# The de Rham / Spencer Double Complex and the Geometry of Forms on Supermanifolds

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## Geometry VS Supergeometry: Why Forms?

- **Geometry:** well known facts:
  - geometric theory of differential forms related to integration on manifolds;
  - many applications to physics, now become standard;

# Motivations & Premises

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- **Supergeometry:** less known facts:
  - richer mathematical theory of forms related to integration on supermanifolds & having quite unique features;
  - important applications to physics (*e.g.* SUSY theories / BV quantization...)

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1. S.L. Cacciatori, S. N., R. Re, *The Universal de Rham / Spencer Double Complex on a Supermanifold*, Doc. Math. **27**, 489-518 (2022)
2. S. Noja, R. Re, *A Note on Super Koszul Complex and the Berezinian*, Ann. Mat. Pura Appl. (1923 - ) **201** (1), 403-421 (2022)
3. S. N., *On the Geometry of Forms on Supermanifolds*, ArXiv:2111.12841
4. S. N., *On BV Supermanifolds and the Super Atiyah Class*, ArXiv:2202.08136

# Differential Forms: Geometry VS Supergeometry

## de Rham Complex over a Manifold

In a **commutative** setting the de Rham complex of a manifold $M$ is bounded from above by the dimension of the manifold

$$0 \longrightarrow \mathcal{O}_M \longrightarrow \Omega^1_M \longrightarrow \ldots\ldots \longrightarrow \Omega^{\dim M}_M \longrightarrow 0$$

...this is so for differential 1-forms are anticommuting.

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## Top-Forms and Integration over $M$

- You can integrate sections of $\Omega^{\dim M}_M$ over $M$.
- More in general, you can integrate sections of $\Omega^{\dim M - p}_M$ over submanifolds of codimension $p$ in $M$.

# Differential Forms: Geometry VS Supergeometry

## Our Setting

Here we work over a supermanifold $\mathcal{M}$ of dimension $p|q$, meaning that we have a system of $p$ **even** local coordinate and $q$ **odd** local coordinate, which we write as $x_1, \dots, x_p | \theta_1, \dots, \theta_q$.

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We took the de Rham differential $d : \mathcal{O}_{\mathcal{M}} \to \Omega^1_{\mathcal{M}}$ to be an **odd** derivation.

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It follows that:

- $dx_1, \ldots, dx_p$ are **odd**;
- $d\theta_1, \ldots, d\theta_q$ are **even**.

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## de Rham Complex over a Supermanifold

We took the de Rham differential $d : \mathcal{O}_{\mathcal{M}} \to \Omega^1_{\mathcal{M}}$ to be an **odd** derivation.

It follows that:

- $dx_1, \dots, dx_p$ are **odd**;
- $d\theta_1, \dots, d\theta_q$ are **even**.

Since $d\theta_i$ is even for any $i$, we have that $d\theta_i^n \neq 0$ for any $i$.

It follows that the de Rham complex is **not** bounded from above:

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow \Omega^1_{\mathcal{M}} \longrightarrow \dots \longrightarrow \Omega^k_{\mathcal{M}} \longrightarrow \dots$$

# Differential Forms: Geometry VS Supergeometry

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## Differential Forms and Integration over Supermanifolds

Differential forms can be integrated only over ordinary (non super) submanifolds of a supermanifold.

1. Let $\iota : M \longrightarrow \mathcal{M}$ be an immersion of an ordinary $k$-dimensional manifold in a $p|q$-dimensional supermanifolds $\mathcal{M}$.
2. Let $\omega \in \Omega^k_{\mathcal{M}}$ be a degree $k$ differential form.
3. It make sense to integrate $\iota^* \omega$ over $M$ (notice that the integral can be zero).

If $\mathcal{M}$ is a supermanifold of dimension $p|q$, differential forms can be integrated over ordinary submanifolds $M$ of codimension $k|q$, with $0 \leq k \leq p$ in $\mathcal{M}$.

# Differential Forms: Geometry VS Supergeometry

## Differential Forms and Integration

- (Geometric) integration of differential forms over supermanifolds does **not** involve odd directions.
- Need to look for something different if want odd directions be part of integration process.

# Integral Forms

## Integral Forms and their Complex

Let $\mathcal{M}$ be a supermanifold of dimension $p|q$. An integral form of degree $p-k$ is a section of the sheaf

$$\Sigma^{p-k}_{\mathcal{M}} := \mathcal{B}er(\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} S^k \Pi \mathcal{T}_\mathcal{M}$$

Where we have that

- $\mathcal{B}er(\mathcal{M})$ is the Berezinian sheaf of $\mathcal{M}$ (...in a moment...);
- $S^k \Pi \mathcal{T}_\mathcal{M}$ is the $k$-fold (super)symmetric product of the "parity-shifted" tangent sheaf, so that $\Pi \mathcal{T}_\mathcal{M} \cong (\Omega^1_\mathcal{M})^*$.

$$(\mathcal{T}_\mathcal{M})_0 \ni \partial_x \xmapsto{\Pi} \pi\partial_x \in (\Pi\mathcal{T}_\mathcal{M})_1 \quad \rightsquigarrow \quad (dx)^* = \pi\partial_x$$

$$(\mathcal{T}_\mathcal{M})_1 \ni \partial_\theta \xmapsto{\Pi} \pi\partial_\theta \in (\Pi\mathcal{T}_\mathcal{M})_0 \quad \rightsquigarrow \quad (d\theta)^* = \pi\partial_\theta$$

# Integral Forms

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This can be structured into an actual complex:

$$\ldots \to \mathcal{B}er(\mathcal{M}) \otimes S^k \Pi \mathcal{T}_{\mathcal{M}} \to \ldots \to \mathcal{B}er(\mathcal{M}) \otimes \Pi \mathcal{T}_{\mathcal{M}} \to \mathcal{B}er(\mathcal{M}) \to 0.$$

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This can be structured into an actual complex:

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The differential of this complex is a bit tricky:

$$\delta(\mathcal{D}(x) f \otimes_k \pi\partial^I) = -\sum_a (-1)^{|x_a||f|+|\pi\partial^I|} \mathcal{D}(x)(\partial_a f) \otimes_k \partial_{\pi\partial_a}(\pi\partial^I)$$

where $\mathcal{D}(x) \in \mathcal{B}er(\mathcal{M})$, $f \in \mathcal{O}_{\mathcal{M}}$, $\pi\partial^I \in S^k \Pi \mathcal{T}_{\mathcal{M}}$.

