Derived algebraic geometry and Enumerative Geometry

Global Portuguese Mathematicians
IST

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Summary

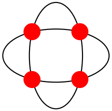
1. Motivations for derived algebraic geometry

2. What can it do for enumerative geometry?

Algebraic geometry

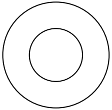
Geometry shaped by solutions of system of polynomial equations

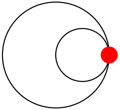
$2x^2 + y^2 = 1$



$x^2 + 2y^2 = 1$ Intersections

in $\mathbb{R}^2$





Zoology. General Pattern?

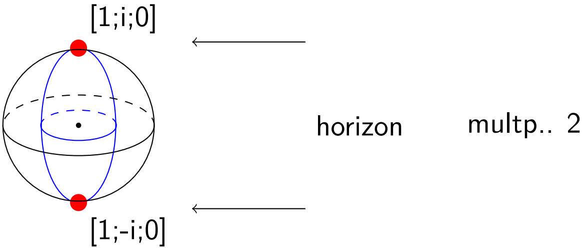
Discovery (Bezout):

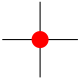
- replace $\mathbb{R}$ by $\mathbb{C}$
- add horizon points to the plane (Projective Plane $\mathbb{P}^2$),

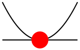
$$\Rightarrow \text{ regular pattern}$$

$$\sharp\{C \cap D\} = \text{degree } c.\text{degree } d \ (\textit{counted with multiplicities})$$

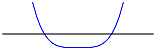
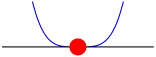
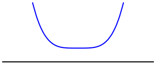
Example:





mult. 1 (transversal), $(y = x^2)$  mult. 2

Geometric Meaning: continuity under small perturbations

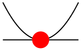
 $\leftarrow$ $(y = x^4)$  $\rightarrow$ 

$1, -1, i, -i$

$\sqrt[2]{i}, \sqrt[2]{-i}, -\sqrt[2]{i}, -\sqrt[2]{-i}$

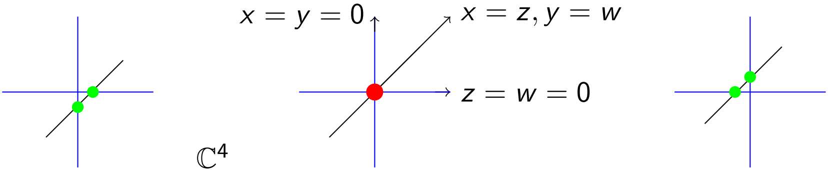
Need Algebraic Formula $P_C = 0, P_D = 0$ intersecting at single point,

$$geo.\ multp. = dim\ [\mathbb{C}[x,y]/_{P_C} \otimes_{\mathbb{C}[x,y]} \mathbb{C}[x,y]/_{P_D}]$$

Example: $(y = x^2)$ 

$$dim_{\mathbb{C}}[\mathbb{C}[x,y]/(y) \otimes_{\mathbb{C}[x,y]} \mathbb{C}[x,y]/(y-x^2)] \simeq\ dim_{\mathbb{C}}\mathbb{C}[x]/(x^2) = 2$$

Problem in higher dimensions:



Geometry says multp. 2; Algebra says 3

$$dim_{\mathbb{C}} \left[\mathbb{C}[x,y,z,w]/_{(xz,xw,yz,yw)} \otimes_{\mathbb{C}[x,y,z,w]} \mathbb{C}[x,y,z,w]/_{(x-z,y-w)}\right] = 3$$

Problem in lower dimensions: Intersection of 0 with itself in $\mathbb{C}$.

$-\delta$ ● ● δ 0 ●

Continuity says multp. 0; Algebra says 1

$$dim_{\mathbb{C}} \left[\mathbb{C}[x]/(x) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x)\right] = 1$$

Serre's discovery: $\otimes$ alone misses subtle geometric information.
$\rightsquigarrow$ Introduce a new operation $\otimes^{\mathbb{L}}$ that corrects $\otimes$.

Invention: To account for the corrections, the *output* of $\otimes^{\mathbb{L}}$ is no longer a single vector space but rather a chain of vector spaces,

$$[\cdots \longrightarrow \underbrace{V_{-2}}_{Layer-2} \xrightarrow{d_2} \underbrace{V_{-1}}_{Layer-1} \xrightarrow{d_1} \underbrace{V_0}_{Layer 0}],\ d^i d^{i-1} = 0,$$

Each extra layer adds a correction. H^i = complexity at level i.

