Cutting-Gluing of TQFTs in the Symplectic Cohomological Formalism (BV-BFV Formalism)

Joint program with G. Canepa, A. S. Cattaneo, P. Mnev, N. Reshetikhin, M. Schiavina, Ö. Tetik, K. Wernli

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Content

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1. a d-manifold Σ (a model for space–time),
2. a space of fields F_Σ (e.g. differential forms, maps between manifolds, sections of a vector bundle, ...),
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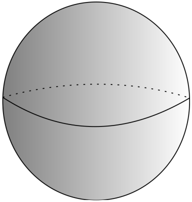
1. $\Sigma =$  $\Sigma =$
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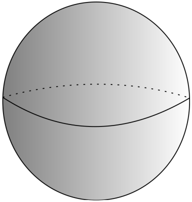
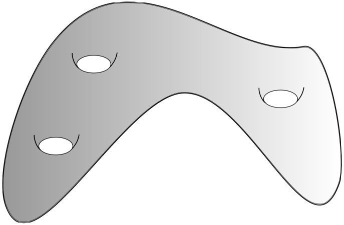
1. $\Sigma =$  $\Sigma =$ 
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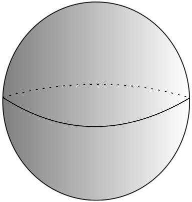
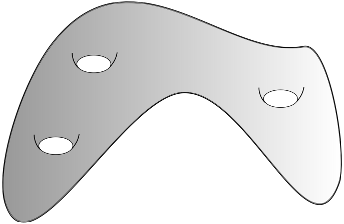
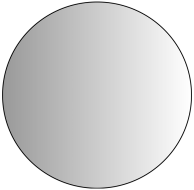
1. $\Sigma =$ 

$\Sigma =$ 

2. $\Sigma =$ 

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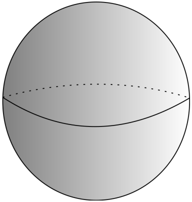
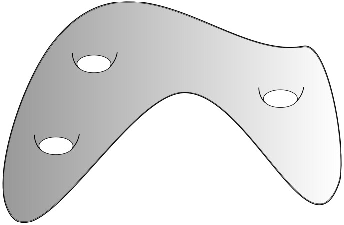
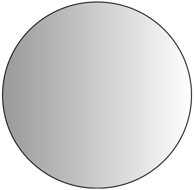
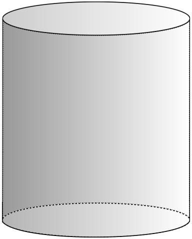
1. $\Sigma =$  $\Sigma =$ 

2. $\Sigma =$  $\Sigma =$ 

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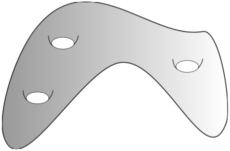
$\Sigma =$  $= =$

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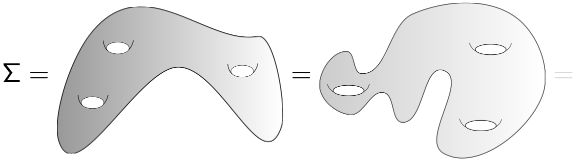

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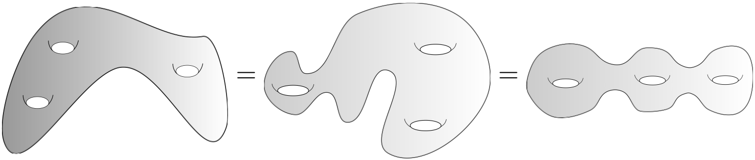
$\Sigma =$ 

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It basically says that we can cut Σ into smaller pieces, compute things there, and paste it back together.

