Symplectic mapping class groups of K3 surfaces.

(X, ω) is a symplectic compact closed manifold.

ω is $d\omega = 0$ + non-degenerate.

the group of symplectomorphisms.

$$\mathrm{Symp}(X, \omega) = \{ f \in \mathrm{Diff}(X) \mid f^* \omega = \omega \}.$$

$$\downarrow$$

$$\pi_k \mathrm{Symp}(X, \omega)$$

$$\downarrow$$

$$\underline{\pi_0 \mathrm{Symp}(X, \omega)}$$

$$\mathrm{Symp}(X, \omega) \to \mathrm{Diff}(X)$$

$$\pi_k \mathrm{Symp}(X, \omega) \to \pi_k \mathrm{Diff}$$

$$\mathrm{Ker}[\pi_0 \mathrm{Symp} \to \pi_0 \mathrm{Diff}] = K(X, \omega)$$

smoothly trivial symplectic mapping class group.

Examples

1. If $\dim X = 2$, that is,

Moser's trick: $\mathrm{Symp}(X,\omega) \simeq \mathrm{Diff}(X)$ $\Rightarrow$ $\pi_0 \mathrm{Symp} = \pi_0 \mathrm{Diff}$

homotopy equivalent

$K(X,\omega)$ is trivial.

2. If $X = \mathbb{P}^2$, and ω is F.-S. form, then

$\mathrm{Symp}(\mathbb{P}^2,\omega) \simeq PU(3)$ is connected $\quad$ $\pi_0 \mathrm{Symp}(\mathbb{P}^2)$ is trivial.

homotopy equivalent

$K(\mathbb{P}^2,\omega)$ is also trivial.

Gromov, the proof is via J-holomorphic curves.

3. Thm (Sheridan-Smith: JCM 18)

There is an algebraic K3 surface (X,ω) such that $K(X,\omega)$ is infinitely generated.

The proof is via Floer theory + mirror symmetry.

Let X be a K3 surface. Choose a polarisation $\kappa \in H^2(X;\mathbb{R})$.
(All K3's are Kähler.)

Associated to κ, we consider the set of roots:

$$\Delta_\kappa = \left\{ \delta \in H_2(X;\mathbb{Z}) \;\middle|\; \delta^2 = -2,\ (\kappa,\delta) = 0 \right\}.$$

Let ω be a Kähler in the class κ, that is, $[\omega] = \kappa \in H^2(X;\mathbb{R})$.

<u>Thm:</u> If Δ_κ is infinite, then $K(X,\omega)$ is infinitely generated.

There is a surjective homomorphism

$$K(X,\omega) \to \bigoplus_{\delta \in \Delta_\kappa} \mathbb{Z}$$

The proof goes via algebraic geometry + Seiberg-Witten invariants.

Four-dimensional Dehn twists.

If (X,ω) is a symplectic manifold with a Lagrangian sphere $L \subset X$

$L \cong S^2$

$\omega|_L = 0$.

there is a symplectomorphism $\tau^2 : (X,\omega) \to (X,\omega)$, squared Dehn twist.

τ^2 is supported in $U(L)$

$\tau^2|_L = id$

Dehn twist can be defined as a monodromy map as follows:

$(\mathbb{CP}^3, \omega)$, and let X be a quartic surface, e.g. $X = \{ z_0^4 + \dots + z_3^4 = 0 \}$.

$\downarrow$

$\mathbb{CP}^N = \{$ the projective space of all quartics $\}$.

the locus of singular quartics.

(it's of complex codimension 1)

Each X_t carries a Kähler form (coming from $\mathbb{CP}^3$).

X_t, using Moser's trick, gives us a monodromy map $\tau^2 : X_0 \to X_0$.

singular quartics.

family of algebraic surfaces.

the (non-smooth) central fiber has a double-point singularity $\Rightarrow$ there is a vanishing cycles

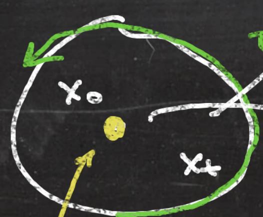
Sketch of the proof: (for quartics)

If (X,ω) is a smooth quartic in $\mathbb{P}^3$, then it has a non-trivial Dehn twist.

Let $\mathcal{X} = \{X_t\}_{t\in\Delta}$ such that if $t \neq 0$, then X_t is a smooth quartic; if $t = 0$, then X_0 is a quartic with a double point.

(Δ the unit disk of $\mathbb{C}$)

τ^2 is non-trivial.



the singular fiber.

We can resolve X_0: There exists a family $\{Y_t\}_{t\in\Delta}$ such that each Y_t is smooth

+

$$\begin{tikzcd} Y_t \arrow[r] \arrow[dr] & X_t \arrow[d] \\ & \Delta - \{0\} \end{tikzcd}$$

there is isomorphism of family $\{Y_t\}_{t\in\Delta\setminus\{0\}}$ and $\{X_t\}_{t\in\Delta-\{0\}}$.

There is a morphism $\mathcal{F}: \{Y_t\}_{t\in\Delta} \to \{X_t\}_{t\in\Delta}$ such that

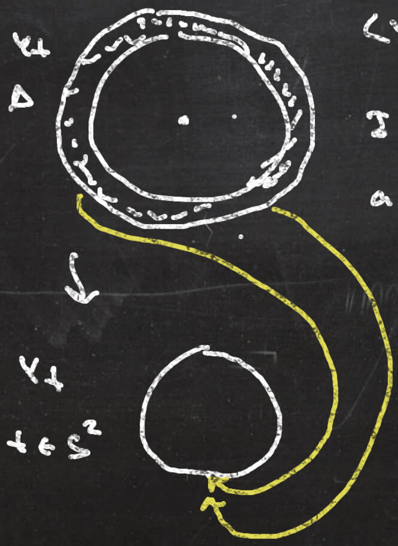
$\mathcal{F}: \{Y_t\}_{t\in\Delta-0} \to \{X_t\}_{t\in\Delta-0}$ is an isomorphism

$+$

$\mathcal{F}: Y_0 \to X_0$ is a contraction of a sphere of self-intersection (-2).

We'll work with $\{Y_t\}$: After the resolution, we choose a Kähler form on Y_0.

$\{Y_t\}_{t\in\Delta}$ is a family of Kähler surfaces but the cohomology class is not constant.



$\{Y_t\}_{t\in\partial\Delta}$ has the same monodromy as $\{X_t\}_{t\in\partial\Delta}$.

If τ^2 is trivial, then we can extend $\{Y_t\}_{t\in\Delta}$ to a bundle over S^2.

We can count holomorphic spheres in the fibers of $\{Y_t\}_{t\in S^2}$.

We count spheres in the homology class of the exceptional (-2)-curve; and the counting gives 1.

If Y_t is a family of Kähler K3's (parametrised by S^2), and $t \in S^2$

$E \in H_2(Y_t;\mathbb{Z})$ with $E^2 = -2$, then

$$FGW(Y_t, E) = \left\{ \begin{array}{c} \text{the number of } J\text{-holomorphic spheres} \\ \text{in the class } E \text{ in the fibres of } Y_t \end{array} \right\}.$$

(w.r.t. to a chosen fibrewise almost-complex structure).

$$\boxed{FGW(Y_t, E) = FGW(Y_t, -E)} \qquad c_1(X) = 0.$$

$\quad \|$ $\qquad\qquad\qquad\qquad$ $\|$ Witten (no Taubes thm. needed.)

$$FSW(Y_t, E) \overset{\text{easy}}{=} FSW(Y_t, -E)$$

$E \quad , \quad \underline{\underline{K - E}}$