Example:

$$\mathbb{C}[x]/(x) \otimes^{\mathbb{L}}_{\mathbb{C}[x]} \mathbb{C}[x]/(x) \simeq [0 \to \underbrace{\mathbb{C}}_{deg\ -1} \to^0 \underbrace{\mathbb{C}}_{deg 0} \to 0], \quad H^{-1} = \mathbb{C},$$

$$H^0(- \otimes^{\mathbb{L}} -) = \text{old } \otimes \quad , i > 0\ H^{-i} = \text{corrections ("Tor's")}$$

Serre's formula p is an isolated intersection point,

$$\sum_{i\geq 0}(-1)^i dim_{\mathbb{C}} H^{-i}(-\otimes^{\mathbb{L}}-) = geo.\ mult.$$

Previous examples;

- Lower dimension: 1 - 1+ 0 - 0 + 0.... = 0
- Higher dimension: $3 - 1 + 0 - 0 + 0.... = 2$;

Problem: chains of vector spaces are out of the classical dictionary geometry $\leftrightarrow$ algebra.

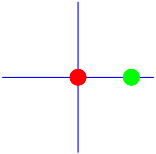
(**Toen-Vezzosi, Lurie**) Enhancement of classical geometry where new infinitesimal information can live in higher layers.

For Free: Extra Layers of tangent information $\Rightarrow$

tangent ($\mathbb{T}$)/cotangent ($\mathbb{T}^*$) complexes.

$H^0(\mathbb{T})=$ usual tangent, $H^i(\mathbb{T})=$ code the singularities.

Illustration: $C = \{f(X,Y) := XY = 0\} \subseteq \mathbb{C}^2$



$f : \mathbb{C}^2 \to \mathbb{C}$

$p = {\color{green}\bullet} \leftrightarrow dim\, T^{*,cl}_{C,p} = 1$

$p = {\color{red}\bullet} \leftrightarrow dim\, T^{*,cl}_{C,p} = 2$

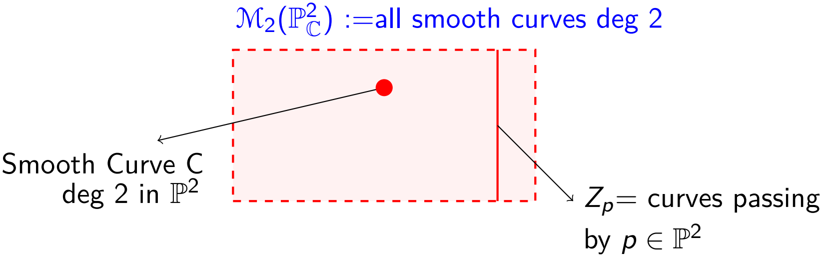
$$\mathbb{T}^*_{C,p} \simeq [\, 0 \longrightarrow \underbrace{\mathbb{C}}_{deg\,-1} \xrightarrow{1 \mapsto df_p} \underbrace{\mathbb{C}.dx \bigoplus \mathbb{C}.dy}_{deg 0} \longrightarrow 0 \,]$$

- $p = {\color{green}\bullet} \leftrightarrow H^0 = \mathbb{C} = T^{*,cl}_{C,p} =$ usual cotangent space, $H^{-1} = 0$.
- $p = {\color{red}\bullet} \leftrightarrow H^0 = \mathbb{C} \oplus \mathbb{C}$, $H^{-1} \simeq \mathbb{C}$, singularity.

Information in higher degrees controls lack of smoothness.

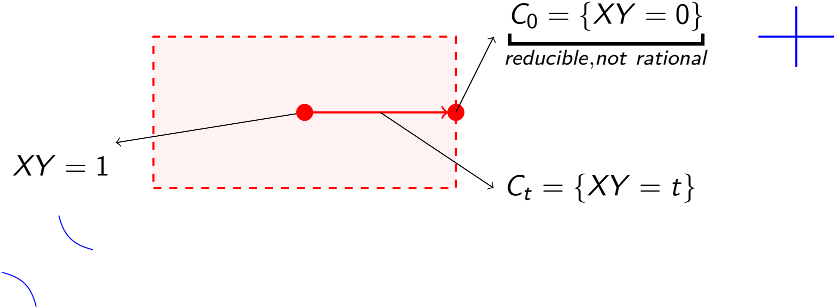
Easy Example: How many lines pass by 2 diff pts in the plane?