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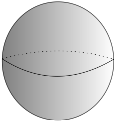
1. $\Sigma =$  $\Longleftrightarrow$
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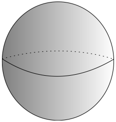
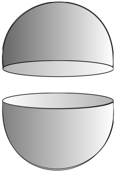
1. $\Sigma =$  $\Longleftrightarrow$ 
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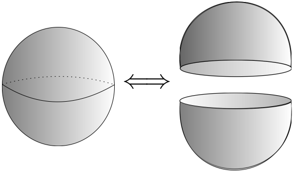
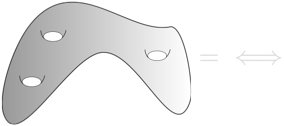
1. $\Sigma =$ 
2. $\Sigma =$ 

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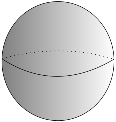
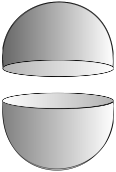
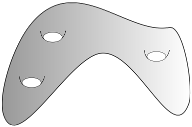
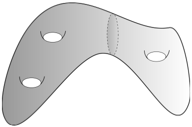
1. $\Sigma =$  $\Longleftrightarrow$ 
2. $\Sigma =$  $=$  $\Longleftrightarrow$

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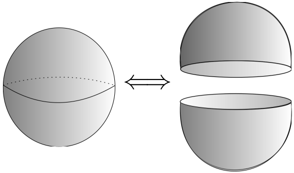
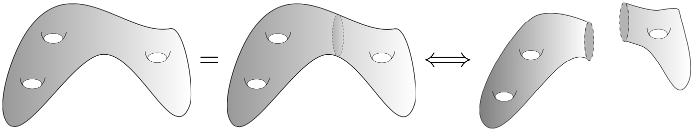
1. $\Sigma =$ 
2. $\Sigma =$ 

Local functional

Locality also tells us, roughly, that we can express e.g. the action as an integral:

$$S_\Sigma[\varphi] = \int_\Sigma \mathscr{L}(\varphi, \partial\varphi, \dots), \qquad \varphi \in F_\Sigma.$$

If $\mathscr{L}$ is invariant under local transformation of a Lie group G, we say it has *symmetry*, and call it a *gauge theory*.

Example: For *electromagnetism* we have $\mathscr{L} = -\frac{1}{4} F \wedge *F$ where $F = \mathrm{d}A$ for some 1-form A. Then if we transform $A \to A' = A + \mathrm{d}\lambda$ for some function λ, we get $F' = \mathrm{d}A' = \mathrm{d}(A + \mathrm{d}\lambda) = \mathrm{d}A + \underbrace{\mathrm{dd}\lambda}_{=0} = F$ and hence

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Quantum field theory

For the quantum construction we are mainly interested in the *partition function*

$$Z_\Sigma = \int_{\varphi \in F_\Sigma} e^{\frac{i}{\hbar} S_\Sigma[\varphi]} \mathscr{D}[\varphi].$$

One can make sense of this expression by considering the *perturbative expansion* in terms of *Feynman graphs*:

$$Z_\Sigma \underset{\hbar \to 0}{\approx} \sum_{\text{crit. pts. } \varphi_0 \text{ of } S_\Sigma} |\det \partial^2 S_\Sigma[\varphi_0]|^{-\frac{1}{2}} \sum_\Gamma \hbar^{k_\Gamma} \int_{C_\Gamma(\Sigma)} Z_\Gamma,$$

where $\underset{\hbar \to 0}{\approx}$ means in the $\hbar \to 0$ asymptotic up to some prefactors, $C_\Gamma(\Sigma)$ is a suitable configuration space of the vertex set of Γ in Σ and Z_Γ some differential form.

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To be able to perform the perturbative expansion (stationary phase expansion) we need to make sure that the critical points of S_Σ are all *isolated*, which is never the case for gauge theories.

Thus, to compute Z_Σ, we need to replace the action S_Σ by another functional whose critical points are all isolated without changing the value of Z_Σ.

There are different methods for dealing with this issue (e.g. Faddeev–Popov, BRST).

