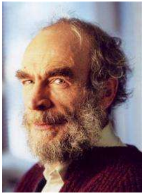

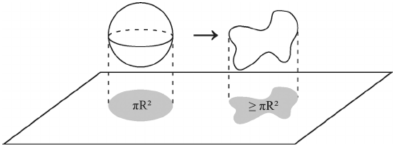

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The existence of a normalized symplectic capacity is equivalent to Gromov's nonsqueezing theorem.

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- First embedded contact homology capacity c_1^{ECH} (Hutchings 2011) - only in dimension 4.

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Moreover equality holds if, and only if, X is symplectomorphic to a ball.

Minimal action

If X is a compact and convex set of $\mathbb{R}^{2n}$ with smooth boundary, let $A_{min}(X)$ denote the shortest period of a closed characteristic on ∂X.

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Strong Viterbo $\Rightarrow$ Viterbo $\Rightarrow$ Weak Viterbo $\Rightarrow$ Mahler

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Definition

A toric domain $X_\Omega \subset \mathbb{C}^n$ is a set of the form $X_\Omega = \mu^{-1}(\Omega)$ where $\Omega \subset \mathbb{R}^n_{\geq 0}$ is star-shaped with respect with 0 and

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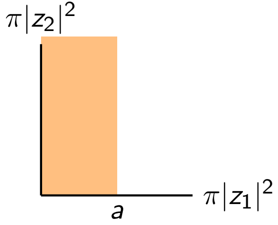

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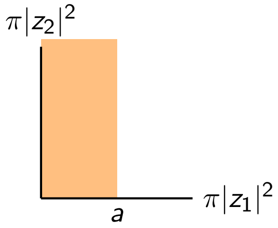

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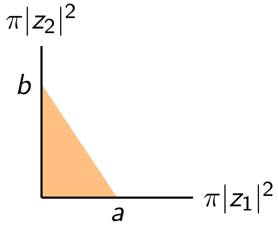

$E(a, b) := \left\{(z_1, z_2) \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} \leq 1\right\}$

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A toric domain $X_\Omega \subset \mathbb{R}^{2n}$ is called *monotone* if for each point $p \in \partial\Omega \setminus \{x_i = 0, \text{ for some i}\}$, the normal vector $\nu = (\nu_1, \ldots, \nu_n)$ satifies $\nu_i \geq 0$ for every i.

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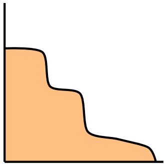

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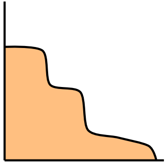

Remark

If X_Ω is monotone, then it can be approximated by a toric domains which are bounded by the coordinate hyperplanes and a the graph of a non-negative smooth function $f : \Omega' \subset \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ whose partial derivatives are all negative.

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Monotone toric domains are not necessarily convex.

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Proposition

A 4-dimensional toric domain X_Ω is (strictly) monotone if, and only if, $(\partial X_\Omega, \alpha_0)$ is dynamically convex.

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Theorem (Gutt–Hutchings–R. 2020)

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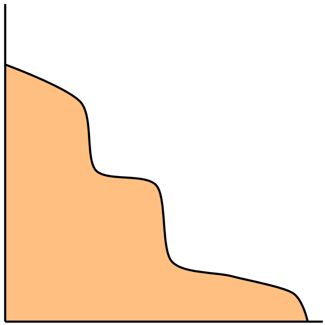
For a monotone toric domain $X_\Omega \subset \mathbb{R}^{2n}$,

$$c_{Gr}(X_\Omega) = c_1^{CH}(X_\Omega).$$

Proof



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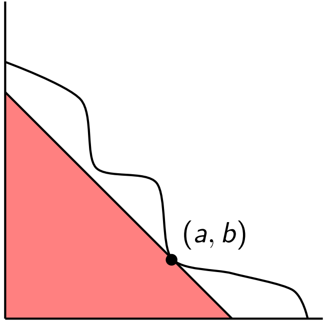

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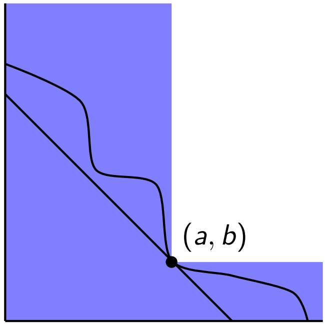