Example: What is the number N_2 of smooth plane curves of degree 2 that pass though $p_1, ..., p_5$ distinct points in $\mathbb{P}^2$, no 3 colinear?



$$N_d := Vol(\underbrace{Z_{p_1} \cap \cdots \cap Z_{p_{3d-1}}}_{\textit{Curves passing by all the points}}) = \int^{alg}_{\mathcal{M}_2(\mathbb{P}^2_{\mathbb{C}})} \underbrace{\omega_{p_1}}_{\text{"Poincare duals" } Z_{p_1}} \wedge \cdots \wedge \omega_{p_{3d-1}}$$

Problem: $\mathcal{M}_2(\mathbb{P}^2_{\mathbb{C}})$ not compact $\rightsquigarrow$ problems with the integral.



Solution: For N_2 there is a simple candidate for a compactification of $\mathcal{M}_2(\mathbb{P}^2_{\mathbb{C}})$:

$$\underbrace{\{aX^2 + bXY + cY^2 + dX + eY + f\}}_{\text{all curves in } P^2} / \mathbb{C}^* \simeq \mathbb{P}^5_{\mathbb{C}}$$

Higher degrees

Gromov-Witten numbers. $N_d := \sharp$ of *rational curves* (ie, parametrized by $\mathbb{P}^1$) of degree d passing by $3d-1$ points in $\mathbb{P}^2_{\mathbb{C}}$ in general position. (3d-1 to get finite)

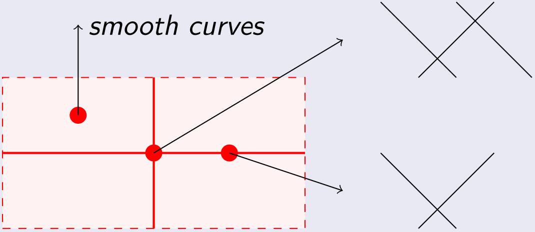
Kontsevich's Recursion:

$$N_d = \Sigma_{d_A+d_B=d} N_{d_A} N_{d_B} d_A^2 d_B (d_B \binom{3d-4}{3d_A-2} - d_A \binom{3d-4}{3d_A-1})$$

Theorem (Gromov-Kontsevich Orbifold Compactification by stable maps)

There exists a nice smooth compact algebraic orbifold whose points are parametrized curves of degree d (stable maps)

$\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2_{\mathbb{C}}, d) =$



Recursion $\leftrightarrow$ *skeleton of the boundary* $\qquad N_d = \int^{alg}_{\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2_{\mathbb{C}}, d)}$

Next step: Replace the plane $\mathbb{P}^2_{\mathbb{C}}$ by general X (smooth projective)?

Problem: $\overline{\mathcal{M}}_{0,n}(X,d)$ no longer smooth. Very singular. Has pieces of different dimensions. Naive $\int$ fails.

Solution: Behrend-Fantechi (Chow), Givental-Lee (K-theory).
Virtual fundamental classes

$$\underbrace{\int_{\times}^{\text{virtual}}}_{\text{new number}} := \int_{\overline{\mathcal{M}}_{0,n}(X,d)^{\text{good dim.}}}^{\text{alg}} + \text{ hand corrections for different dim.}$$

- Get the good numbers;
- Interpretation as volume is lost
- Very difficult to handle the corrections and prove recursive behavior.

New solution: Correct the lack of smoothness of $\overline{\mathcal{M}}_{0,n}(X,d)$ via derived geometry:

Theorem (Schurg-Toen-Vezzosi, Lurie)

The space $\overline{\mathcal{M}}_{0,n}(X,d)$ has a non-trivial structure of derived-orbifold.

$$\mathbb{R}\overline{\mathcal{M}}_{0,n}(X,d)$$

Proof: Lurie's master result: the representability theorem.

Theorem (Mann-R.)

The integrals

$$\int_{\mathbb{R}\overline{\mathcal{M}}_{0,n}(X,d)}^{alg,\ K-theoretic}$$

are well-defined and verify the recursive relations.. Moreover,

$$\int_{\mathbb{R}\overline{\mathcal{M}}_{0,n}(X,d)}^{alg,\ K-theoretic} = \int_{\times}^{virtual,\ Givental\text{-}Lee}$$

Proof: Brane actions for the ∞-operad of stable curves + h-descent for perfect complexes.

- Interpretation as a volume remains.
- Easier to recover recursive relations;

In Progress[Mann-R.]:

$$\underbrace{\int_{\mathbb{R}\overline{\mathcal{M}}_{0,n}(X,d)}^{alg,\ Chow}}_{is\ defined} = \int_{\times}^{virtual,\ \text{Behrend-Fantechi}}$$

(GRR for derived Orbifolds)

New directions (Yu-Porta): Use this strategy to define GW-invariants in rigid geometry and prove Mirror Symmetry

Obrigado