We consider a gauge formalism developed by *Batalin* and *Vilkovisky* (BV formalism):

$$\int_{F_\Sigma} e^{\frac{i}{\hbar}S_\Sigma} \longrightarrow \int_{\mathcal{L}\subset\mathcal{F}_\Sigma} e^{\frac{i}{\hbar}\mathcal{S}_\Sigma}$$

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- a *BV space of fields* $\mathcal{F}_\Sigma$ ($\mathbb{Z}$-graded supermanifold),
- a *BV symplectic form* ω_Σ of degree -1,
- a *BV action functional* $\mathcal{S}_\Sigma$ of degree 0 satisfying the *CME*

$$(\mathcal{S}_\Sigma, \mathcal{S}_\Sigma) = 0.$$

We call the triple $(\mathcal{F}_\Sigma, \omega_\Sigma, \mathcal{S}_\Sigma)$ a *BV theory*.

We are also interested in the Hamiltonian vector field Q_Σ of $\mathcal{S}_\Sigma$ of degree $+1$ which, by definition, satisfies $(Q_\Sigma)^2 = 0$ (cohomological) and $L_{Q_\Sigma}\omega_\Sigma = 0$ (symplectic).

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In finite dimensions, one can canonically define a second order differential operator Δ acting on half-densities on $\mathcal{F}_\Sigma$ such that $\Delta^2 = 0$, called *BV Laplacian*.

Theorem (Batalin–Vilkovisky)

For half-densities f, g we have:

1. *if $f = \Delta g$ (BV exact), then $\int_{\mathcal{L}} f = 0$ for any Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}_\Sigma$,*
2. *if $\Delta f = 0$ (BV closed), then $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{L}_t} f = 0$ for any continuous family of Lagrangian submanifolds $(\mathcal{L}_t)$.*

For the case of QFT, we want $f = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_\Sigma}\rho$, where ρ is some non-vanishing, Δ-closed reference half-density. Hence we want that the *QME* holds:

$$\Delta \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_\Sigma}\rho = 0 \Longleftrightarrow (\mathcal{S}_\Sigma, \mathcal{S}_\Sigma) - 2\mathrm{i}\hbar\Delta\mathcal{S}_\Sigma = 0.$$

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In finite dimensions, one can canonically define a second order differential operator Δ acting on half-densities on $\mathcal{F}_\Sigma$ such that $\Delta^2 = 0$, called *BV Laplacian*.

Theorem (Batalin–Vilkovisky)

For half-densities f, g we have:

1. *if $f = \Delta g$ (BV exact), then $\int_{\mathcal{L}} f = 0$ for any Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}_\Sigma$,*
2. *if $\Delta f = 0$ (BV closed), then $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{L}_t} f = 0$ for any continuous family of Lagrangian submanifolds $(\mathcal{L}_t)$.*

For the case of QFT, we want $f = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\mathcal{S}_\Sigma}\rho$, where ρ is some non-vanishing, Δ-closed reference half-density. Hence we want that the *QME* holds:

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The BV formalism only works if Σ is closed ($\partial\Sigma = \varnothing$).

We want to be able to deal with gauge theories in the case where Σ has boundary ($\partial\Sigma \neq \varnothing$).

$\Longrightarrow$ Use the *BV-BFV formalism* developed by *Cattaneo–Mnev–Reshetikhin*

It is a quantum gauge formalism which couples the *Batalin–Vilkovisky bulk theory* to the *Batalin–Fradkin–Vilkovisky boundary theory* compatible with cutting and gluing in the sense of Atiyah–Segal.

There we consider the quadruple $(\mathcal{F}^\partial_{\partial\Sigma}, \omega^\partial_{\partial\Sigma} = \delta\alpha^\partial_{\partial\Sigma}, Q^\partial_{\partial\Sigma}, \mathcal{S}^\partial_{\partial\Sigma})$ on the boundary $\partial\Sigma$ which is connected to the bulk theory in a coherent way. We have:

- $\mathcal{F}^\partial_{\partial\Sigma}$ is a $\mathbb{Z}$-graded supermanifold,
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Choosing a *polarization* (involutive Lagrangian distribution) $\mathcal{P}$ on $\partial\Sigma$, we assume a splitting

$$\mathcal{F}_\Sigma = \mathcal{B}_{\partial\Sigma} \times \mathcal{Y},$$

where $\mathcal{B}_{\partial\Sigma} := \mathcal{F}^\partial_{\partial\Sigma}/\mathcal{P}$ denotes the (smooth) leaf space for $\mathcal{P}$, and $\mathcal{Y}$ a symplectic complement (ω_Σ constant on $\mathcal{B}_{\partial\Sigma}$).