# The Berezinian

## Definition (Berezinian - via Transition Functions)

The Berezinian sheaf $\mathcal{B}er(\mathcal{M})$ is the locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $\delta_{0,p+q}|\delta_{0,p+q+1}$, whose local generators $\{\mathcal{D}_{U_i}(x_{U_i}, \theta_{U_i})\}$ transforms as

$$\mathcal{D}_{U_j \lfloor_{U_i \cap U_j}}(x_{U_j}|\theta_{U_j}) = \mathcal{D}_{U_i \lfloor_{U_i \cap U_j}}(x_{U_i}|\theta_{U_i}) Ber(\mathcal{J}ac(\varphi_{ij}))$$

with

$$Ber(\mathcal{J}ac(\varphi_{ij})) = \det(A - BD^{-1}C)\det(D)^{-1},$$

where $\mathcal{J}ac(\varphi_{ij})$ is the super Jacobian of the change of coordinates

$$\begin{aligned} \varphi_{ij} : U_i\lfloor_{U_i \cap U_j} &\longrightarrow U_j\lfloor_{U_i \cap U_j} \\ x_{U_i}|\theta_{U_i} &\longmapsto \varphi_{ij,0}(x|\theta) = x_{U_j} \,|\, \varphi_{ij,1}(x|\theta) = \theta_{U_j}, \end{aligned}$$

and where we have posed

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := \left(\begin{array}{c|c} \partial_x \varphi_{ij,0} & \partial_\theta \varphi_{ij,0} \\ \hline \partial_x \varphi_{ij,1} & \partial_\theta \varphi_{ij,1} \end{array}\right) = \mathcal{J}ac(\varphi_{ij}).$$

# The Berezinian

## Definition (Berezinian - via Koszul Complex)

We call the Berezinian sheaf of $\mathcal{M}$ is the locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $\delta_{0,p+q}|\delta_{0,p+q+1}$ defined by the only non-trivial cohomology class of the Koszul complex of $\Omega^1_{\mathcal{M}}$,

$$\mathcal{B}er(\mathcal{M}) := \mathcal{E}xt^p_{S^\bullet(\Omega^1_{\mathcal{M}})^*}(\mathcal{O}_{\mathcal{M}}, S^\bullet(\Omega^1_{\mathcal{M}})^*),$$

In particular $\mathcal{B}er(\mathcal{M})$ is locally generated on an open set $U$ by the class

$$\mathcal{B}er(\mathcal{M})(U) = [dx_1 \dots dx_p \otimes \partial_{\theta_1} \dots \partial_{\theta_q}] \cdot \mathcal{O}_{\mathcal{M}}(U).$$

## Remarks

- The analogous construction for ordinary manifolds yields the *canonical sheaf* $\Omega^p_{\mathcal{M}}$, generated by $[dx_1 \dots dx_p]$;
- Make a sense out of the catch-phrase for Berezin integral:

**"Berezin integral = integrate the $x$'s and derive the $\theta$'s"**

# What Are Integral Forms Good For?

## Integral Forms and their Complex

Let $\mathcal{M}$ be a supermanifold of dimension $p|q$. An integral form of degree $p-k$ is a section of the sheaf

$$\Sigma^{p-k}_{\mathcal{M}} := \mathcal{B}er(\mathcal{M}) \otimes_{\mathcal{O}_{\mathcal{M}}} S^k \Pi \mathcal{T}_{\mathcal{M}}$$

They fit into the complex

$$\ldots \to \mathcal{B}er(\mathcal{M}) \otimes S^k \Pi \mathcal{T}_{\mathcal{M}} \to \ldots \to \mathcal{B}er(\mathcal{M}) \otimes \Pi \mathcal{T}_{\mathcal{M}} \to \mathcal{B}er(\mathcal{M}) \to 0.$$

**Integral forms of degree $p-k$ can be integrated over sub-supermanifolds of codimension $k|0$ in $\mathcal{M}$.**

- Can integrate sections of $\mathcal{B}er(\mathcal{M})$ over $\mathcal{M}$;
- Can integrate sections of $\mathcal{B}er(\mathcal{M}) \otimes \Pi \mathcal{T}_{\mathcal{M}}$ over a sub-supermanifold of codimension $1|0$ in $\mathcal{M}$
- Can integrate sections of $\mathcal{B}er(\mathcal{M}) \otimes S^2 \Pi \mathcal{T}_{\mathcal{M}}$ over a sub-supermanifold of codimension $2|0$ in $\mathcal{M}$...

# Where To Go?

## Where we are...

On a supermanifold $\mathcal{M}$ of dimension $p|q$, for $k \geq 0$ :

- **Differential Forms:** control integration on sub-manifolds of codim. $k|q$;
- **Integral Forms:** control integration on sub-supermanifold of codim. $k|0$;

## Where to go...

1. Will try to make these notions "go together"
   - ...coming out of a certain spectral sequence of a double complex...

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- **Differential Forms:** control integration on sub-manifolds of codim. $k|q$;
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## Where to go...

1. Will try to make these notions "go together"
   - ...coming out of a certain spectral sequence of a double complex...
2. ...as to show that they indeed compute the same invariant related to the supermanifold
   - ...which is the de Rham cohomology of the reduced space

# $\mathcal{D}$-Modules

**Definition (The Sheaf $\mathcal{D}_{\mathcal{M}}$)**

We define $\mathcal{D}_{\mathcal{M}}$ to be the subsheaf of $\mathcal{E}nd_k(\mathcal{O}_{\mathcal{M}})$ generated by $\mathcal{O}_{\mathcal{M}}$ and $\mathcal{T}_{\mathcal{M}}$, and we call it the *sheaf $\mathcal{D}_{\mathcal{M}}$ of differential operators* on $\mathcal{M}$.