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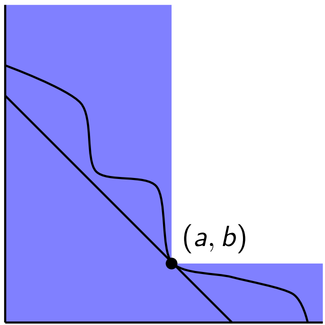

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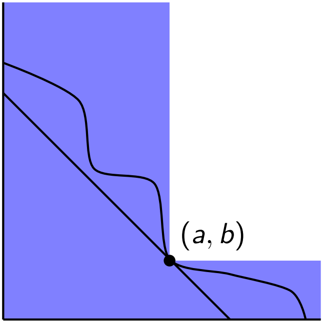

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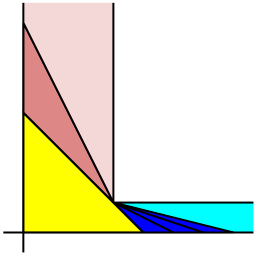
In dimension 4, let

$$Z_2(a, b) := \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 \leq a \text{ or } \pi|z_2|^2 \leq b\}.$$

Proof

$Z_2(a,b)$ is a concave toric domain with weight sequence $(a+b, a, b, a, b, a, b, \ldots)$.



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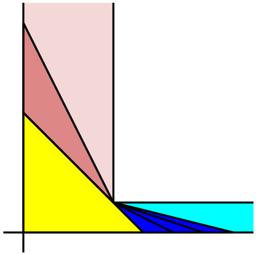

Theorem (Cristofaro-Gardiner)

If X_Ω is a concave toric domain with weight sequence $(w_1, w_2, \dots)$. Then

$$X_\Omega \overset{s}{\hookrightarrow} Z^4(r) \iff \bigsqcup_i B(w_i) \overset{s}{\hookrightarrow} Z^4(r).$$

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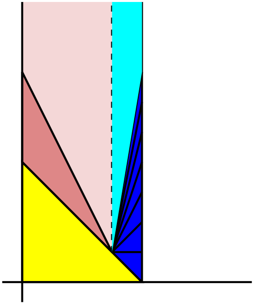

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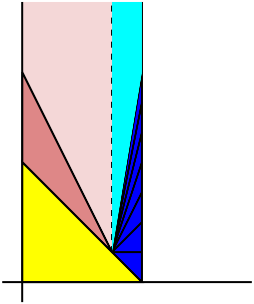

Corollary

$$Z_2(a,b) \overset{s}{\hookrightarrow} Z^4(a+b).$$

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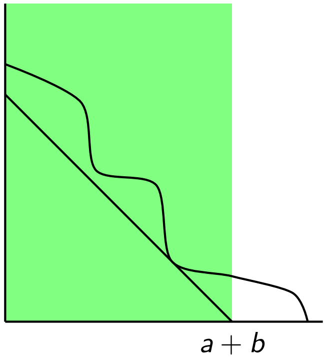

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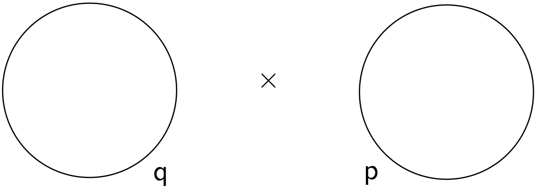

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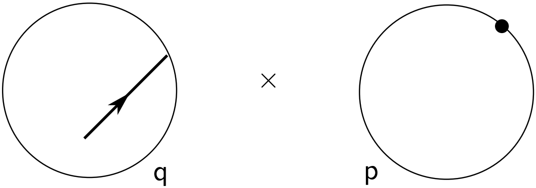

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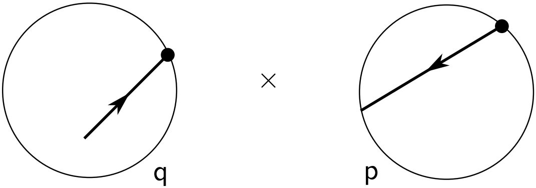