1. CME $\Rightarrow$ *modified* CME:

$$Q_\Sigma(\mathcal{S}_\Sigma) = \pi_\Sigma^*(2\mathcal{S}^\partial_{\partial\Sigma} - \iota_{Q^\partial_{\partial\Sigma}}\alpha^\partial_{\partial\Sigma}) \Longleftrightarrow \frac{1}{2}\iota_{Q_\Sigma}\iota_{Q_\Sigma}\omega_\Sigma = \pi_\Sigma^*\mathcal{S}^\partial_{\partial\Sigma},$$

where $\pi_\Sigma\colon \mathcal{F}_\Sigma \to \mathcal{F}^\partial_{\partial\Sigma}$ is a surjective submersion.

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$$(\hbar^2\Delta_\mathcal{Y} + \Omega_{\partial\Sigma})Z_\Sigma = 0.$$

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$$\Omega_{\partial\Sigma} = \text{ordered standard quantization of } \mathcal{S}^\partial_{\partial\Sigma}.$$

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Globalization of split AKSZ theories

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AKSZ-BV theories

AKSZ (*Alexandrov–Kontsevich–Schwarz–Zaboronsky*) theories are a particular type of BV theories.

Fix the following data:

- a d-manifold Σ,
- an *exact Hamiltonian dg symplectic manifold* $(\mathcal{M}, \omega = \mathrm{d}\alpha, \Theta)$ of degree $d-1$,
- a BV space of fields $\mathcal{F}_\Sigma = \mathrm{Map}(T[1]\Sigma, \mathcal{M})$.

Using ω, one can construct a BV symplectic form ω_Σ on $\mathcal{F}_\Sigma$ which is locally given by

$$\omega_\Sigma = \int_\Sigma \omega_{\mu\nu}(\Phi)\delta\Phi^\mu \wedge \delta\Phi^\nu.$$

Moreover, we can construct a BV action by $\mathcal{S}_\Sigma$ on $\mathcal{F}_\Sigma$ which is locally given by

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One can check that *constant* maps $x\colon \Sigma \to \mathcal{M}$ are solutions to $\delta \mathcal{S}_\Sigma = 0$.

Definition (Split AKSZ theory)

We say that an AKSZ theory is *split* if $\mathcal{M} = T^*[d-1]M$ for some graded manifold M.

Then, considering *superfields* $(\boldsymbol{P}, \boldsymbol{X})\colon T[1]\Sigma \to \mathcal{M}$, we get

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To compute Z_Σ, we want to perturb around constant background fields $x \in \mathcal{M}$.

Question (?)

How can we put the perturbative expansions around different points $x \in \mathcal{M}$ together, which corresponds to *globalization* of the AKSZ model?

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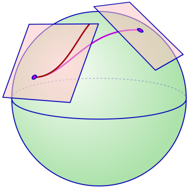
Formal geometry

Since we are looking at maps to some target manifold $\mathcal{M}$, we want to work with coordinates in a formal neighborhood at each point $x \in \mathcal{M}$. Moreover, we want to pass from one coordinate chart to another by changing the base point $\Rightarrow$ need a connection to formulate a covariant setting.