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## Locally...

We have a local trivialization of $\mathcal{D}_{\mathcal{M}}\lfloor_U$, where $x_i \in \mathcal{O}_{\mathcal{M}}\lfloor_U$ and $\partial_i \in \mathcal{T}_{\mathcal{M}}\lfloor_U$, satisfying the **Clifford-Weyl algebra**: $[x_i, x_j] = 0$, $[\partial_i, \partial_j] = 0$, $[x_i, \partial_j] = \delta_{ij}$. One has

$$U \longmapsto \mathcal{D}_{\mathcal{M}}(U) := \{D_U \text{ is a differential operator on } \mathcal{O}_{\mathcal{M}}(U)\}.$$

with $D_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_{\mathcal{M}}\lfloor_U \partial_x^\alpha$ where $\partial_x^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$.

## Definition (Left/Right $\mathcal{D}_{\mathcal{M}}$-Modules)

We say that a sheaf $\mathcal{E}$ is a sheaf of *left/right $\mathcal{D}_{\mathcal{M}}$*-modules (a $\mathcal{D}_{\mathcal{M}}$-module, for short) if $\mathcal{E}(U)$ is endowed with a *left/right $\mathcal{D}_{\mathcal{M}}(U)$*-module structure for any open set $U$, which is compatible with the restriction morphisms.

# Universal de Rham Complex I

## Definition (Universal de Rham Sheaf of $\mathcal{M}$)

Given a supermanifold $\mathcal{M}$, we call the sheaf $\Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}}$ the universal de Rham sheaf of $\mathcal{M}$.

## Definition (The Operator $D$)

For $\omega \otimes F \in \Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}}$ such that $\omega$ and $F$ are homogeneous, we let $D$ be the operator

$$
\begin{aligned}
D : \Omega^\bullet_{\mathcal{M}} \otimes_{\mathbb{K}} \mathcal{D}_{\mathcal{M}} &\longrightarrow \Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}} \\
\omega \otimes F &\longmapsto D(\omega \otimes F) := d\omega \otimes F + \textstyle\sum_a (-1)^{|\omega||x_a|} dx_a \omega \otimes \partial_{x_a} \cdot F,
\end{aligned}
$$

where the index $a$ runs over all of the even and odd coordinates.

# Universal de Rham Complex II

## Lemma (Properties of $D$)

*The operator $D$ has the following properties:*

1. *it is globally well-defined i.e. it is invariant under generic change of coordinates;*
2. *it is $\mathcal{O}_{\mathcal{M}}$-defined, i.e. it induces an operator $D : \Omega^{\bullet}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}} \to \Omega^{\bullet}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}}$;*
3. *it is nilpotent, i.e. $D^2 = 0$.*

## Theorem (Homology of the Universal de Rham Complex)

*Let $\mathcal{M}$ be a supermanifold and let $(\Omega^{\bullet}_{\mathcal{M},odd}, D)$ be the universal de Rham complex of $\mathcal{M}$. Then there exists a canonical isomorphism*

$$H_{\bullet}(\Omega^{\bullet}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}}, D) \cong \mathcal{B}er(\mathcal{M}),$$

*where $\mathcal{B}er(\mathcal{M})$ is the Berezinian sheaf of $\mathcal{M}$.*

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## Nods to the Proof

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$$H(dx^I \otimes \partial^J f) := \sum_a (-1)^{|x_a|(|\omega|+|\partial^J|+1)} \partial_{dx_a} dx^I \otimes [\partial^J, x_a] f.$$

- the homotopy fails on $[dz_1 \dots dz_p \otimes \partial_{\theta_1} \dots \partial_{\theta_q}]$, which generates $\mathcal{B}er(\mathcal{M})$.

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## Corollary ($\mathcal{B}er(\mathcal{M})$ is a Right $\mathcal{D}_{\mathcal{M}}$-Module / Lie Derivative)

*Let $\mathcal{M}$ be a supermanifold. The right action*

$$H_\bullet(\Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}}, D) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}} \longrightarrow H_\bullet(\Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}}, D)$$

*is uniquely characterized by $\mathcal{D}(x) \cdot \partial_a := [dz_1 \ldots dz_p \otimes \partial_{\theta_1} \otimes \ldots \partial_{\theta_q}] \cdot \partial_a = 0$ for any $a$, and it is given by the Lie derivative on $\mathcal{B}er(\mathcal{M})$.*

# Lie Derivative and Right $\mathcal{D}_{\mathcal{M}}$-modules

**Theorem ($\mathcal{D}_{\mathcal{M}}$-Modules and $\mathcal{O}_{\mathcal{M}}$-Modules)**

*$\mathcal{E}$ is a sheaf of right $\mathcal{D}_{\mathcal{M}}$-module on $\mathcal{M}$ $\Longleftrightarrow$ $\mathcal{E}$ is a sheaf of $\mathcal{O}_{\mathcal{M}}$-modules endowed with a* **flat right connection** $\Delta_R$.

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## Definition ((Flat) Right Connection on $\mathcal{E}$)

A **right connection** on $\mathcal{E}$ a $\mathbb{C}$-bilinear morphism $\Delta_R : \mathcal{E} \otimes_{\mathbb{C}} \mathcal{D}^{(1)}_{\mathcal{M}} \to \mathcal{E}$ such that the following are satisfied for any $f \in \mathcal{O}_{\mathcal{M}}$, $X \in \mathcal{T}_{\mathcal{M}}$ and $e \in \mathcal{E}$ :

1. $\Delta_R(e \otimes f) = ef$;
2. $\Delta_R(e \otimes X \circ f) = \Delta_R(e \otimes X)f$; (Leibniz rule)
3. $\Delta_R(e \otimes fX) = \Delta_R(ef \otimes X)$,

where by definition $X \circ f := X(f) + (-1)^{|X||f|} f \circ X$. A right connection is **flat** if

$$\Delta_R(- \otimes [X, Y]) = [\Delta_R(- \otimes X), \Delta_R(- \otimes Y)].$$

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## The Canonical and Berezinian Sheaf are Right $\mathcal{D}_{\mathcal{M}}$-Modules

- If $\omega^{top}$ is a section of $\mathcal{K}_M$, then one poses $\Delta_R(\omega^{top} \otimes X) = -\mathcal{L}_X(\omega)$.