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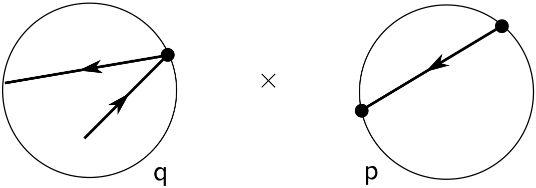

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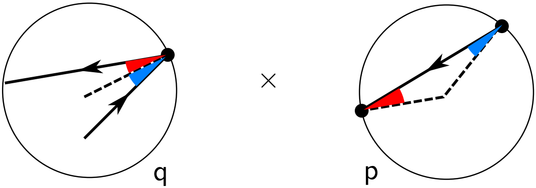

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The Arnold-Liouville theorem

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  Then there exists a symplectomorphism $\Phi : (U, \omega) \to (\phi(U) \times \mathbb{T}^n, \omega_0)$ such that the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & \phi(U) \times \mathbb{T}^n \\ \downarrow F & & \downarrow \pi_1 \\ F(U) & \xrightarrow{\phi} & \phi(U) \end{array}$$

The Arnold-Liouville theorem in action

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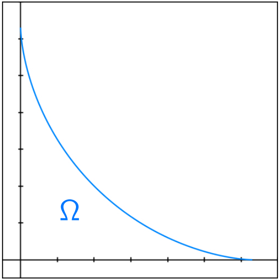

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$$J(\mathrm{q},\mathrm{p}) = \frac{1}{3}\mathrm{Re}(\mathrm{p}^3) + \mathbf{p} \times \nabla U(\mathrm{q}).$$

The Toda lattice

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + e^{q_n - q_1}.$$

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$$a_i = e^{\frac{1}{2}(q_i - q_{i+1})}, \quad b_i = -p_i.$$

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Hamiltonian system:

$$\begin{aligned} \dot{b}_i &= a_i^2 - a_{i-1}^2 \\ \dot{a}_i &= \frac{1}{2} a_i (b_{i+1} - b_i). \end{aligned}$$

Lax pair formulation

There exists a Lax pair (L, B) such that the Hamiltonian system above is equivalent to $\dot{L} = [B, L]$,

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \dots & a_n \\ a_1 & b_2 & a_2 & \dots & 0 \\ 0 & a_2 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & 0 & \dots & b_n \end{pmatrix}.$$

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Theorem (Toda)

The spectrum of L is invariant under the flow.

Action-angle coordinates

Theorem (Flaschka–McLaughlin, van Moerbeke, Moser)

Let $\Delta(\lambda) = \det(L - \lambda I)^2 - 2$ *and let* $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{2n}$ *be the roots of* $\Delta(\lambda)^2 - 4$.

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$$\int_{\lambda_{2i}}^{\lambda_{2i+1}} \cosh^{-1}\left|\frac{\Delta(\lambda)}{2}\right| d\lambda.$$

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Theorem (Ostrover–R.–Sepe)

$\Delta^{n-1} \times \mathbb{R}^{n-1}$ *has a toric action whose moment map is given by*

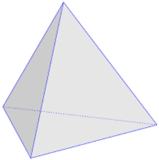
$$\Delta^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}_{\geq 0}$$
$$(q, p) \mapsto (p_{i_1} - p_{i_2}, p_{i_2} - p_{i_3}, \ldots, p_{i_{n-1}} - p_{i_n}),$$

where $p_{i_1} \geq p_{i_2} \geq \cdots \geq p_{i_n}$.

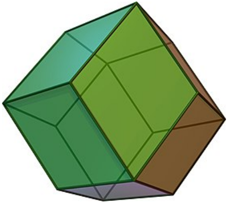
The ball

Corollary

The ball is symplectomorphic to the Lagrangian product of



and



(n-simplex) *(rhombic dodecahedron)*

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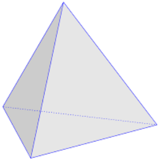

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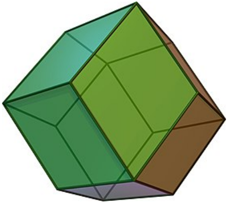

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Observation: One can independently verify that the Lagrangian product above satisfies the equality in Viterbo's conjecture.