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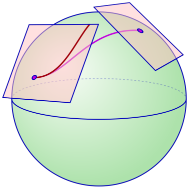

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We consider a map $\phi\colon T\mathcal{M} \to \mathcal{M}$, $(x,y) \mapsto \phi_x(y)$, for $y \in T_x\mathcal{M}$ such that

- $\phi_x(0) = 0$,
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We use ϕ to map tensor fields (functions, multivector fields, differential forms) on $\mathcal{M}$ to formal tensor fields in the vertical fibers parametrized by the base:

$$\sigma \mapsto \Xi(\sigma), \quad \Xi(\sigma)(x) := \mathrm{Taylor}_y(\phi_x^{-1})_*\sigma.$$

In case of a function we have

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Grothendieck connection

Hence, if $\sigma = \phi^* f$ and R is the vector field on the fiber induced by the formal map ϕ as before, we get

$$(\mathrm{d}_x + R)\sigma = 0.$$

In fact, a tensor field σ lies in the image of Ξ if and only if

$$\mathrm{d}_x \sigma + L_R \sigma = 0.$$

The connection $D_{\mathrm{G}} := \mathrm{d}_x + R$ is *flat* and is called *Grothendieck connection*.

Using *homological perturbation theory*, one can show that

$$H^k_{D_{\mathrm{G}}} = \begin{cases} \Xi(C^\infty(\mathcal{M})) \cong C^\infty(\mathcal{M}), & k = 0 \\ 0, & k > 0 \end{cases}$$

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differential Classical Master Equation

Consider again an AKSZ theory with source Σ and target $\mathcal{M}$.

Define the corresponding *formal global action*:

$$\tilde{\mathcal{S}}_\Sigma := \int_\Sigma \sum_i \widehat{\boldsymbol{P}}_i \wedge \mathrm{d}_\Sigma \widehat{\boldsymbol{X}}^i + \int_\Sigma \Xi(\Theta)(\widehat{\boldsymbol{P}}, \widehat{\boldsymbol{X}}) + \int_\Sigma \sum_k R^k \wedge \widehat{\boldsymbol{P}}_k,$$

where $(\widehat{\boldsymbol{P}}, \widehat{\boldsymbol{X}})$ denotes a supermap $T[1]\Sigma \to T_x\mathcal{M}$ and R^k is the 1-form part of R.
We get the *differential CME*:

$$\mathrm{d}_x \tilde{\mathcal{S}}_\Sigma + \frac{1}{2}(\tilde{\mathcal{S}}_\Sigma, \tilde{\mathcal{S}}_\Sigma) = 0.$$

It captures:

- CME,
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If we put everything together, on the quantum level, we get the following theorem:

Theorem (Cattaneo–M.–Wernli)

For the global state $\tilde{Z}_\Sigma$ given by the quantization of a formal global split AKSZ theory, the modified differential QME (mdQME) holds:

$$\nabla_{\mathrm{G}}\tilde{Z}_\Sigma = \left(\mathrm{d}_x - \mathrm{i}\hbar\Delta_{\mathcal{Y}} + \frac{\mathrm{i}}{\hbar}\Omega_{\partial\Sigma}\right)\tilde{Z}_\Sigma = 0.$$

We call ∇_{G} the *quantum Grothendieck BFV operator (qGBFV operator)*. Can be also regarded as a *connection* on the total state space $\mathcal{H}_{tot} = \coprod_x \mathcal{H}_x$.

Moreover, we get:

Theorem (Cattaneo–M.–Wernli)

The qGBFV operator ∇_{G} squares to zero ($(\nabla_{\mathrm{G}})^2 = 0$).

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Change of data

Our construction depends on the choice of:

- a formal map ϕ,
- representatives of residual fields (low energy fields),
- a propagator.

We can show that $\tilde{Z}_\Sigma$ changes in a controlled way under change of data.

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For families of BFV operators (Ω_t) and global states $(\tilde{Z}_t)$ parametrized by $t \in [0,1]$:

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The Poisson Sigma Model and its globalization

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Poisson Sigma Model (PSM)

The Poisson Sigma Model is an example of a 2-dimensional split AKSZ theory with:

- $\mathcal{M} = T^*[1]M$ for a Poisson manifold (M, π),
- BV action functional given by

$$\mathcal{S}_\Sigma = \int_\Sigma \sum_i \boldsymbol{\eta}_i \wedge \mathrm{d}_\Sigma \boldsymbol{X}^i + \frac{1}{2} \int_\Sigma \sum_{ij} \pi^{ij}(\boldsymbol{X}) \boldsymbol{\eta}_i \wedge \boldsymbol{\eta}_j,$$

where $(\boldsymbol{\eta}, \boldsymbol{X})\colon T[1]\Sigma \to T^*[1]M$ are superfields.