# Lie Derivative and Right $\mathcal{D}_{\mathcal{M}}$-modules

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1. $\Delta_R(e \otimes f) = ef$;
2. $\Delta_R(e \otimes X \circ f) = \Delta_R(e \otimes X)f$;
3. $\Delta_R(e \otimes fX) = \Delta_R(ef \otimes X)$,

where by definition $X \circ f := X(f) + (-1)^{|X||f|} f \circ X$. A right connection is **flat** if

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## The Canonical and Berezinian Sheaf are Right $\mathcal{D}_{\mathcal{M}}$-Modules

- If $\omega^{top}$ is a section of $\mathcal{K}_M$, then one poses $\Delta_R(\omega^{top} \otimes X) = -\mathcal{L}_X(\omega)$.
- If $\mathcal{D}$ is a section of $\mathcal{B}er(\mathcal{M})$, then one poses $\Delta_R(\mathcal{D} \otimes X) = -\mathcal{L}_X(\mathcal{D})$

# Lie Derivative and Right $\mathcal{D}_{\mathcal{M}}$-modules

## Ordinary Manifolds

$$0 \longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow \Omega^1_M \longrightarrow \dots \longrightarrow \Omega^{\dim M}_M \longrightarrow 0$$

- *left* and *right* $\mathcal{D}_M$-module in the same complex

## Supermanifolds

**Differential Forms**: $0 \longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow \Omega^1_M \longrightarrow \dots$

**Integral Forms**: $\dots \longrightarrow \mathcal{B}er(\mathcal{M}) \otimes \Pi \mathcal{T}_{\mathcal{M}} \longrightarrow \mathcal{B}er(\mathcal{M}) \longrightarrow 0$

- *left* $\mathcal{D}_{\mathcal{M}}$-module $\rightsquigarrow$ differential forms;
- *right* $\mathcal{D}_{\mathcal{M}}$-module $\rightsquigarrow$ integral forms.

# Universal Spencer Complex I

## Definition (Universal Spencer Sheaf of $\mathcal{M}$)

Given a supermanifold $\mathcal{M}$, we call the sheaf $\mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M}})^* \cong \mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{S}^{\bullet}\Pi\mathcal{T}_{\mathcal{M}}$ the universal Spencer sheaf of $\mathcal{M}$.

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## Definition (The Operator $\delta$)

For $F \otimes \tau \in \mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M}})^*$ such that $F$ and $\tau$ are homogeneous, we let $\delta$ be the operator

$$F \otimes \tau \longmapsto \delta(F \otimes \tau) := (-1)^{|\tau|} F \textstyle\sum_a \partial_a \otimes \langle dx_a, \tau \rangle - (-1)^{|\tau|} F \otimes \mathfrak{e}_x(\tau)$$

where the index $a$ runs over all of the even and odd coordinates.

## Definition (Operator $\mathfrak{e}_x$)

Let $\tau \in (\Omega^{\bullet}_{\mathcal{M}})^*$. We define the operator $\mathfrak{e}_x$ to be such that

$$\tau \longmapsto \mathfrak{e}_x(\tau) := \textstyle\sum_a \langle dx_a, \mathfrak{L}_{\partial_a}(\tau) \rangle$$

# Universal Spencer Complex II

## Lemma (Properties of $\delta$)

*The operator $\delta$ has the following properties:*

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2. *it is $\mathcal{O}_{\mathcal{M}}$-defined, i.e. it induces an operator $\delta : \mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M}})^* \to \mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M}})^*$;*
3. *it is nilpotent, i.e. $\delta^2 = 0$.*

## Theorem (Homology of Universal Spencer Complex)

*Let $\mathcal{M}$ be a supermanifold and let $\mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M}})^*$ be its universal Spencer complex. There exists a canonical isomorphism*

$$H_{\bullet}(\mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M}})^*, \delta) \cong \mathcal{O}_{\mathcal{M}}.$$

# Universal Spencer Complex III

## Nods to the Proof

- Construct a homotopy $K$ for $\delta$, that is we need $\delta K + K\delta = 1$

$$K(F \otimes \tau) = (-1)^{|\tau|} \sum_a [F, x_a] \otimes \pi\partial_a \cdot \tau.$$

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- the homotopy fails on functions, that is on $\mathcal{O}_{\mathcal{M}}$.

## What Did We Get?

- $H_\bullet(\Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}}, D) \cong \mathcal{B}er(\mathcal{M})$ ($\rightsquigarrow$ right $\mathcal{D}_{\mathcal{M}}$-module)
- $H_\bullet(\mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^\bullet)^*_{\mathcal{M}}, \delta) \cong \mathcal{O}_{\mathcal{M}}$ ($\rightsquigarrow$ left $\mathcal{D}_{\mathcal{M}}$-module)

...we now aim at relating the two constructions!

# Virtual Forms Double Complex

## Definition (Sheaf of Virtual Forms)

Let $\mathcal{M}$ be a supermanifold, we call $\Omega^{\bullet}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M}})^*$ the sheaf of virtual forms.

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## Corollary

*Let $\mathcal{M}$ be a supermanifold and let $D \otimes 1$ and $1 \otimes \delta$ act on the sheaf of virtual superforms. Then $D \otimes 1$ and $1 \otimes \delta$ commute with each other, i.e.*

$$[1 \otimes \delta, D \otimes 1] := (1 \otimes \delta) \circ (D \otimes 1) - (D \otimes 1) \circ (1 \otimes \delta) = 0.$$

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## Definition (The operators $\hat{d}$ and $\hat{\delta}$ on Virtual Forms)

Let $\omega \otimes F \otimes \tau \in \Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{D}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} (\Omega^\bullet_{\mathcal{M}})^*$ be a virtual superform. We define:

$$\hat{d}(\omega \otimes F \otimes \tau) := (D \otimes 1)(\omega \otimes F \otimes \tau),$$

$$\hat{\delta}(\omega \otimes F \otimes \tau) := (-1)^{|\omega|+|F|+|\tau|}(1 \otimes \delta)(\omega \otimes F \otimes \tau).$$

# Virtual Forms Double Complex

## Definition (Virtual Superforms Double Complex)

We define ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}} := (\Omega^{\bullet}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M}})^{*}, \hat{d}, \hat{\delta})$ to be the virtual forms double complex. We define the bi-degrees of the double complex so that the differential $\hat{d}$ moves *vertically* and $\hat{\delta}$ moves *horizontally*.

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## Definition (Spectral Sequences $E^{\Omega}_r$ and $E^{\Sigma}_r$)

Let ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}}$ be the virtual superform double complex of $\mathcal{M}$. We call

1. $(E^{\Omega}_r, d^{\Omega}_r)$ the spectral sequence w.r.t. the *vertical* filtration;
2. $(E^{\Sigma}_r, d^{\Sigma}_r)$ the spectral sequence w.r.t. the *horizontal* filtration.

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## Definition (Spectral Sequences $E_r^{\Omega}$ and $E_r^{\Sigma}$)

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2. $(E_r^{\Sigma}, d_r^{\Sigma})$ the spectral sequence w.r.t. the *horizontal* filtration.

## Definition (Čech-Virtual Forms Triple Complex)

Let $\mathcal{M}$ be a supermanifold and $\mathcal{U}$ an open cover of $\mathcal{M}$. We call the triple complex ${}_{\mathcal{T}}\mathcal{V}^{\bullet\bullet\bullet}_{\mathcal{M}} := (\check{C}^{\bullet}(\mathcal{U}, \Omega^{\bullet}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M},odd})^*, \check{\delta}, \hat{d}, \hat{\delta})$ Čech-virtual superforms triple complex or Čech-virtual superforms complex for short.

# Virtual Forms Double Complex

**Theorem (Differential Forms from ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}}$)**

*Let $\mathcal{M}$ be a supermanifold and let $(E^{\Omega}_r, d^{\Omega}_r)$ be the spectral sequence of the double complex ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}}$ defined as above. Then*

1. $E^{\Omega}_1 \cong \Omega^{\bullet}_{\mathcal{M}}$;

# Virtual Forms Double Complex

**Theorem (Differential Forms from ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}}$)**

*Let $\mathcal{M}$ be a supermanifold and let $(E_r^{\Omega}, d_r^{\Omega})$ be the spectral sequence of the double complex ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}}$ defined as above. Then*

1. $E_1^{\Omega} \cong \Omega^{\bullet}_{\mathcal{M}}$;
2. $E_2^{\Omega} = E_{\infty}^{\Omega} \cong \mathbb{K}_{\mathcal{M}}$, *for $\mathcal{M}$ real or complex supermanifold*

*where $\mathbb{K}_{\mathcal{M}}$ is the constant sheaf valued in the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).*

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**Theorem (Integral Forms from ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}}$)**

*Let $\mathcal{M}$ be a supermanifold and let $(E_r^{\Sigma}, d_r^{\Sigma})$ be the spectral sequence of the double complex ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}}$ defined as above. Then*

1. $E_1^{\Sigma} \cong \mathcal{B}er(\mathcal{M}) \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M},odd})^{*}$;

# Virtual Forms Double Complex

**Theorem (Differential Forms from ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}}$)**

*Let $\mathcal{M}$ be a supermanifold and let $(E_r^{\Omega}, d_r^{\Omega})$ be the spectral sequence of the double complex ${}_{\mathcal{D}}\mathcal{V}^{\bullet\bullet}_{\mathcal{M}}$ defined as above. Then*

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*where $\mathbb{K}_{\mathcal{M}}$ is the constant sheaf valued in the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).*

# Poincaré Lemmas for Diff. & Int. Forms

## Theorem (Poincaré Lemma for Diff. Forms)

*The de Rham complex $\Omega^\bullet_{\mathcal{M}}$ is a right resolution of $\mathbb{K}_{\mathcal{M}}$, i.e.*

$$H^k_d(\Omega^\bullet_{\mathcal{M}}) \cong \begin{cases} \mathbb{K}_{\mathcal{M}} & k = 0 \\ 0 & k > 0. \end{cases}$$

*In particular, any closed form is locally exact.*

...exactly the same as over an *ordinary* manifold! The homotopy can be constructed in the same way:

$$h(\omega) = \int_{t=0}^{t=1} dt \, (\iota_{\partial_t} G^*_t(\omega)) \in \Omega^{k-1}_{\mathcal{M}}.$$

# Poincaré Lemmas for Diff. & Int. Forms

## Theorem (Poincaré Lemma for Int. Forms)

*The cohomology of the complex of integral forms $\Sigma^\bullet_{\mathcal{M}}$ is given by*

$$H^k_\delta(\Sigma^{p-\bullet}_{\mathcal{M}}) \cong \begin{cases} \mathbb{K}_{\mathcal{M}} & k = 0 \\ 0 & k \neq 0. \end{cases}$$

*In particular, $H^0_\delta(\Sigma^{p-\bullet}_{\mathcal{M}})$ is generated by the section $s_0 = \mathcal{D}\,\theta_1 \ldots \theta_q \otimes \pi\partial_{x_1} \ldots \pi\partial_{x_p}$, where $\mathcal{D}$ is a generating section of the Berezinian sheaf.*

Once again the homotopy involves and integral!

$$h(\mathcal{D}f \otimes F) := (-1)^{|f|+|F|}\mathcal{D}\sum_b (-1)^{|f|(|x_b|+1)} \left( \int_{t=0}^{t=1} dt\, t^{Q_s} x_b G_t^* f \right) \otimes \pi\partial_b F,$$