The classical PSM action S_Σ has the same form as the BV action where superfields $(\boldsymbol{\eta}, \boldsymbol{X})$ are replaced by vector bundle maps

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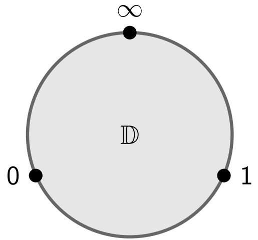
Kontsevich's star product using the PSM

Kontsevich's star product is induced by the PSM on the disk $\mathbb{D}$ with boundary condition $\eta = 0$ (*Cattaneo–Felder*):

$$f \star g(x) = \int_{X(\infty)=x} f(X(0))g(X(1))e^{\frac{i}{\hbar}S_{\mathbb{D}}}, \quad f, g \in C^{\infty}(U), \quad U \subset \mathbb{R}^n.$$

where $0, 1, \infty$ are cyclically ordered points on the boundary of the disk.



Globalization of Kontsevich's star product

Kontsevich's formula uses local coordinates but can be globalized using formal geometry (*Cattaneo–Felder–Tomassini*).

There, one derives the formulae locally by field theory and globalizes algebraically afterwards.

We give an approach where the globalization is directly contained in the field-theoretic construction.

One starts with the Grothendieck connection D_{G} and uses Kontsevich's *formality map* $\mathcal{U}$ to quantize it to a derivation $\mathcal{D}_{\mathrm{G}} = D_{\mathrm{G}} + O(\hbar)$ (*deformed Grothendieck connection*).

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One can check that $\mathcal{D}_\mathrm{G}$ is not a differential. In fact, we get

$$(\mathcal{D}_\mathrm{G})^2 = [F, \ ]_\star,$$

$$F = \sum_{n=0}^{\infty} \frac{\hbar^{n+2}}{(n+2)!} \mathcal{U}_{n+2}(R, R, \Xi(\pi), \ldots, \Xi(\pi)) \in O(\hbar^2).$$

There exists a 1-form γ on M with values in $\widehat{\mathrm{Sym}}(T^*M)[\![\hbar]\!]$ such that

$$\mathcal{D}_\mathrm{G}\gamma + \frac{1}{2}\gamma \star \gamma = F.$$

If one defines $\bar{\mathcal{D}}_\mathrm{G} := \mathcal{D}_\mathrm{G} + [\gamma, \ ]_\star$, we can check that $(\bar{\mathcal{D}}_\mathrm{G})^2 = 0$.

Its cohomology $H^\bullet_{\bar{\mathcal{D}}_\mathrm{G}}$ is identified with globally defined functions.

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Globalization in the BV-BFV formalism

As before, we can use the formal global action for the PSM:

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The difference is that on some boundary components $\partial_0^\ell \Sigma \subset \partial\Sigma$ we can have additional boundary conditions $\widehat{\eta} = 0$ instead of a polarization.

$\Longrightarrow$ failure of the general method.

Proposition (Cattaneo–M.–Wernli)

Consider the global state $\tilde{Z}_\Sigma$ for the formal global PSM $\tilde{\mathcal{S}}_\Sigma$. Then we have a failure of the mdQME by curvature terms of the deformed Grothendieck connection:

$$\nabla_{\mathrm{G}} \tilde{Z}_\Sigma = \exp\left(\frac{\mathrm{i}}{\hbar}\int_{\partial_0\Sigma := \bigsqcup_\ell \partial_0^\ell \Sigma} F\right)\tilde{Z}_\Sigma.$$

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We call this the *twisted theory*.