# Triple Complex and Equivalence of Cohomologies

## What is the Triple Complex good for?

Using the Čech-Virtual superforms complex, one can prove prove that differential forms and integral forms computes exactly the same topological invariants related to a real supermanifold $\mathcal{M}$, namely the (co)homology $\check{H}^\bullet(\mathcal{M}, \mathbb{R}_\mathcal{M})$.
We define

$$H^\bullet_{d\mathcal{R}}(\mathcal{M}) := H_d(\Omega^\bullet_\mathcal{M}(\mathcal{M})) \quad \text{and} \quad H^\bullet_{\mathcal{S}p}(\mathcal{M}) := H_\delta(\Sigma^\bullet_\mathcal{M}(\mathcal{M})).$$

## Theorem (Equivalence of Cohomology of Diff. & Int. Forms)

*Let $\mathcal{M}$ be a real supermanifold. The cohomology of differential forms $H^\bullet_{d\mathcal{R}}(\mathcal{M})$ and the cohomology of integral forms $H^\bullet_{\mathcal{S}p}(\mathcal{M})$ are isomorphic:*

$$H^\bullet_{d\mathcal{R}}(\mathcal{M}) \cong \check{H}^\bullet(\mathcal{M}, \mathbb{R}_\mathcal{M}) \cong H^\bullet_{\mathcal{S}p}(\mathcal{M}).$$

*In particular the complex of differential and integral forms are quasi-isomorphic.*

# Where do we go from here?

$$\hat{d} \circlearrowright \; \Omega^{\bullet}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{D}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} (\Omega^{\bullet}_{\mathcal{M}})^{*} \; \circlearrowleft \hat{\delta}$$

# Where do we go from here?

$$\hat{d} \circlearrowright \; \Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{D}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} (\Omega^\bullet_\mathcal{M})^* \; \circlearrowleft \hat{\delta}$$

## "De-Quantization" (?)

$$\Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{D}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} (\Omega^\bullet_\mathcal{M})^* \overset{\textit{de-quantization}}{\rightsquigarrow} \Omega^\bullet_{\mathcal{M}} \otimes_{\mathcal{O}_\mathcal{M}} S^\bullet \mathcal{T}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} (\Omega^\bullet_\mathcal{M})^*.$$

## ...geometry?

**Is this "approximation" canonically related to any geometrical object?**

# Where do we go from here?

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## ...geometry?

**Is this "approximation" canonically related to any geometrical object?**

There exists a natural construction on a (super)manifold leading to this approximation?

# Total Space of $\Omega^1_{\mathcal{M}}$ as a Supermanifold

## Definition (Total Cotangent / BV Supermanifold)

Let $\mathcal{M}$ be a supermanifold. We set $(X, \mathcal{O}_X)$ with

$$X := \mathrm{Tot}(\Omega^1_{\mathcal{M}}) \qquad \mathcal{O}_X := (\Omega^\bullet_{\mathcal{M}})^*$$

This means that functions on $X$ are polynomial functions on the fibers $\mathcal{T}^*_{\mathcal{M},x}$.
If $(U, x_a := \chi_i|\vartheta_\alpha)$ is a local chart $\mathcal{M}$, then $(\pi^{-1}(U), x_a, p_a := q_\alpha|p_i)$ is a chart for $X$ with

$$p_a := (-1)^{|x_a|}\partial_{dx_a}.$$

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| $X$ | *even* | *odd* |
|---|---|---|
| *base coordinates* | $\chi_i \leftrightsquigarrow$ *fields* | $\vartheta_\alpha \leftrightsquigarrow$ *ghosts* |
| *fiber coordinates* | $q_\alpha \leftrightsquigarrow$ *anti-ghosts* | $p_i \leftrightsquigarrow$ *anti-fields* |

## Forms on X: the sheaf $\Omega^1_X$

On $X = (\mathrm{Tot}(\Omega^1_{\mathcal{M}}), (\Omega^\bullet_{\mathcal{M}})^*)$ we have the following canonical exact sequence

$$0 \longrightarrow \pi^*\Omega^1_{\mathcal{M}} \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/\mathcal{M}} \longrightarrow 0.$$

# Geometry of Forms on $X$

## Forms on X: the sheaf $\Omega^1_X$

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$$dx_a = dz_b \left(\frac{\partial x_a}{\partial z_b}\right), \quad dp_a = \left(\frac{\partial z_b}{\partial x_a}\right) dq_b + (-1)^{|x_a|+|z_b|} d\left(\frac{\partial z_b}{\partial x_a}\right) q_b.$$

- The transition functions shows that

$$0 \longrightarrow \pi^*\Omega^1_{\mathcal{M}} \longrightarrow \Omega^1_X \longrightarrow \pi^*\mathcal{T}_{\mathcal{M}} \longrightarrow 0.$$

- For $\mathcal{M}$ smooth, this is (non-canonically) split so that $\Omega^1_X \cong \pi^*\Omega^1_{\mathcal{M}} \oplus \pi^*\mathcal{T}_{\mathcal{M}}$.

$$\Omega^\bullet_X \cong S^\bullet_{\mathcal{O}_X}(\Omega^1_X) \cong \Omega^\bullet_{\mathcal{M}} \otimes S^\bullet\mathcal{T}_{\mathcal{M}} \otimes (\Omega^\bullet_{\mathcal{M}})^*$$

# Odd Symplectic Forms on Total Cotangent Space

## Definition (Odd Symplectic Forms on $X$)

We call $\omega := \sum_a dx_a dp_a \in (\Omega^2_X)_1$ the odd symplectic form of the supermanifold $X$.

## Observation

The odd symplectic form is **invariant** and **nilpotent**:

$$\eta := (-1)^{|x_a|+1} dx_a p_a \quad \rightsquigarrow \quad \omega = d\eta$$

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## Lemma ($d$ commutes with $\omega$)

*Let $d$ and $\omega$ be the de Rham differential and the multiplication by the odd symplectic form, then $[\, d, \omega \,] = 0$. In particular the triple $(\Omega^\bullet_X, \omega, d)$ is a double complex, called the* **deformed** *de Rham complex.*

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This observation has been first made by Ševera (Lett.Math.Phys. **78** 55 (2006))

# Odd Symplectic Forms on Total Cotangent Space

## Theorem (Integral Forms & BV Laplacian)

*Let $E_i^\omega$ the spectral sequence of the deformed de Rham complex. Then*

(1) *the first page of the spectral sequence $E_i^\omega$ is isomorphic to the integral forms on $\mathcal{M}$, i.e.*

$$E_1^\omega \cong \pi^* \mathcal{B}er(\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} (\Omega^\bullet_\mathcal{M})^*;$$

*In turn, this is isomorphic to semi-densities on $X$, i.e. $E_1^\omega \cong \mathcal{D}ens^{1/2}(X)$.*

(2) *the second differential $\delta_2$ of the spectral sequence $E_i^\omega$ is zero. In particular the second page of the spectral sequence is $E_2^\omega = E_1^\omega$.*

(3) *the third differential $\delta_3$ of the spectral sequence $E_i^\omega$ is*

$$\Delta_2^{\mathcal{BV}}(\mathcal{D} \otimes f) := \mathcal{D} \otimes \sum_a \frac{\partial^2}{\partial x_a \partial p_a} f$$

*In particular, the spectral sequence converges at page three and we have $E_3^\omega \cong \mathbb{R}_\mathcal{M} \cong E_\infty^\omega$. A representative of this homology class if given by*

$$[dx_1 \dots dx_n \otimes dp_{n+1} \dots dp_{n+m}] \otimes x_{n+1} \dots x_{n+m} p_1 \dots p_n \in \pi^* \mathcal{B}er(\mathcal{M}) \otimes (\Omega^n_\mathcal{M})^*.$$

# Integral Forms and BV Laplacians

**The BV Laplacian acting on semi-densities reflects the action of the Spencer differential $\hat{\delta}$ acting on integral forms.**

# Total Cotangent Space and Complex Supermanifolds

Let us consider $\mathcal{M}$ a complex supermanifold and look at forms on $X = \mathrm{Tot}(\Omega^1_{\mathcal{M}})$

$$0 \longrightarrow \pi^* \Omega^1_{\mathcal{M}} \longrightarrow \Omega^1_X \longrightarrow \pi^* \mathcal{T}_{\mathcal{M}} \longrightarrow 0.$$

Splitting of this exact sequence is controlled by $Ext^1_X(\pi^* \mathcal{T}_{\mathcal{M}}, \pi^* \Omega^1_{\mathcal{M}})$.

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## Theorem (Projection to $\mathcal{M}$)

*Let $\mathcal{M}$ be a complex supermanifold, let $X$ as above and let $\pi : X \to \mathcal{M}$ be the projection map. Then one has the following natural isomorphism*

$$Ext^1_X(\pi^*\mathcal{T}_{\mathcal{M}}, \pi^*\Omega^1_{\mathcal{M}}) \cong H^1(\mathcal{M}, \mathcal{T}^*_{\mathcal{M}} \otimes \mathcal{E}nd(\mathcal{T}_{\mathcal{M}})).$$

*Cocycles are elements of the form $-(dg_{ij})g_{ij}^{-1}$, for $g_{ij}$ transition functions of $\mathcal{T}_{\mathcal{M}}$.*

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## Super Atiyah Class

$$\begin{aligned} \mathfrak{At}_{\mathcal{T}_{\mathcal{M}}} : H^0(\mathcal{M}, \mathcal{E}nd(\mathcal{T}_{\mathcal{M}})) &\longrightarrow H^1(\mathcal{M}, \mathcal{T}^*_{\mathcal{M}} \otimes \mathcal{E}nd(\mathcal{T}_{\mathcal{M}})) \\ id_{\mathcal{T}^*_{\mathcal{M}} \otimes \mathcal{T}_{\mathcal{M}}} &\longmapsto \mathfrak{At}(\mathcal{T}_{\mathcal{M}}) := \delta(id_{\mathcal{T}^*_{\mathcal{M}} \otimes \mathcal{T}_{\mathcal{M}}}). \end{aligned}$$

# Total Cotangent Space and Complex Supermanifolds

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- Representatives are again given by $-(dg_{ij})g_{ij}^{-1}$;
- $\mathfrak{At}(\mathcal{T}_{\mathcal{M}}) = 0$ if and only if $\mathcal{M}$ admits an affine connection (same as usual!).

# Total Cotangent Space and Complex Supermanifolds

## Theorem (Projection to $\mathcal{M}$)

*Let $\mathcal{M}$ be a complex supermanifold, let $X$ as above and let $\pi : X \to \mathcal{M}$ be the projection map. Then one has the following natural isomorphism*

$$Ext^1(\pi^* \mathcal{T}_{\mathcal{M}}, \pi^* \Omega^1_{\mathcal{M}}) \cong H^1(\mathcal{M}, \mathcal{T}^*_{\mathcal{M}} \otimes \mathcal{E}nd(\mathcal{T}_{\mathcal{M}})).$$

*In particular, the extension*

$$0 \longrightarrow \pi^* \Omega^1_{\mathcal{M}} \longrightarrow \Omega^1_X \longrightarrow \pi^* \mathcal{T}_{\mathcal{M}} \longrightarrow 0$$

*is split if and only if $\mathfrak{At}(\mathcal{T}_{\mathcal{M}})$ is trivial if and only if $\mathcal{M}$ admits an affine connection.*

# Total Cotangent Space and Complex Supermanifolds

## Theorem (Projection to $\mathcal{M}$)

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**...this is related to $\mathcal{M}$ being "split"...**

# Total Cotangent Space and Complex Supermanifolds

## Structural Exact Sequence for $\mathcal{M}$

$$0 \longrightarrow \mathcal{J}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}_{red}} \longrightarrow 0.$$

# Total Cotangent Space and Complex Supermanifolds

## Structural Exact Sequence for $\mathcal{M}$

$$0 \longrightarrow \mathcal{J}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}_{red}} \longrightarrow 0.$$

1. $\mathcal{J}_{\mathcal{M}}$, the *nilpotent sheaf*, is a sheaf of ideals generated by all the nilpotent sections in $\mathcal{O}_{\mathcal{M}}$.
2. $\mathcal{F}_{\mathcal{M}} := \mathcal{J}_{\mathcal{M}} / \mathcal{J}^2_{\mathcal{M}}$, the *fermionic sheaf*, is a locally-free sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules.