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Let $\tilde{Z}_\Sigma^\gamma$ be the quantization of the twisted theory $\tilde{\mathcal{S}}_\Sigma^\gamma$ and let $\tilde{\nabla}_{\mathrm{G}}^\gamma$ be a twisted version of the qGBFV operator. Then a twisted version of the mdQME holds:

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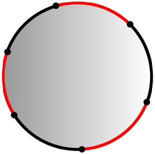
Let $\tilde{Z}^{\gamma}_{\Sigma}$ be the quantization of the twisted theory $\tilde{\mathcal{S}}^{\gamma}_{\Sigma}$ and let $\tilde{\nabla}^{\gamma}_{\mathrm{G}}$ be a twisted version of the qGBFV operator. Then a twisted version of the mdQME holds:

$$\tilde{\nabla}^{\gamma}_{\mathrm{G}} \tilde{Z}^{\gamma}_{\Sigma} = 0.$$

Mixed boundary structure

Moreover, the quantization for *mixed boundary structures* leads to *corner* terms:

$\partial\Sigma =$ 

This is important for the quantization of the *relational symplectic groupoid (RSG)*.
$\Longrightarrow$ Can construct a global version of Kontsevich's star product, using the BV-BFV formalism by gluing the disk out of *canonical relations*:

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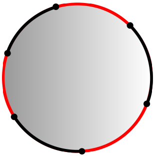
$\partial\Sigma =$ 

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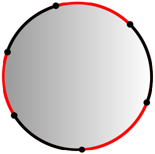
$\partial\Sigma =$ 

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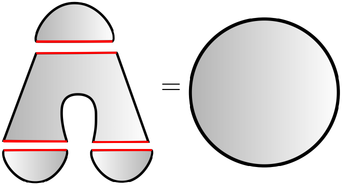

Higher codimension

Question: What happens when we consider stratified manifolds and their higher codimension submanifolds (e.g. manifolds with corners, etc.)?

For codimension k theories, we speak of $BV\text{-}BF^kV$.

The classical theory is easily formulated by iteration of the previous process, i.e. we consider $\mathcal{F}^{\partial^k}$, S^{∂^k}, ω^{∂^k}, Q^{∂^k}, etc.

The shift (ghost number) of the symplectic structure is always raised by $+1$. Hence, the ghost number of ω^{∂^k} is $k-1$.

The mCME in codimension k is then

$$Q^{\partial^k}(S^{\partial^k}) = \pi^*_{\partial^k}(2S^{\partial^{k+1}} - \iota_{Q^{\partial^{k+1}}}\alpha^{\partial^{k+1}}),$$

where $\pi_{\partial^k}\colon \mathcal{F}^{\partial^k} \to \mathcal{F}^{\partial^{k+1}}$.
See also e.g. Cattaneo–Mnev–Reshetikhin (2012), Canepa–Cattaneo (2022).

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The Quantum case is much more difficult.

We would like to have (up to coefficients) something like

$$(\Delta + \Omega_\partial + \Omega_{\partial\partial} + \ldots + \Omega_{\partial^k})Z = 0.$$

Not clear what the above equation should mean. The objects of the categories assigned to higher codimensions (≥ 2) are not explicitly constructed. Before, i.e. in the codimension 1 case, it was the category of *chain complexes* Ch (usually the category of *vector spaces* in the setting of TQFTs). Denote by $\mathrm{Alg}_{\mathbb{P}_k}^{\mathbb{E}_d}(\mathrm{Ch})$ the category of $\mathbb{P}_k$-algebras over $\mathbb{E}_d$-algebras in chain complexes Ch and by $\mathrm{dgCat}_{(\infty,k)}$ the category of differential graded (∞,k)-categories. Then the deformation quantization part gives something like $\mathrm{Alg}_{\mathbb{BD}_k}^{\mathbb{E}_{n-k}}(\mathrm{Ch})$ acting on $\mathrm{dgCat}_{(\infty,k-1)}$ coming from the geometric quantization part. In particular, we have something like

$$Q_\hbar^{\partial^k} = [\mathcal{S}_\hbar^{\partial^k},\ ]_{\mathbb{E}_k}, \qquad k \geq 2.$$

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Thank you!