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## Projected and Split Supermanifolds

1. $\mathcal{M}$ is *projected* if $\mathcal{O}_{\mathcal{M}} \cong \mathcal{O}_{\mathcal{M}_{red}} \oplus \mathcal{J}_{\mathcal{M}} \leftrightsquigarrow$ the above sequence splits;
2. $\mathcal{M}$ is *split* if $\mathcal{O}_{\mathcal{M}} \cong \wedge^{\bullet}_{\mathcal{M}_{red}} \mathcal{F}_{\mathcal{M}} \leftrightsquigarrow \mathcal{O}_{\mathcal{M}}$ is a sheaf of exterior $\mathcal{O}_{\mathcal{M}_{red}}$-algebras.

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## Obstruction Theory (to Splitting $\mathcal{M}$)

$$\text{Even Obstructions} : H^1(\mathcal{M}, \mathcal{H}om_{\mathcal{O}_{\mathcal{M}_{red}}}(\wedge^{2i+2}\mathcal{F}^*_{\mathcal{M}}, \mathcal{T}_{\mathcal{M}_{red}}))$$

$$\text{Odd Obstructions} : H^1(\mathcal{M}, \mathcal{H}om_{\mathcal{O}_{\mathcal{M}_{red}}}(\wedge^{2i+1}\mathcal{F}^*_{\mathcal{M}}, \mathcal{F}^*_{\mathcal{M}}))$$

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## Obstruction Theory

$$\text{Fundamental Obstruction:} \quad H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}_{red}} \otimes_{\mathcal{O}_{\mathcal{M}_{red}}} \wedge^2 \mathcal{F}_{\mathcal{M}}))$$

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## A Non-Projected Supermanifold

$$X^2 + Y^2 + Z^2 + \Theta_1 \Theta_2 = 0 \quad \subseteq \quad \mathbb{CP}^{2|2}$$

# Connections and Splittings

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## Theorem (Donagi & Witten (2014))

*The restriction $\mathcal{T}_{\mathcal{M}}\lfloor_{\mathcal{M}_{red}}$ of the tangent sheaf to $\mathcal{M}_{red}$ induces the following decomposition*

$$\mathfrak{At}(\mathcal{T}_{\mathcal{M}})\lfloor_{\mathcal{M}_{red}} = \mathfrak{At}(\mathcal{T}_{\mathcal{M}_{red}}) \oplus \omega_{\mathcal{M}} \oplus \mathfrak{At}(\mathcal{J}_{\mathcal{M}}/\mathcal{J}^2_{\mathcal{M}}).$$

*where:*

1. $\mathfrak{At}(\mathcal{T}_{\mathcal{M}_{red}}) \in H^1(\mathcal{M}, \mathcal{H}om(S^2\mathcal{T}_{\mathcal{M}_{red}}, \mathcal{T}_{\mathcal{M}_{red}})) =$ *the Atiyah class of* $\mathcal{T}_{\mathcal{M}_{red}}$*;*
2. $\omega_{\mathcal{M}} \in H^1(\mathcal{M}, \mathcal{T}_{\mathcal{M}_{red}} \otimes \wedge^2 \mathcal{F}_{\mathcal{M}}) =$ **first obstruction class to splitting** $\mathcal{M}$*;*
3. $\mathfrak{At}(\mathcal{F}_{\mathcal{M}}) \in H^1(\mathcal{M}, \mathcal{H}om(\mathcal{T}_{\mathcal{M}_{red}} \otimes \mathcal{F}_{\mathcal{M}}, \mathcal{F}_{\mathcal{M}})) =$ *Atiyah class of* $\mathcal{F}_{\mathcal{M}}$*.*

# Total Cotangent Space and Complex Supermanifolds

## Theorem

*The extension exact sequence for* $\Omega^1_X$

$$0 \longrightarrow \pi^*\Omega^1_{\mathcal{M}} \longrightarrow \Omega^1_X \longrightarrow \pi^*\mathcal{T}_{\mathcal{M}} \longrightarrow 0$$

*is split if and only if* $\mathfrak{At}(\mathcal{T}_{\mathcal{M}})$ *is trivial if and only if* $\mathcal{M}$ *admits an affine connection.*

*For this to be true, necessary conditions to be satisfied are:*

- $\mathcal{T}_{\mathcal{M}_{red}}$ *admits a holomorphic connection;*
- $\mathcal{F}_{\mathcal{M}}$ *admits a holomorphic connection;*
- $\mathcal{M}$ *is split.*

## That's a lot to ask for..

A class of complex supermanifolds satisfying the above are complex Lie supergroups.

# Comments & Outlooks

1. For a complex (compact Kähler) manifold $\mathfrak{At}(\mathcal{T}_X)$ contains informations on all Chern classes $c_k(X)$ : if $\mathfrak{At}(\mathcal{T}_X) = 0$ then $c_k(\mathcal{T}_X) = 0$ for every $k$.

$\mathfrak{At}(\mathcal{T}_{\mathcal{M}})$ contains informations on all the Chern classes of $\mathcal{T}_{\mathcal{M}_{red}}$ and $\mathcal{F}_{\mathcal{M}}$ but also on all the obstructions to splitting $\mathcal{M}$!

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2. Odd symplectic geometry in the holomorphic category?

**Theorem (Schwarz)**

*Every odd symplectic supermanifold $(\mathcal{M}, \omega)$ is globally symplectomorphic to a total cotangent space supermanifold $X$.*

**Couple of Questions...**

- Does this hold true in the holomorphic category?
- Can one find an example of complex supermanifold admitting a (globally defined holomorphic) odd symplectic form which does not fall in Schwarz's class?

# I am done!

**... THANKS A LOT...!**